Generalization of Some Results on Cordial Graphs

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Abstract. A graph is said to be cordial if it has a 0-1 labeling that satisfies certain properties. The purpose of this paper is to generalize some known theorems and results of cordial graphs. Specifically, we show that certain combinations of paths, cycles, stars and null graph are cordials. Finally, we prove that the tours grids are cordial if and only if its size is not congruent to 2 (mod4).
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1 Introduction

It is well known that graph theory has applications in many other fields of study, including physics, chemistry, biology, communication, psychology, sociology, economics, engineering, operations research, and especially computer science.

One area of graph theory of considerable recent research is that of graph labeling. In a labeling of a particular type, the vertices are assigned values from a given set, the edges have a prescribed induced labeling, and the labelings must satisfy certain properties. An excellent reference on this subject is the survey by Gallian [5].

Two of the most important types of labelings are called graceful and harmonious. Graceful labelings were introduced independently by Rosa [8] in 1966 and Golomb [6] in 1972, while harmonious labelings were first studied by Graham and Sloane [7] in 1980. A third important type of labeling, which contains aspects of both of the other two, is called cordial and was introduced by Cahit [1] in 1990. Whereas the label of an edge vw for graceful and harmonious labeling is given respectively by $|f(v) - f(w)|$ and $f(v) + f(w)$ (modulo the number of edges), cordial labelings use only labels 0 and 1 and the induced label $(f(v) + f(w))$ (mod2), which of course equals $|f(v) - f(w)|$. Because arithmetic modulo2 is an integral part of computer science, cordial labelings have close connections with that field.

More precisely, cordial graphs are defined as follows.
Let $G = (V, E)$ be a graph, let $f : V \to \{0, 1\}$ be a labeling of its vertices, and let $f^* : E \to \{0, 1\}$ is the extension of f to the edges of G by the formula $f^*(vw) = f(v) + f(w)$ (mod 2). (Thus, for any edge e, $f^*(e) = 0$ if its two vertices have the same label and $f^*(e) = 1$ if they have different labels).

Let v_0 and v_1 be the numbers of vertices labeled 0 and 1 respectively, and let e_0 and e_1 be the corresponding numbers of edge. Such a labeling is called cordial if both $|v_0 - v_1| \leq 1$ and $|e_0 - e_1| \leq 1$. A graph is called cordial if it has a cordial labeling.

Seoud, Diab and Elsakhawi [9], determined that the join of the cycle C_n and the star $K_{1,m}$ is cordial for all n and odd m except $n \equiv 3 \pmod 4$ and $(n, m) = (3, 1)$. In section 3, we extend this result to show that the join of the cycle C_n and the star $K_{1,m}$ is cordial for all $n > 3$ and all m except $n \equiv 3 \pmod 4$, if $n \equiv 3 \pmod 4$ and $n > 3$, then the join of the cycle C_n and the star $K_{1,m}$ is cordial for all even m, where $m \geq 2$, and if $n \equiv 3$ (mode 4) and odd m, then the join of the cycle C_n and the star $K_{1,m}$ is not cordial. also, we show that the union of the cycle C_n and the star $K_{1,m}$ is cordial for all n and all m. In section 4, we generalize the result due to Seoud, Diab and Elsakhawi [9], which state that the join of the cycle C_n and the null graph N_m is cordial when n is odd and when n is even and m is odd to the join of the cycle C_n and the null graph N_m is cordial for all m and n except $m \equiv 3 \pmod 4$ and n odd and $m \equiv 2 \pmod 4$ and n even. Also, we show that the union of the cycle C_n and the null graph N_m is cordial for all n and m if and only if n is not congruent to 2(mod 4). In section 5, we show that the join and the union of the graph P_n and the null graph N_m are cordial for all n and all m. Finally, in section 6, we prove that the Tours grid $C_n \times C_n$ is cordial for all $n \geq 3$ and $m \geq 3$ if and only if $2nm$ is not congruent to 2(mod 4).

