On Potentially $(K_5 - C_4)$-graphic Sequences *

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Abstract

In this paper, we characterize the potentially $(K_5 - C_4)$-graphic sequences where $K_5 - C_4$ is the graph obtained from K_5 by removing four edges of a 4 cycle C_4. This characterization implies a theorem due to Lai [6].
Key words: graph; degree sequence; potentially $(K_5 - C_4)$-graphic sequences
AMS Subject Classifications: 05C07

1 Introduction

An n-term nonincreasing nonnegative integer sequence $\pi = (d_1, d_2, \cdots, d_n)$ is said to be graphic if it is the degree sequence of a simple graph G of order n; such a graph G is referred to as a realization of π. We denote by $\sigma(\pi)$ the sum of all the terms of π. K_n is the complete graph on n vertices. C_n is the cycle of length n. $K_n - C_4$ is the graph obtained from K_n by removing 4 edges of a 4 cycle C_4. Let H be a simple graph. A graphic sequence π is said to be potentially H-graphic if it has a realization G containing H as a subgraph.

Given a graph H, what is the maximum number of edges of a graph with n vertices not containing H as a subgraph? This number is denoted

*Project Supported by NNSF of China(10271105), NSF of Fujian(Z0511034), Science and Technology Project of Fujian, Project of Fujian Education Department and Project of Zhangzhou Teachers College.

$ex(n, H)$, and is known as the Turán number. This problem was proposed for $H = C_4$ by Erdös [1] in 1938 and generalized by Turán [16]. In terms of graphic sequences, the number $2ex(n, H) + 2$ is the minimum even integer l such that every n-term graphical sequence π with $\sigma(\pi) \geq l$ is forcibly H-graphical. In [3], Gould, Jacobson and Lehel considered the following variation of the classical Turán-type extremal problems: determine the smallest even integer $\sigma(H, n)$ such that every n-term positive graphic sequence $\pi = (d_1, d_2, \cdots, d_n)$ with $\sigma(\pi) \geq \sigma(H, n)$ has a realization G containing H as a subgraph. They proved that $\sigma(pK_2, n) = (p-1)(2n-p)+2$ for $p \geq 2$; $\sigma(C_4, n) = 2[\frac{3n-1}{2}]$ for $n \geq 4$. In [5,6], Lai determined the values $\sigma(K_4 - e, n)$ for $n \geq 4$ and $\sigma(K_5 - C_4, n)$ for $n \geq 5$. Yin, Li, and Mao [14] determined the values $\sigma(K_{r+1} - e, n)$ for $r \geq 3$ and $r + 1 \leq n \leq 2r$ and $\sigma(K_5 - e, n)$ for $n \geq 5$. Recently, Yin and Li [15] determined $\sigma(K_{r+1} - e, n)$. Erdös, Jacobson and Lehel [2] showed that $\sigma(K_k, n) \geq (k-2)(2n-k+1)+2$ and conjectured that the equality holds. They proved the conjecture is true for $k = 3$ and $n \geq 6$, i.e., $\sigma(K_3, n) = 2n$ for $n \geq 6$. The conjecture was confirmed in [3], [7], [8], [9] and [10].

Motivated by the above problems, we consider the following problem: given a graph H, characterize the potentially H-graphic sequences without zero terms. In [11], Luo characterized the potentially C_k-graphic sequences for each $k = 3, 4, 5$. Recently, Luo and Warner [12] characterized the potentially K_4-graphic sequences. In [13], Eschen and Niu characterized the potentially $(K_4 - e)$-graphic sequences.

In this paper, we characterize the potentially $(K_5 - C_4)$-graphic sequences without zero terms. This characterization implies a theorem due to Lai [6].

2 Preparations

Let $\pi = (d_1, d_2, \cdots, d_n)$ be a nonincreasing positive integer sequence. Then $\pi' = (d_1 - 1, d_2 - 1, \cdots, d_{d_n} - 1, d_{d_n+1}, \cdots, d_{n-1})$ is the residual sequence obtained by laying off d_n from π. We denote the nonincreasing sequence π' by $(d'_1, d'_2, \cdots, d'_{n-1})$. From here on, denote π' the residual sequence obtained by laying off d_n from π and all the graphic sequences have no zero terms. In order to prove our main result, we need the following results.

Theorem 2.1 [3] If $\pi = (d_1, d_2, \cdots, d_n)$ is a graphic sequence with a realization G containing H as a subgraph, then there exists a realization G' of π containing H as a subgraph so that the vertices of H have the largest degrees of π.

The following corollary is obvious.

Corollary 2.2 Let H be a simple graph. If π' is potentially H-graphic, then π is potentially H-graphic.

We will use Corollary 2.2 repeatedly in the proofs of our main results.

Lemma 2.3 (Kleitman and Wang [4]) π is graphic if and only if π' is graphic.

3 Potentially $(K_5 - C_4)$-graphic sequences

Our main result is as follows:

Theorem 3.1 Let $\pi = (d_1, d_2, \cdots, d_n)$ be a graphic sequence with $n \geq 5$. Then π is potentially $(K_5 - C_4)$-graphic if and only if the following conditions hold:

(1) $d_1 \geq 4$.

(2) $d_5 \geq 2$.