2 Terminology and notations

We introduce some notation and terminology for a graph with $4r$ vertices [2, 3, 4], we let L_{4r} denote the labeling 00110011...0011, M_r denote the labeling 0101...01 (zero-one repeated r-times) if r is even and 0101...010 (zero-one repeated r-times) if r is odd, O_r denotes the labelling 0000...0000 (zero repeated r- times) and I_r denotes the labelling 111...1111 (one repeated r- times). In most cases, we then modify this by adding symbols at one end or the other (or both). Thus $01L_{4r}$ denotes the labeling 0100110011...0011 of either C_{4r+2} or P_{4r+2}. For specific labeling L and M of $G \cup H$, where G and H are paths or cycles or stars or null graphs, we let $[L; M]$ denote the joint labeling.

Additional notation that we use is the following.
For a given labeling of the join $G + H$, we let v_i and e_i (for i = 0,1) be the numbers of labels that are i as before, we let x_i and a_i be the corresponding quantities for G, and we let y_i and b_i be those for H. It follows that $v_0 = x_0 + y_0, v_1 = x_1 + y_1, e_0 = a_0 + b_0 + x_0 y_0 + x_1 y_1$ and $e_1 = a_1 + b_1 + x_0 y_1 + x_1 y_0$, thus, $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and

$e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1)$. When it comes to the proof, we only need to show that, for each specified combination of labeling, $|v_0 - v_1| \leq 1$ and $|e_0 - e_1| \leq 1$.

3 Joins of Unions of Cycles and Stars

Seoud, Diab and Elsakhawi [9] have proved that the join of the cycle C_n and the star $K_{1,m}$ is cordial for all odd m and n except $n \equiv 3 \pmod 4$ and $(n, m) = (3, 1)$. In this section, we generalize this results as follows:

1- The join of the cycle C_n and the star $K_{1,m}$ is cordial for all $n > 3$ and all m except $n \equiv 3 \pmod 4$,

2- If $n \equiv 3 \pmod 4$ and $n > 3$, then the join of the cycle C_n and the star $K_{1,m}$ is cordial for all even m, where $m \geq 2$,

3- If $n \equiv 3 \pmod 4$ and odd m, then the join of the cycle C_n and the star $K_{1,m}$ is not cordial.

Moreover, we prove that the union of the cycle C_n and the star $K_{1,m}$ is cordial for all n and all m.

Theorem 3.1. The join of the cycle C_n and the star $K_{1,m}$ is cordial for all m and $n > 3$ except $n \equiv 3 \pmod 4$.

Proof. The labelings that we use are given in Table 3.1, along with the corresponding values of x_i and a_i or y_i and b_i (for $i = 0, 1$). We let $m = 2s + j$ (for $j = 0, 1$) and $n = 4r + i$ (for $i = 0, 1, 2$). For given values of i and j with $0 \leq i \leq 2$ and $0 \leq j \leq 1$, we use the labeling A_i, A'_i or A''_i for cycle C_n and B_j, B'_j or B''_j for the star $K_{1,m}$ as given in Table 3.1. Using Table 3.1 and the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1)$, we can compute the values shown in the last two columns of Table 3.2. Since these are all 0,1, or -1, the theorem follows.

$n=4r+i$, $i=0,1,2$	Labeling of C_n	x_0	x_1	a_0	a_1
$i=0$	$A_0=L_{4r}$	$2r$	$2r$	$2r$	$2r$
$i=1$	$A_1=1L_{4r}$	$2r$	$2r+1$	$2r+1$	$2r$
	$A'_1=L_{4r}0$	$2r+1$	$2r$	$2r+1$	$2r$
$i=2$	$A_2=0L_{4r}0$	$2r+2$	$2r$	$2r+2$	$2r$
	$A'_2=01L_{4r}$	$2r+1$	$2r+1$	$2r$	$2r+2$
	$A''_2=0L_{4r}1$	$2r+1$	$2r+1$	$2r+2$	$2r$

$j=2s+j$, $j=0,1$	Labeling of $K_{1,m}$	y_0	y_1	b_0	b_1
$j=0$	$B_0=1M_{2s}$	s	$s+1$	s	s
	$B'_0=0M_{2s}$	$s+1$	s	s	s
	$B''_0=011M_{2s-2}$	s	$s+1$	$s-1$	$s+1$
$j=1$	$B_1=11M_{2s}$	s	$s+2$	$s+1$	s
	$B'_1=01M_{2s}$	$s+1$	$s+1$	s	$s+1$

Table 3.1. Labelings of Cycles and Stars.

$n=4r+i$, $i=0,1,2$	$m=4s+j$, $j=0,1,2,3$	C_n	$K_{1,m}$	v_0-v_1	e_0-e_1
0	0	A_0	B_0	-1	0
1	0	A_1	B'_0	0	0
2	0	A''_2	B''_0	-1	0
0	1	A_0	B'_1	0	-1
1	1	A_1	B'_1	-1	0
2	1	A''_2	B'_1	0	1

Table 3.2. Combinations of labelings.

We note in the Table 3.1, the labelings of stars that the first lable with respect to the center of the star and others labelings respect to vertices of $K_{1,m}$, for example $B_0 = 1M_{2s}$ means that we label the center of $K_{1,m}$ by 1 and others vertices by M_{2s}.

Theorem 3.2. If $n \equiv 3 \pmod 4$ and $n > 3$, then $C_n + K_{1,m}$ is cordial for all even m, where $m \geq 2$.

Proof. Let $n = 4r+3$ and $r \geq 1$, then we label the vertices of C_n as $A_3 = L_{4r}100$, i.e., $x_0 = 2r+2, x_1 = 2r+1, a_0 = 2r+3$ and $a_1 = 2r$. For labelings the vertices of $K_{1,m}$, where $m > 2$ is an even number, we have two cases;

Case 1. $m \equiv 0 \pmod 4$. i.e., $m = 4s$. We label the center of $K_{1,m}$ by 0 and the others vertices by 111 $L_{rs-4}0$, i.e., $y_0 = 2s, y_1 = 2s+1, b_0 = 2s-1$

band $b_1 = 2s+1$. Hence from the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1) = 0$.
Case 2. $m \equiv 2 \pmod 4$. i.e., $m = 4s+2$. We label the center of $K_{1,m}$ by 0 and the others vertices by 1111 $L_{4s-4}00$, i.e., $y_0 = 2s+1, y_1 = 2s+2, b_0 = 2s$ and $b_1 = 2s+2$. Hence from the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1) = 0$. Therefore, we obtain from last cases that $v_0 - v_1 = 0$ and $e_0 - e_1 = 0$. This means that $C_n + K_{1,m}$ is cordial for all even $m \geq 4$.
In case of $m = 2$, we label the vertices of $K_{1,2}$ as 011, i.e., $y_0 = 1, y_1 = 2, b_0 = 0$ and $b_1 = 2$. Hence $v_0 - v_1 = 0$ and $e_0 - e_1 = 0$. This completes the proof of this theorem.
Example 3.1. If $n \equiv 3 \pmod 4$ and $n \geq 3$, then $C_n + P_1$ is not cordial.
Solution. The solution follows directly from the following theorem [5], which state that: "If G is a graph with n vertices and m edges and every vertex has odd degree, then$G$ is not cordial when $n + m = 2 \pmod 4$)".
We note that, if $n = 3$, then $C_3 + P_1 = K_4$ is not cordial.
Lemma 3.1. If $n \equiv 3 \pmod 4$ and odd m, the $C_n + K_{1,m}$ is not cordial.
Proof. The labelings that we use are given in Table 3.3, along with the corresponding values of x_i and a_i or y_i and b_i (for $i = 0, 1$). We let $n = 4r+i$ (for $i = 0, 1, 2, 3$) and $m = 2s + j$ (for $j = 0, 1$). For given values of i and j with $0 \leq i \leq 3$ and $0 \leq j \leq 1$, we use the labeling A_i, A'_i or A''_i for the cycle C_m and B_j, B'_j for the star $K_{1,m}$ as given in Table 3.3. Using Table 3.3 and the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1)$, we can compute the values shown in the last two columns of Table 3.4. Since these are all 0.1, or -1, the theorem follows.