(3) $\pi \neq ((n-2)^2, 2^{n-2})$ for $n \geq 6$, where the symbol x^y stands for y consecutive terms x.

(4) $\pi \neq (n-k, k+i, 2^i, 1^{n-i-2})$ where $i = 3, 4, \cdots, n-2k$ and $k = 1, 2, \cdots, [\frac{n-1}{2}] - 1$.

(5) If $n = 6$, then $\pi \neq (4, 2^5)$.

(6) If $n = 7$, then $\pi \neq (4, 2^6)$.

Proof: First we assume that π is potentially $(K_5 - C_4)$-graphic. In this case the necessary conditions (1) and (2) are obvious. we are going to prove the conditions (3) – (6) by way of contradiction.

If $\pi = ((n-2)^2, 2^{n-2})$ where $n \geq 6$ is potentially $(K_5 - C_4)$-graphic, then according to theorem 2.1, there exists a realization G of π containing $K_5 - C_4$ as a subgraph so that the vertices of $K_5 - C_4$ have the largest degrees of π. Then the sequence $\pi^* = (n-4, n-6, 2^{n-5})$ obtained from $G - (K_5 - C_4)$ must be graphic and there must be no edge between two vertices with degree $n-4$ and $n-6$ for the realization of π^*, which is impossible. Thus, $\pi = ((n-2)^2, 2^{n-2})$ where $n \geq 6$ is not potentially $(K_5 - C_4)$-graphic. Hence, (3) holds.

If $\pi = (n-k, k+i, 2^i, 1^{n-i-2})$ where $i = 3, 4, \cdots, n-2k$ and $k = 1, 2, \cdots, [\frac{n-1}{2}] - 1$ is potentially $(K_5 - C_4)$-graphic, then according to theorem 2.1, there exists a realization G of π containing $K_5 - C_4$ as a subgraph so that the vertices of $K_5 - C_4$ have the largest degrees of π. Then the sequence $\pi^* = (n-k-4, k+i-2, 2^{i-3}, 1^{n-i-2})$ obtained from $G - (K_5 - C_4)$ must be graphic and there must be no edge between two vertices with degree $n-k-4$ and $k+i-2$ for the realization of π^*. Thus, π^* satisfies: $(n-k-4) + (k+i-2) \leq 2(i-3) + (n-i-2)$, that is, $0 \leq (-2)$, which is a contradiction. Hence, (4) holds.

If $\pi = (4, 2^5)$ is potentially $(K_5 - C_4)$-graphic, then according to theorem 2.1, there exists a realization G of π containing $K_5 - C_4$ as a subgraph so that the vertices of $K_5 - C_4$ have the largest degrees of π. Then the sequence $\pi^* = (2)$ obtained from $G - (K_5 - C_4)$ must be the degree sequence of a simple graph, which is a contradiction. Thus, $\pi = (4, 2^5)$ is not potentially $(K_5 - C_4)$-graphic. Hence, (5) holds.

If $\pi = (4, 2^6)$ is potentially $(K_5 - C_4)$-graphic, then according to theorem 2.1, there exists a realization G of π containing $K_5 - C_4$ as a subgraph so that the vertices of $K_5 - C_4$ have the largest degrees of π. Then the sequence $\pi^* = (2^2)$ obtained from $G - (K_5 - C_4)$ must be the degree sequence of a simple graph, which is a contradiction. Thus, $\pi = (4, 2^6)$ is not potentially $(K_5 - C_4)$-graphic. Hence, (6) holds.

Now we prove the sufficient condition. Suppose the graphic sequence π satisfies the conditions (1) – (6). Our proof is by induction on n.

First we prove the sufficient condition for $n = 5$. Since $\pi \neq (4^2, 2^3)$, then π is one of the following sequences:

(4^5), $(4^3, 3^2)$, $(4^2, 3^2, 2)$, $(4, 3^4)$, $(4, 3^2, 2^2)$, $(4, 2^4)$. It is easy to see that they are all potentially $(K_5 - C_4)$-graphic. Therefore, π is potentially $(K_5 - C_4)$-graphic for $n = 5$.

We now suppose that the sufficient condition holds for $(n-1) \geq 5$. We will prove that it holds for n. Let $\pi = (d_1, d_2, \cdots, d_n)$ be a graphic sequence with n terms that satisfies the conditions (1) – (6). We only need to show that π is potentially $(K_5 - C_4)$-graphic. If π' satisfies the assumption, then π' is potentially $(K_5 - C_4)$-graphic by the induction hypothesis. Therefore, π is potentially $(K_5 - C_4)$-graphic by Corollary 2.2. Thus, we consider the following cases:

Case 1: If $\pi' = (4, 2^5)$, then $\pi = (5, 3, 2^5)$ or $\pi = (5, 2^5, 1)$. It is easy to see that both of them are potentially $(K_5 - C_4)$-graphic.

Case 2: If $\pi' = (4, 2^6)$, then $\pi = (5, 3, 2^6)$ or $\pi = (5, 2^6, 1)$. It is easy to see that both of them are potentially $(K_5 - C_4)$-graphic.

Case 3: $\pi' = ((n-3)^2, 2^{n-3})$ where $n - 1 \geq 6$.

If $d_n = 2$, then $\pi = ((n-2)^2, 2^{n-2})$, which is contradict to condition(3).