$n = 4r + i$, $i = 0, 1, 2, 3$	Labeling of C_n	x_0	x_1	a_0	a_1
$i = 0$	$A_0 = L_{4r}$	$2r$	$2r$	$2r$	$2r$
$i = 1$	$A_1 = 1L_{4r}$	$2r$	$2r+1$	$2r+1$	$2r$
$i = 2$	$A_2 = L_{4r}10$	$2r+1$	$2r+1$	$2r+2$	$2r$
$i = 3$	$A_3 = L_{4r}001$	$2r+2$	$2r+1$	$2r+1$	$2r+2$

$m = 2s + j$, $j = 0, 1$	Labeling of $K_{1,m}$	y_0	y_1	b_0	b_1
$j = 0$	$B_0 = 1M_{2s}$	s	$s+1$	s	s
	$B'_0 = 0M_{2s}$	$s+1$	s	s	s
	$B''_0 = 011M_{2s-2}$	s	$s+1$	$s-1$	$s+1$
$j = 1$	$B_1 = 11M_{2s}$	s	$s+2$	$s+1$	s
	$B'_1 = 01M_{2s}$	$s+1$	$s+1$	s	$s+1$

Table 3.3. Labelings of Cycles and Stars.

$n=4r+i$, $i=0,1,2,3$	$m=2s+j$, $j=0,1$	C_n	$K_{1,m}$	v_0-v_1	e_0-e_1
0	0	A_0	B_0	-1	0
1	0	A_1	B_0'	0	1
2	0	A_2	B_0''	-1	0
3	0	A_3	B_0	0	-1
0	1	A_0	B_1'	0	-1
1	1	A_1	B_1'	-1	0
2	1	A_2	B_1'	0	1
3	1	A_3	B_1	-1	0

Table 3.4. Combinations of labelings.

4 Joins Unions of Cycles and Null Graphs

Seoud, Diab and Elsakhawi [9], proved that the join of the cycle C_n and the null graph N_m is cordial when n is odd and when n is even and m is odd. In this section, we extend the above result to the following result $C_n + N_m$ is cordial if and only if for all n and all m except when $n \equiv 3 \pmod 4$ and odd m or when $n \equiv 2 \pmod 4$ and even m. Moreover, we prove that the union of the cycle C_n and the null graph N_m is cordial for all n and all m if and only if n is not congruent to 2(mod 4).

Theorem 4.1. If n is not congruent to 2(mod 4), then $C_n + N_m$ is cordial for all n and all m except when $n \equiv 3 \pmod 4$ and odd m.

Proof. The labelings that we use are given in Table 4.1, along with the corresponding values of x_i and a_i or y_i and b_i (for $i = 0,1$). We let $m = 4s+j$ (for $j = 0,1,2,3$) and $n = 4r + i$ (for $i = 0,1,3$). For given values of i and j with $0 \le i \le 3, i \ne 2$ and $0 \le j \le 3$, we use the labeling A_i for the cycle C_n and B_j for the null graph N_m as given in Table 4.1. Using Table 4.1 and the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1)$, we can compute the values shown in the last two columns of Table 4.2. Since these are all 0,1, or -1, the theorem follows.

$n = 4r+i$, $i = 0,1,3$	Labeling of XC_n	x_0	x_1	a_0	a_1
$i=0$	$A_0 = L_{4r}$	$2r$	$2r$	$2r$	$2r$
$i=1$	$A_1 = 1L_{4r}$	$2r$	$2r+1$	$2r+1$	$2r$
$i=3$	$A_3 = L_{4r}011$	$2r+1$	$2r+2$	$2r+1$	$2r+2$

$m = 4s+j$, $j = 0,1,2,3$	Labeling of N_m	y_0	y_1	b_0	b_1
$j=0$	$B_0 = L_{4s}$	$2s$	$2s$	0	0
$j=1$	$B_1 = 0L_{4s}$	$2s+1$	$2s$	0	0
$j=2$	$B_2 = 0L_{4s}1$	$2s+1$	$2s+1$	0	0
$j=3$	$B_3 = 001L_{4s}$	$2s+2$	$2s+1$	0	0

Table 4.1. Labelings of Paths and Null graphs.

$n = 4r+i$, $i = 0,1,3$	$m = 4s+j$, $j = 0,1,2,3$	C_n	N_m	$v_0 - v_1$	$e_0 - e_1$
0	0	A_0	B_0	0	0
0	1	A_0	B_1	1	0
0	2	A_0	B_2	0	0
0	3	A_0	B_3	1	0
1	0	A_1	B_0	-1	1
1	1	A_1	B_1	0	0
1	2	A_1	B_2	-1	1
1	3	A_1	B_3	0	0
3	0	A_3	B_0	-1	-1
3	2	A_3	B_2	-1	-1

Table 4.2. Combinations of labelings.

Theorem 4.2. If $n \equiv 2 \pmod 4$, then $C_n + N_m$ is cordial for all odd m.
Proof. Let $n = 4r+2$ and $r \geq 1$, then we label the vertices of C_n as $A_2 = 0L_{4r}0$, i.e., $x_0 = 2r+2, x_1 = 2r, a_0 = 2r+2$ and $a_1 = 2r$. Let $m = 2s+1$, then we label the vertices of N_m as $1_{s+1}O_s$, i.e., $y_0 = s, y_1 = s+1, b_0 = 0$ and $b_1 = 0$. Hence $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 2 - 1 = 1$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) + (x_0 - x_1)(y_0 - y_1) = 2 - 2 = 0$. Therefore $C_n + \overline{K}_m$ is cordial for all $n = 2 \pmod 4$ and odd m, the theorem follows.
Example 4.1. $C_3 + N_m$ is cordial for all even m.
Proof. The following labeling suffice: $C_3 + N_m : [001 : L_{4s}]$, if $m = 0 \pmod 4)$ and $C_3 + N_m : [001 : 0L_{4s}1]$, if $m = 2 \pmod 4$.
Lemma 4.1. If $n \equiv 2 \pmod 4$ and even m, then $C_n + N_m$ is not cordial.
Proof. It is easy to verify that the graph $C_n + N_m$ is an Eulerian graph, which its size of congruent to 2 (mod 4) and from Cahit's theorem (which

is mentioned in the proof of lemma 3.1), we obtain that $C_n + N_m$is not cordial.

Lemma 4.2. If $n \equiv 3 \pmod 4$ and odd m, then $C_n + N_m$ is not cordial.

Proof. It is easy to verify that the degree of all vertices of the graph $C_n + N_m$ when $n \equiv 3 \pmod 4$ and odd m are odd and $|V| + |E|$ is congruent to 2(mod 4), where $|V|$ is the order of $C_n + N_m$ and $|E|$ is a size of $C_n + N_m$ (which is mentioned in example 3.1). Hence $C_n + N_m$ is not cordial.

Theorem 4.3. The join of the cycle C_n and the null graph N_m is cordial for all n and all m if and only if n is not congruent to 3(mod 4) and odd m, or when n is not congruent to 2(mod 4) and even m.

Proof. The proof follows directly from theorem 4.1, example 4.1 and last lemmas.

Theorem 4.4. The union of the cycle C_n and the null graph N_m is cordial for all n and all m except $n = 2 \pmod 4$.