If $d_n = 1$, then $\pi = (n-2, n-3, 2^{n-3}, 1)$. We are going to prove that π is potentially $(K_5 - C_4)$-graphic. First we show it is true for $n = 6$. In this case, $\pi = (4, 3, 2^3, 1)$. It is easy to see that π is potentially $(K_5 - C_4)$-graphic. Now we prove that π is potentially $(K_5 - C_4)$-graphic for $n \geq 7$. It is enough to show $\pi_1 = (n-5, n-6, 2^{n-6}, 1)$ is graphic and there exist no edge between two vertices with degree $n-5$ and $n-6$ for the realization of π_1. Hence it is enough to show $\pi_2 = (n-6, 1^{n-6})$ is graphic. Clearly, π_2 has a realization consisting of $n-6$ edges and these edges have only one vertex in common.

Thus, $\pi = (n-2, n-3, 2^{n-3}, 1)$ is potentially $(K_5 - C_4)$-graphic for $n \geq 6$.

Case 4: $\pi' = (n-1-k, k+i, 2^i, 1^{n-i-3})$ where $i = 3, 4, \cdots, n-1-2k$ and $k = 1, 2, \cdots, [\frac{n-2}{2}] - 1$.

If $d_n = 2$, then $n - i - 3 = 0$ and $\pi = (n-k, k+i+1, 2^{i+1})$, which is contradict to condition(4).

If $d_n = 1$, then $\pi = (n-k', k'+i, 2^i, 1^{n-i-2})$, which is contradict to condition(4) .

Case 5: $d_n \geq 4$. In this case, π' satisfies the conditions $(1)-(6)$. Thus, π' is potentially $(K_5 - C_4)$-graphic. Therefore, π is potentially $(K_5 - C_4)$-graphic by Corollary 2.2.

Case 6: $d_n = 3$.

If $d_1 \geq 5$, then π' satisfies the conditions $(1) - (6)$. Thus, π' is potentially $(K_5 - C_4)$-graphic. Therefore, π is potentially $(K_5 - C_4)$-graphic by Corollary 2.2.

If $d_1 = 4$, there are two subcases: $d_4 = 4$ and $d_4 = 3$.

Subcase 1: $d_4 = 4$. In this case, $d_1 = d_2 = d_3 = d_4 = 4$. Obviously, π' satisfies the conditions $(1)-(6)$. Thus, π' is potentially $(K_5 - C_4)$-graphic. Therefore, π is potentially $(K_5 - C_4)$-graphic by Corollary 2.2.

Subcase 2: $d_4 = 3$.

Subcase 2.1: $d_3 = 4$. Then $\pi = (4^3, 3^{n-3})$. Since $\sigma(\pi)$ is even, n must be odd. We are going to prove that π is potentially $(K_5 - C_4)$-graphic. It is easy to see that $\pi = (4^3, 3^4)$ is potentially $(K_5 - C_4)$-graphic. If $n \geq 9$, then $(4^3, 3^{n-3})$ has a realization containing a $K_5 - C_4$ (see Figure 1).

Thus, $\pi = (4^3, 3^{n-3})$ where n is odd is potentially $(K_5 - C_4)$-graphic.

Subcase 2.2: $d_3 = 3$.

If $d_2 = 4$, then $\pi = (4^2, 3^{n-2})$. Since $\sigma(\pi)$ is even, n must be even. We are going to prove that π is potentially $(K_5 - C_4)$-graphic. It is easy to see that $\pi = (4^2, 3^4)$ and $\pi = (4^2, 3^6)$ are potentially $(K_5 - C_4)$-graphic. If $n \geq 10$, then $(4^2, 3^{n-2})$ has a realization containing a $K_5 - C_4$ (see Figure 2).

Thus, $\pi = (4^2, 3^{n-2})$ where n is even is potentially $(K_5 - C_4)$-graphic.

If $d_2 = 3$, then $\pi = (4, 3^{n-1})$. Since $\sigma(\pi)$ is even, n must be odd . We are going to prove that π is potentially $(K_5 - C_4)$-graphic. It is easy to see that $\pi = (4, 3^6)$ is potentially $(K_5 - C_4)$-graphic. If $n \geq 9$, then $(4, 3^{n-1})$ has a realization containing a $K_5 - C_4$ (see Figure 3).

Thus, $\pi = (4, 3^{n-1})$ where n is odd is potentially $(K_5 - C_4)$-graphic.

Case 7: $d_n = 2$ and $\pi' \neq ((n-3)^2, 2^{n-3})$ where $n-1 \geq 6$, $\pi' \neq (n-1-k, k+i, 2^i, 1^{n-i-3})$ where $i = 3, 4, \cdots, n-1-2k$ and $k = 1, 2, \cdots, [\frac{n-2}{2}]-1$. $\pi' \neq (4, 2^5)$, $\pi' \neq (4, 2^6)$.

If $d_1 \geq 5$, then π' satisfies the conditions $(1) - (6)$. Thus, π' is potentially $(K_5 - C_4)$-graphic. Therefore, π is potentially $(K_5 - C_4)$-graphic by Corollary 2.2.

If $d_1 = 4$, there are three subcases: $d_2 = 4$, $d_2 = 3$ and $d_2 = 2$.

Subcase 1: $d_2 = 4$.

If $d_3 = 4$, then π' satisfies the conditions $(1) - (6)$. Thus, π' is potentially $(K_5 - C_4)$-graphic. Therefore, π is potentially $(K_5 - C_4)$-graphic by Corollary 2.2.

If $d_3 = 3$, then $\pi = (4^2, 3^a, 2^{n-2-a})$ where $a \geq 1$ and $n-2-a \geq 1$. Since $\sigma(\pi)$ is even, a must be even. We are going to prove that π is potentially $(K_5 - C_4)$-graphic.

First, we consider $\pi = (4^2, 3^2, 2^{n-4})$. It is easy to see that $\pi = (4^2, 3^2, 2^2)$ and $\pi = (4^2, 3^2, 2^3)$ are potentially $(K_5 - C_4)$-graphic. If $n \geq 8$, then $(4^2, 3^2, 2^{n-4})$ has a realization containing a $K_5 - C_4$ (see Figure 4). Thus, we are done.

Then we consider $\pi = (4^2, 3^a, 2^{n-2-a})$ where $a \geq 4$ and $n-2-a \geq 1$. It is easy to see that $\pi = (4^2, 3^4, 2)$ and $\pi = (4^2, 3^4, 2^2)$ are potentially $(K_5 - C_4)$-graphic. If $a = 4$ and $n \geq 9$, then $(4^2, 3^4, 2^{n-6})$ has a realization containing a $K_5 - C_4$ (see Figure 5). If $a \geq 6$, then $(4^2, 3^a, 2^{n-2-a})$ has a realization containing a $K_5 - C_4$ (see Figure 6).

If $d_3 = 2$, then $\pi = (4^2, 2^{n-2})$. Since $\pi \neq (4^2, 2^4)$, we must have $n \geq 7$. We are going to prove that π is potentially $(K_5 - C_4)$-graphic. It is enough to show $\pi_1 = (2^{n-4})$ is graphic. Clearly, C_{n-4} is a realization of π_1. Thus, we are done.

Subcase 2: $d_2 = 3$. Then $\pi = (4, 3^a, 2^{n-1-a})$ where $a \geq 1$ and $n-1-a \geq 1$. Since $\sigma(\pi)$ is even, a must be even. We are going to prove that π is potentially $(K_5 - C_4)$-graphic.

First, we consider $\pi = (4, 3^2, 2^{n-3})$. It is enough to show $\pi_1 = (2^{n-5}, 1^2)$ is graphic. Clearly, π_1 is graphic. Thus, $\pi = (4, 3^2, 2^{n-3})$ is potentially $(K_5 - C_4)$-graphic.

Second, we consider $\pi = (4, 3^4, 2^{n-5})$. It is easy to see that $\pi = (4, 3^4, 2)$ and $\pi = (4, 3^4, 2^2)$ are potentially $(K_5 - C_4)$-graphic. If $n \geq 8$, then $(4, 3^4, 2^{n-5})$ has a realization containing a $K_5 - C_4$ (see Figure 7). Thus, we are done.

Then we consider $\pi = (4, 3^a, 2^{n-1-a})$ where $a \geq 6$ and $n-1-a \geq 1$. It is easy to see that $\pi = (4, 3^6, 2)$ is potentially $(K_5 - C_4)$-graphic. If $a = 6$ and $n \geq 9$, then $(4, 3^6, 2^{n-7})$ has a realization containing a $K_5 - C_4$ (see Figure 8). If $a \geq 8$ and $n - 1 - a = 1$, then $(4, 3^a, 2)$ has a realization containing a $K_5 - C_4$ (see Figure 9). If $a \geq 8$ and $n - 1 - a \geq 2$, then $(4, 3^a, 2^{n-1-a})$ has a realization containing a $K_5 - C_4$ (see Figure 10). Thus, we are done.

Subcase 3: $d_2 = 2$. Then $\pi = (4, 2^{n-1})$. Since $\pi \neq (4, 2^5)$ and $\pi \neq (4, 2^6)$, we must have $n \geq 8$. We are going to prove that π is potentially $(K_5 - C_4)$-graphic. It is enough to show $\pi_1 = (2^{n-5})$ where $n \geq 8$ is graphic. Obviously, C_{n-5} is a realization of π_1. Thus, $\pi = (4, 2^{n-1})$ is potentially $(K_5 - C_4)$-graphic.

Case 8: $d_n = 1$ and $\pi' \neq ((n-3)^2, 2^{n-3})$, $\pi' \neq (n-1-k, k+i, 2^i, 1^{n-i-3})$ where $i = 3, 4, \cdots, n-1-2k$ and $k = 1, 2, \cdots, [\frac{n-2}{2}] - 1$. $\pi' \neq (4, 2^5)$, $\pi' \neq (4, 2^6)$.

If $d_1 \geq 5$, then π' satisfies the conditions (1) – (6). Thus, π' is potentially $(K_5 - C_4)$-graphic. Therefore, π is potentially $(K_5 - C_4)$-graphic by Corollary 2.2.

If $d_1 = 4$, there are three subcases: $d_2 = 4$, $d_2 = 3$ and $d_2 = 2$.

Subcase 1: $d_2 = 4$. In this case, π' satisfies the conditions (1) – (6). Thus, π' is potentially $(K_5 - C_4)$-graphic. Therefore, π is potentially $(K_5 - C_4)$-graphic by Corollary 2.2.