Proof. Bu using the label of the cycle C_n and the null graph N_m is Table 4.1 of theorem 4.1 and the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) = a_0 - a_1$, we can compute the values shown in the last two columns of Table 4.3. Since these are all 0,1 or -1, the theorem follows

$n = 4r + i$, $i = 0,1,3$	$m = 4s + j$, $j = 0,1,2,3$	C_n	N_m	$v_0 - v_1$	$e_0 - e_1$
0	0	A_0	B_0	0	0
0	1	A_0	B_1	1	0
0	2	A_0	B_2	0	0
0	3	A_0	B_3	1	0
1	0	A_1	B_0	-1	1
1	1	A_1	B_1	0	1
1	2	A_1	B_2	-1	1
1	3	A_1	B_3	0	1
3	0	A_3	B_0	-1	-1
3	1	A_3	B_1	0	-1
3	2	A_3	B_2	-1	-1
3	3	A_3	B_3	0	-1

Table 4.3. Combinations of Labelings.

Lemma 4.4. If $n \equiv 2 \pmod 4$, then $C_n \cup N_m$ is not cordial for all m.

Proof. It is easy to verify that the graph $C_n \cup N_m$ is an Eulerian graph, which has a size of congruent to 2(mod 4) and from Cahit's theorem (which is mentioned in the proof of lemma 4.1), we obtain that $C_n \cup N_m$ is not cordial.

Theorem 4.5. The union of the cycle C_n and the null graph N_m is cordial for all n and all m if and only if n is not to congruent to 2(mod 4).

Proof. The proof follows directly from theorem 6.4 and lemma 6.4.

5 Joins and Union of Paths and Null Graphs

In this section, we prove the join of the path P_n and the null graph N_m is cordial for all n and all m, and the union of the path P_n the null graph N_m is cordial for all n and all m

Theorem 5.1. The join of the P_n and the null graph N_m is cordial for all n and all m.

Proof. The labelings that we use are given in Table 5.1, along with the corresponding values of x_i and a_i or y_i and b_i (for $i = 0, 1$). We let $n = 4r+i$ (for $i = 0, 1, 2, 3$) and $m = 2s + j$ (for $j = 0, 1$). For given values of i and j with $0 \leq i \leq 3$ and $0 \leq j \leq 1$, we use the labelings B_j for the null graph N_m and A_i for the path P_n as given in Table 5.1. Using Table 5.1 and the fact that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (x_0 - x_1)(y_0 - y_1)$, we can compute the values shown in the last two columns of Table 5.2. Since are all 0,1, or -1, the theorem follows.

$m = 2s + j$, $j = 0, 1$	Labeling of N_n	y_0	y_1	b_0	b_1
$j = 0$	$B_0 = O_s I_s$	s	s	0	0
$j = 1$	$B_1 = O_{s+1} I_s$	$s + 1$	s	0	0

$n = 4r + i$, $j = 0, 1, 2, 3$	Labeling of P_n	x_0	x_1	a_0	a_1
$i = 0$	$A_0 = L_{4r}$	$2r$	$2r$	$2r$	$2r - 1$
$i = 1$	$A_1 = 1L_{4r}$	$2r$	$2r + 1$	$2r$	$2r$
$i = 2$	$A_2 = 1L_{4r}0$	$2r + 1$	$2r + 1$	$2r$	$2r + 1$
$i = 3$	$A_3 = L_{4r}011$	$2r + 1$	$2r + 2$	$2r + 1$	$2r + 1$

Table 5.1. Labelings of Paths and Null graphs.

$n=4r+i$, $i=0,1,2,3$	m=2s+j $j=0,1$	P_n	N_m	v_0-v_1	e_0-e_1
0	0	A_0	B_0	0	1
1	0	A_1	B_0	-1	0
2	0	A_2	B_0	0	-1
3	0	A_3	B_0	-1	0
0	1	A_0	B_1	1	1
1	1	A_1	B_1	0	-1
2	1	A_2	B_1	1	-1
3	1	A_3	B_1	0	-1

Table 5.2. Combinations of Labelings.