Subcase 2: $d_2 = 3$. Then $\pi = (4, 3^a, 2^b, 1^{n-1-a-b})$ where $a \geq 1$, $a+b \geq 4$ and $n-1-a-b \geq 1$. Since $\sigma(\pi)$ is even, $n-1-b$ must be even. We are going to prove that π is potentially $(K_5 - C_4)$-graphic.

Subcase 2.1: $a = 1$. Then $\pi = (4, 3, 2^b, 1^{n-2-b})$. It is enough to show $\pi_1 = (2^{b-3}, 1^{n-1-b})$ is graphic. Clearly, π_1 is graphic. Thus, we are done.

Subcase 2.2: $a = 2$. Then $\pi = (4, 3^2, 2^b, 1^{n-3-b})$. It is enough to show $\pi_1 = (2^{b-2}, 1^{n-1-b})$ is graphic. Clearly, π_1 is graphic. Thus, we are done.

Subcase 2.3: $a = 3$. Then $\pi = (4, 3^3, 2^b, 1^{n-4-b})$. First, we consider $\pi = (4, 3^3, 2, 1^{n-5})$ where n is even. It is easy to see that $\pi = (4, 3^3, 2, 1)$ is potentially $(K_5 - C_4)$-graphic. If $n \geq 8$, then $(4, 3^3, 2, 1^{n-5})$ has a realization containing a $K_5 - C_4$ (see Figure 11). Second, we consider $\pi = (4, 3^3, 2^2, 1^{n-6})$ where n is odd. It is easy to see that $\pi = (4, 3^3, 2^2, 1)$ is potentially $(K_5 - C_4)$-graphic. If $n \geq 9$, then $(4, 3^3, 2^2, 1^{n-6})$ has a realization containing a $K_5 - C_4$ (see Figure 12). Third, we consider $\pi = (4, 3^3, 2^3, 1^{n-7})$ where n is even. It is easy to see that $\pi = (4, 3^3, 2^3, 1)$ is potentially $(K_5 - C_4)$-graphic. If $n \geq 10$, then $(4, 3^3, 2^3, 1^{n-7})$ has a realization containing a $K_5 - C_4$ (see Figure 13). Then, we consider $\pi = (4, 3^3, 2^b, 1^{n-4-b})$ where $b \geq 4$. In this case, $(4, 3^3, 2^b, 1^{n-4-b})$ has a realization containing a $K_5 - C_4$ (see Figure 14). Thus, we are done.

Subcase 2.4: $a = 4$. Then $\pi = (4, 3^4, 2^b, 1^{n-5-b})$. There are two subcases: $b \geq 1$ and $b = 0$.

Suppose $b \geq 1$. It is easy to see that $\pi = (4, 3^4, 2, 1^{n-6})$ and $\pi = (4, 3^4, 2^2, 1^{n-7})$ are potentially $(K_5 - C_4)$-graphic (see Figure 15 and Figure 16, respectively). If $b \geq 3$, then $(4, 3^4, 2^b, 1^{n-5-b})$ has a realization containing a $K_5 - C_4$ (see Figure 17). Thus, we are done.

Suppose $b = 0$. Then $\pi = (4, 3^4, 1^{n-5})$. Since $\sigma(\pi)$ is even, $n-5$ must be even. Clearly, $(4, 3^4, 1^{n-5})$ has a realization containing a $K_5 - C_4$ (see Figure 18). Thus, we are done.

Subcase 2.5: $a \geq 5$. Then $\pi = (4, 3^a, 2^b, 1^{n-1-a-b})$ where $a \geq 5$ and $n-1-a-b \geq 1$. There are two subcases: $b \geq 1$ and $b = 0$.

Suppose $b \geq 1$.

If a is even, it is easy to see that $\pi = (4, 3^6, 2, 1^{n-8})$ has a realization containing a $K_5 - C_4$ (see Figure 19) . If $a = 6$ and $b \geq 2$, then $(4, 3^6, 2^b, 1^{n-7-b})$ has a realization containing a $K_5 - C_4$ (see Figure 20). If $a \geq 8$ and $b = 1$, then $(4, 3^a, 2, 1^{n-2-a})$ has a realization containing a $K_5 - C_4$ (see Figure 21). If $a \geq 8$ and $b \geq 2$, then $(4, 3^a, 2^b, 1^{n-1-a-b})$ has a realization containing a $K_5 - C_4$ (see Figure 22).

If a is odd, it is easy to see that $\pi = (4, 3^5, 2, 1^{n-7})$ has a realization containing a $K_5 - C_4$ (see Figure 23). If $a = 5$ and $b \geq 2$, then $(4, 3^5, 2^b, 1^{n-6-b})$

has a realization containing a $K_5 - C_4$ (see Figure 24). If $a \geq 7$ and $b = 1$, then $(4, 3^a, 2, 1^{n-2-a})$ has a realization containing a $K_5 - C_4$ (see Figure 25). If $a \geq 7$ and $b \geq 2$, then $(4, 3^a, 2^b, 1^{n-1-a-b})$ has a realization containing a $K_5 - C_4$ (see Figure 26). Thus, we are done.

Suppose $b = 0$. Then $\pi = (4, 3^a, 1^{n-1-a})$. Since $\sigma(\pi)$ is even, $n - 1$ must be even.

If a is even, it is easy to see that $\pi = (4, 3^6, 1^2)$ is potentially $(K_5 - C_4)$-graphic. If $a = 6$ and $n \geq 11$, then $(4, 3^6, 1^{n-7})$ has a realization containing a $K_5 - C_4$ (see Figure 27). If $a \geq 8$, then $(4, 3^a, 1^{n-1-a})$ has a realization containing a $K_5 - C_4$ (see Figure 28).

If a is odd, it is easy to see that $\pi = (4, 3^5, 1)$ and $\pi = (4, 3^7, 1)$ are potentially $(K_5 - C_4)$-graphic. If $a = 5$ and $n \geq 9$, then $(4, 3^5, 1^{n-6})$ has a realization containing a $K_5 - C_4$ (see Figure 29). If $a = 7$ and $n \geq 11$, then $(4, 3^7, 1^{n-8})$ has a realization containing a $K_5 - C_4$ (see Figure 30). If $a \geq 9$, then $(4, 3^a, 1^{n-1-a})$ has a realization containing a $K_5 - C_4$ (see Figure 31). Thus, we are done.

Subcase 3: $d_2 = 2$. Then $\pi = (4, 2^a, 1^{n-1-a})$ where $a \geq 4$ and $n-1-a \geq 1$. Since $\sigma(\pi)$ is even, $n - 1 - a$ must be even. We are going to prove that π is potentially $(K_5 - C_4)$-graphic. If $a = 4$, then $\pi = (4, 2^4, 1^{n-5})$ where $n - 5$ is even. It is enough to show $\pi_1 = (1^{n-5})$ is graphic. Clearly, π_1 has a realization consisting of $\frac{n-5}{2}$ disjoint edges. Thus, $\pi = (4, 2^4, 1^{n-5})$ is potentially $(K_5 - C_4)$-graphic. If $a \geq 5$, it is enough to show $\pi_1 = (2^{a-4}, 1^{n-1-a})$ is graphic. Clearly, π_1 is graphic. Thus, we are done.

4 Application

Using Theorem 3.1, we give a simple proof of the following theorem due to Lai [6]:

Theorem 4.1 (Lai [6]) For $n \geq 5$, $\sigma(K_5 - C_4, n) = 4n - 4$.

Proof: First we claim that for $n \geq 5, \sigma(K_5 - C_4, n) \geq 4n - 4$. It is enough to show that there exist π_1 with $\sigma(\pi_1) = 4n - 6$, such that π_1 is not potentially $(K_5 - C_4)$-graphic. Take $\pi_1 = ((n-1)^2, 2^{n-2})$, then $\sigma(\pi_1) = 4n - 6$, and it is easy to see that π_1 is not potentially $(K_5 - C_4)$-graphic by Theorem 3.1.

Now we show that if π is an n-term ($n \geq 5$) graphical sequence with $\sigma(\pi) \geq 4n - 4$, then there exist a realization of π containing a $K_5 - C_4$. Hence, it suffices to show that π is potentially $(K_5 - C_4)$-graphic.

If $d_5 = 1$, then $\sigma(\pi) = d_1 + d_2 + d_3 + d_4 + (n-4)$ and $d_1 + d_2 + d_3 + d_4 \leq 12 + (n-4) = n + 8$. Therefore, $\sigma(\pi) \leq 2n + 4 < 4n - 4$, which is a contradiction. Thus, $d_5 \geq 2$.

If $d_1 \leq 3$, then $\sigma(\pi) \leq 3n < 4n - 4$, which is a contradiction. Thus, $d_1 \geq 4$.

Since $\sigma(\pi) \geq 4n-4$, then π is not one of the following: $((n-2)^2, 2^{n-2})$ for $n \geq 6$, $(n-k, k+i, 2^i, 1^{n-i-2})$ where $i = 3, 4, \cdots, n-2k$ and $k = 1, 2, \cdots, [\frac{n-1}{2}]-1$, $(4, 2^5)$, $(4, 2^6)$. Thus, π satisfies the conditions (1) – (6) in Theorem 3.1. Therefore, π is potentially $(K_5 - C_4)$-graphic.

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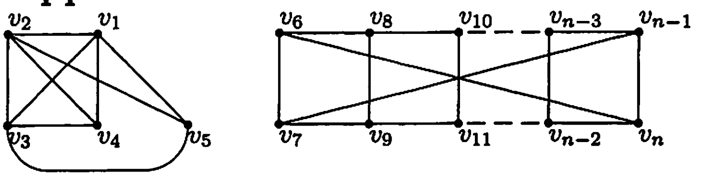
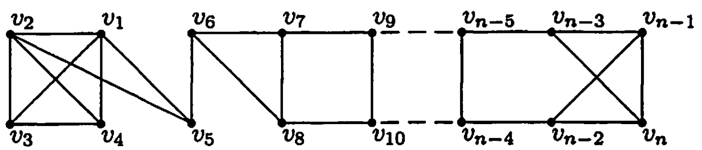
Appendix