Theorem 5.2. The union $P_n \cup N_m$ of the path P_n and the null graph N_m is cordial for all n and all m.
Proof. Using the last labels of the path P_n and the null graph N_m in the Table 5.1 in the last theorem and the fact that $v_0-v_1=(x_0-x_1)+(y_0-y_1)$ and $e_0-e_1=(a_0-a_1)$, we can compute the values shown in the last two columns of Table 5.3. Since these are all 0,1, or -1, the theorem follows

$n=4r+i$, $i=0,1,2,3$	m=2s+j $j=0,1$	P_n	N_m	v_0-v_1	e_0-e_1
0	0	A_0	B_0	0	1
1	0	A_1	B_0	-1	0
2	0	A_2	B_0	0	-1
3	0	A_3	B_0	-1	0
0	1	A_0	B_1	1	1
1	1	A_1	B_1	0	0
2	1	A_2	B_1	1	-1
3	1	A_3	B_1	0	0

Table 5.3. Combinations of Labelings.

6 The Cordiality of Tours Grids

Tours grids are graphs of the form $C_n \times C_m (n \geq 3, \geq 3)$, which has nm vertices and $2nm$ edges. We use the following notation

$$\begin{pmatrix} x & y & . & . & x & x \\ y & y & . & . & x & y \\ . & . & . & . & . & . \\ x & x & . & . & y & y \\ x & y & . & . & x & x \end{pmatrix}_{n\times m}$$

, where x and y are zero or ones.

Example 6.1. $C_3 \times C_3$ is not cordial.
Solution. It is easy to see, that the graph $C_3 \times C_3$ is an Eulerian graph of size 18, which is congruent to 2 (mod 4) and from Cahit's theorem (which is mentioned in last sections), we obtain that $C_3 \times C_3$ is not cordial.

The following theorem generalize the last example as.

Theorem 6.1. The graph $C_n \times C_m$ is not cordial if $2nm$ is congruent to 2(mode 4).
Proof. The proof follows directly from Cahit's theorem and the fact that the degree of all vertices of $C_n \times C_m$ are even and its size $2nm$.

The following corollaries are the special cases of the last theorem.

Corollary 6.1. If $n \equiv 1 \pmod 4$ and m odd (or vice versa), the graph $C_n \times C_m$is not cordial.
Corollary 6.2. If $n \equiv 3 \pmod 4$ and m odd (or vice versa), the graph $C_n \times C_m$is not cordial.
Theorem 6.2. If $n \equiv 1 \pmod 4$ and m even (or vice versa), then the graph $C_n \times C_m$ is cordial.
Proof. Let $n = 4r + 1$ and $m = 2s$, where $r > 1$ and $s > 1$, then the order of the graphs $C_n \times C_m$ is $nm = 6rs$ and its size is $2nm = 12rs + 4s$. Hence we label the vertices of the graph $C_n \times C_m$ as
$\begin{pmatrix} 1 & 1 & . & . & 1 & 1 \\ 0 & 0 & . & . & 0 & 0 \\ . & . & . & . & . & . \\ 1 & 1 & . & . & 1 & 1 \\ 0 & 0 & . & . & 0 & 0 \end{pmatrix}$. It easy to verity that $v_0 = v_1 = 3rs + s$ and $e_0 = e_1 = 6rs + 2s$ for all r and all s. Hence $v_0 - v_1 = 0$ and $e_0 - e_1 = 0$. This means that $C_n \times C_m$ is cordial, the theorem follows.
Theorem 6.3. If $n \equiv 3 \pmod 4$ and m even (or vice versa), then graph $C_n \times C_m$ is cordial.
Proof. Let $n = 4r + 3$ and $m = 2s$, where $r > 1$ and $s > 1$, then the order of $C_n \times C_m$ is $nm = 8rs + 6s$ and its size is $2nm = 16rs + 12s$. Hence we label the vertices of the graph $C_n \times C_m$ as
$\begin{pmatrix} 1 & 1 & . & . & 1 & 1 \\ 0 & 0 & . & . & 0 & 0 \\ . & . & . & . & . & . \\ 1 & 1 & . & . & 1 & 1 \\ 0 & 0 & . & . & 0 & 0 \end{pmatrix}$. It easy to verity that $v_0 = v_1 = 4rs + 3s$ and $e_0 = e_1 = 8rs + 6s$ for all r and all s. Hence $v_0 - v_1 = 0$ and $e_0 - e_1 = 0$. This means that $C_n \times C_m$ is cordial, the theorem follows.
Theorem 6.4. If $n \equiv 0 \pmod 4$, then the graph $C_n \times C_m$ is cordial for all m.
Proof. Let $n = 4r$, where $r > 1$. Hence we have two cases:
Case (1). m is even, i.e., $m = 2s$.
We label the vertices of the graph $C_n \times C_m$ as