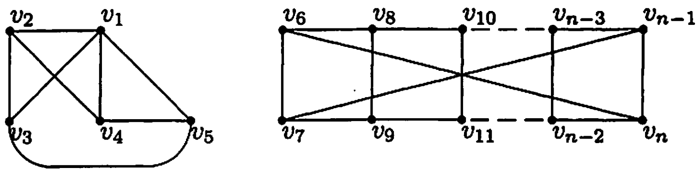
$(4^3, 3^{n-3})$

Figure 1



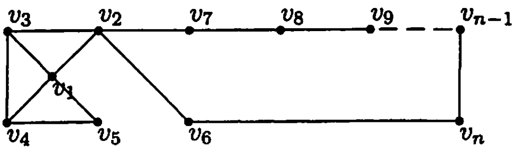
$(4^2, 3^{n-2})$

Figure 2



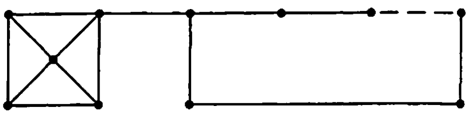
$(4, 3^{n-1})$

Figure 3



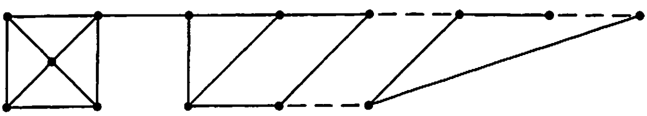
$(4^2, 3^2, 2^{n-4})$

Figure 4



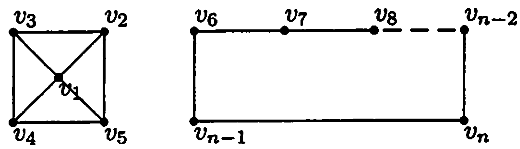
$(4^2, 3^4, 2^{n-6})$

Figure 5



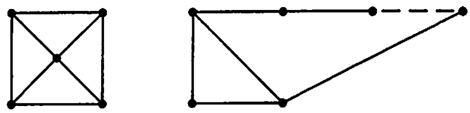
$(4^2, 3^a, 2^{n-2-a})$

Figure 6



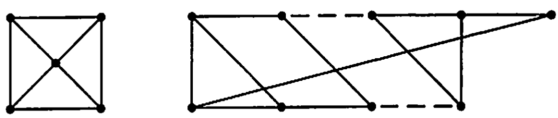
$(4, 3^4, 2^{n-5})$

Figure 7



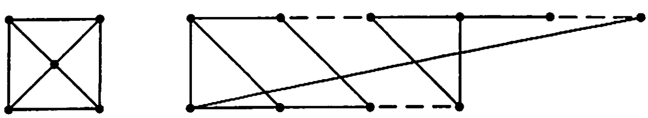
$(4, 3^6, 2^{n-7})$

Figure 8



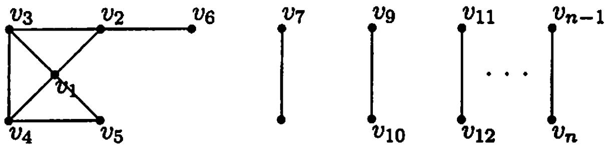
$(4, 3^a, 2)$

Figure 9



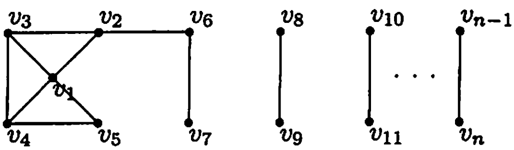
$(4, 3^a, 2^{n-1-a})$

Figure 10



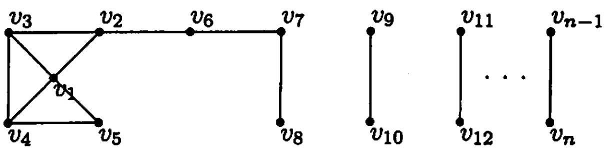
$(4, 3^3, 2, 1^{n-5})$

Figure 11



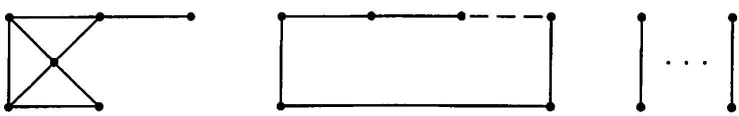
$(4, 3^3, 2^2, 1^{n-6})$

Figure 12



$(4, 3^3, 2^3, 1^{n-7})$

Figure 13

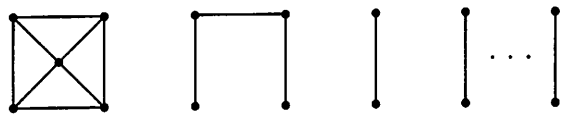


$(4, 3^3, 2^b, 1^{n-4-b})$

Figure 14

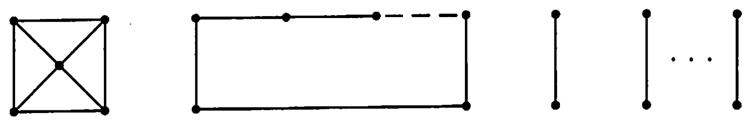
$(4, 3^4, 2, 1^{n-6})$

Figure 15



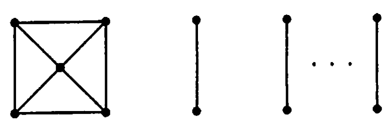
$(4, 3^4, 2^2, 1^{n-7})$

Figure 16



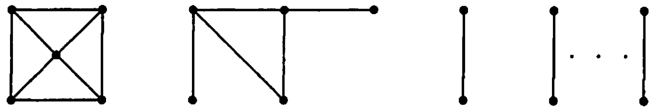
$(4, 3^4, 2^b, 1^{n-5-b})$

Figure 17



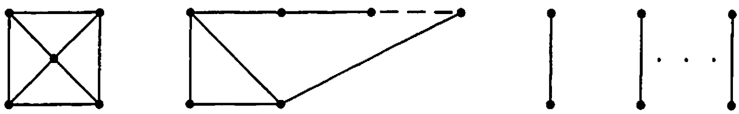
$(4, 3^4, 1^{n-5})$

Figure 18



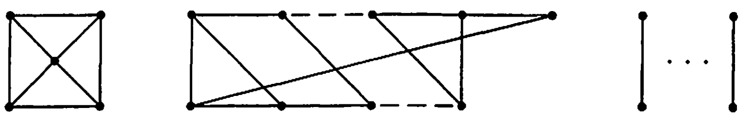
$(4, 3^6, 2, 1^{n-8})$

Figure 19



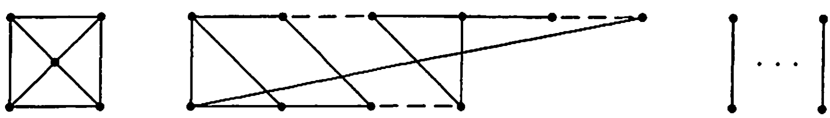
$(4, 3^6, 2^b, 1^{n-7-b})$

Figure 20



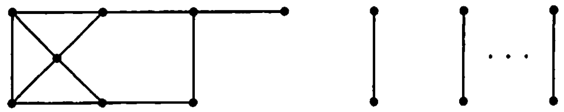
$(4, 3^a, 2, 1^{n-2-a})$

Figure 21



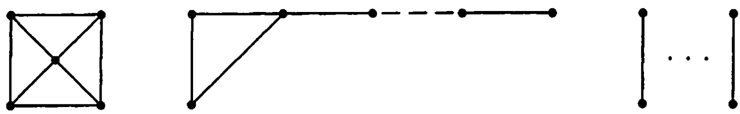
$(4, 3^a, 2^b, 1^{n-1-a-b})$

Figure 22



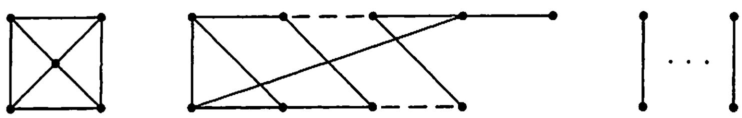
$(4, 3^5, 2, 1^{n-7})$

Figure 23



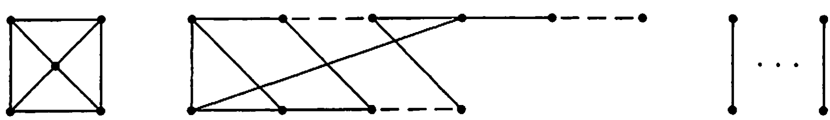
$(4, 3^5, 2^b, 1^{n-6-b})$

Figure 24



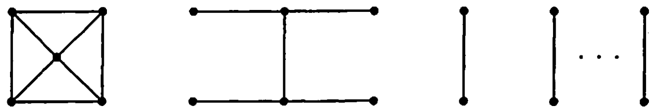
$(4, 3^a, 2, 1^{n-2-a})$

Figure 25



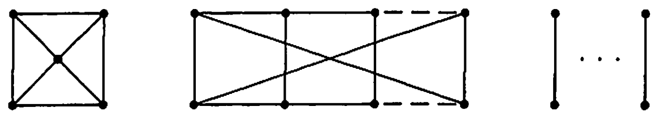
$(4, 3^a, 2^b, 1^{n-1-a-b})$

Figure 26



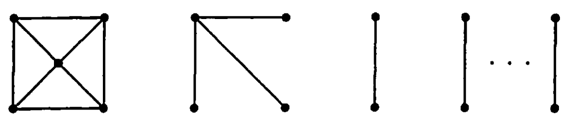
$(4, 3^6, 1^{n-7})$

Figure 27



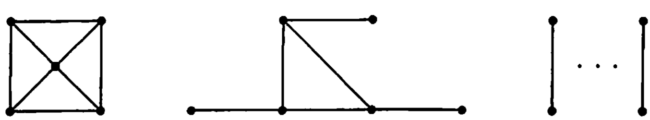
$(4, 3^a, 1^{n-1-a})$

Figure 28



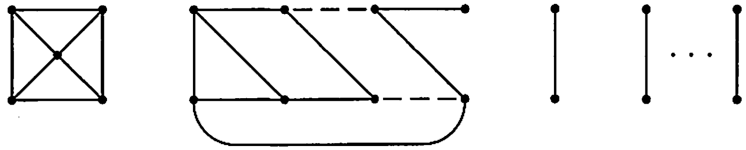
$(4, 3^5, 1^{n-6})$

Figure 29



$(4, 3^7, 1^{n-8})$

Figure 30



$(4, 3^a, 1^{n-1-a})$

Figure 31