$\begin{pmatrix} 1 & 1 & . & . & 1 & 1 \\ 0 & 0 & . & . & 0 & 0 \\ . & . & . & . & . & . \\ 1 & 1 & . & . & 1 & 1 \\ 0 & 0 & . & . & 0 & 0 \end{pmatrix}$. It easy to verity that $v_0 = v_1 = 4rs$ and $e_0 = e_1 = 8rs$ for all r and all s.

Case (2). m is odd, i.e., $m = 2s + 1$.

We label the vertices of the graph $C_n \times C_m$ as

$\begin{pmatrix} 1 & 1 & . & . & 1 & 1 \\ 0 & 0 & . & . & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & . & . & 0 & 0 \\ 0 & 1 & . & . & 0 & 1 \end{pmatrix}$. It easy to verity that $v_0 = v_1 = 4rs + 2r$ and $e_0 = e_1 = 8rs + 4r$ for all r and all s. Hence from the last cases, we obtain that $v_0 - v_1 = 0$ and $e_0 - e_1 = 0$. This means that $C_n \times C_m$ is cordial, the theorem follows.

Theorem 6.5. If $n \equiv 2 \pmod 4$, then the graph $C_n \times C_m$ is cordial for all m.

Proof. Let $n = 4r + 2$, where $r > 1$. Hence we have two cases:

Case (1). m is even, i.e., $m = 2s$.

We label vertices of the graph $C_n \times C_m$ as

$\begin{pmatrix} 1 & 1 & . & . & 1 & 1 \\ 0 & 0 & . & . & 0 & 0 \\ . & . & . & . & . & . \\ 1 & 1 & . & . & 1 & 1 \\ 0 & 0 & . & . & 0 & 0 \end{pmatrix}$. It easy to verity that $v_0 = v_1 = 4rs + 2r$ and $e_0 = e_1 = 8rs + 4s$ for all r and all s.

Case (2). m is odd, i.e., $m = 2s + 1$.

We label the vertices of the graph $C_n \times C_m$ as

$\begin{pmatrix} 1 & 1 & . & . & 1 & 1 \\ 0 & 0 & . & . & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & . & . & 0 & 0 \\ 0 & 1 & . & . & 0 & 1 \end{pmatrix}$. It easy to verity that $v_0 = v_1 = 4rs + 2r + 2s + 1$ and $e_0 = e_1 = 8rs + 4r + 4s + 2$ for all r and all s. Hence from the last cases, we obtain that $v_0 - v_1 = 0$ and $e_0 - e_1 = 0$. This means that $C_n \times C_m$ is cordial, the theorem follows.

From the last facts, we can establish the following theorem.

Theorem 6.6. The tours grids $C_n \times C_m$ is cordial for all $n \geq 3$ and $m \geq 3$ if and only if $2nm$ is not vongruent to $2 \pmod 4$.

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