The Independence Number of the Cartesian Product of Graphs

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Abstract

We study the independence number of the Cartesian product of binary trees and more general bipartite graphs. We give necessary and sufficient conditions on bipartite graphs under which certain upper and lower bonds on the independence number of the product are equal. A basic tool will be an algorithm for finding the independence number of a binary tree.

1 Introduction

In several recent papers ([2], [3],[4], and [5]) the authors investigate the independence number of the Cartesian product of graphs. Our purpose here is to continue this study, concentrating on the independence number of the Cartesian product of binary trees and more general bipartite graphs. One technique used to find the independence number of Cartesian products is to calculate lower and upper bounds on the independence number and then investigate all possibilities in between. To simplify this process, we give necessary and sufficient conditions for the equality of lower and upper bounds for certain classes of bipartite graphs and consider when the diagonal method (defined below) yields the independence number of the Cartesian product. We also define an algorithm for finding the independence number of a binary tree and prove it gives a maximum independent set.

Notation 1 *For vertices g and h of a graph, $g \sim h$ will mean that g and h are adjacent.*

For graphs G and H, the Cartesian product, $G \,\Box\, H$, is the graph with vertex set $G \times H$ and edge set E defined by $(gh, g'h') \in E(G \,\Box\, H)$ iff $g = g'$ and $h \sim h'$ or $g \sim g'$ and $h = h'$. A set, S, of vertices in a graph is called an *independent set of vertices* if no pair of vertices in S is adjacent. A set, U, of edges in a graph is called an *independent set of edges,* or *matching,* if no pair of edges in U is coincident. The size of a maximum independent set of vertices in a graph G is called the *independence number* and will be denoted by $\alpha(G)$; the size of a maximum matching of a graph G is called the *matching number* and will be denoted $\tau(G)$. A *k-independent* set is the disjoint union of k independent sets of vertices; the size of the maximum k-independent set in the graph G will be denoted $\alpha_k(G)$. We define $\overline{\alpha}(G) = \max\{\alpha(G\backslash S)\}$, where the maximum is taken over all maximum independent sets S.

Whenever W is a set of vertices in a graph G,we shall write $w = |W|$. For an induced subgraph W of G, we write $|W| = |V(W)|$.

Remark 2 ***We summarize a few relevant results:***

1. [7] For any graphs G and H,

$$\alpha(G\Box H) \leq \min\{\alpha(G)\,|H|\,,\ \alpha(H)\,|G|\ \}$$

2. [4] For any graphs G and H,

$$\alpha(G\Box H) \leq \tau(G)\alpha_2(H) + (|H| - 2\tau(H))\ \alpha(G)$$

3. [2] If C_n represents a cycle on n vertices, then

$$\alpha(C_{2k+1}\Box C_{2m+1}) = k(2m+1),\ \textit{where} 1 \leq k \leq m$$

4. [4] If $G = V_1 + V_2$ and $H = W_1 + W_2$ are bipartite graphs such that $\alpha(G) = |V_1|$ and $\alpha(H) = |W_1|$ then

$$\alpha(G\Box H) = |V_1|\,|W_1| + |V_2|\,|W_2|$$

We consider several techniques used in the literature. For a vertex $h \in H$, let $G^h = \{(g,h) : g \in G\}$. If S is an independent set of vertices in $G\Box H$ and $H = \{h_1, h_2, ..., h_n\}$, then we represent S as $S =< S^1, S^2, ..., S^n >$, where $S^i = S \cap G^{h_i}$. Another technique, useful in finding lower bounds for $\alpha(G\Box H)$, is to use a diagonal procedure [4]. Let G and H be graphs, set $G_1 = G$ and $H_1 = H$, and pick A_1 and B_1, maximal independent sets in G_1 and H_1 respectively. Let $G_2 = G_1\backslash A_1$ and $H_2 = H_1 \setminus B_1$ and select maximal independent sets A_2 in G_2 and B_2 in H_2. Continue until we arrive at graphs G_k and H_k such that $V(G_k) = A_k$ or $V(H_k) = B_k$. Then $\bigcup_{k=1}^{n} (A_i \times B_i)$ is a maximal independent set in $G\Box H$. Now, let $\lambda(G\Box H) = \max\{\sum |A_i|\,|B_i|\}$, where the maximum is over all possible selections. Then $\lambda(G\Box H) \leq \alpha(G\Box H)$. This gives a lower bound for

$\alpha(G\Box H)$. For bipartite graphs, $G = V_1 + V_2$ and $H = W_1 + W_2$, there are three approaches to finding the largest maximum independent set using the diagonal procedure.

a. Bipartite approach: returns $((V_1 \times W_1) \cup (V_2 \times W_2))$.

b. Greedy approach: pick the largest independent set at each step in the diagonal procedure.

c. Alternate approach: use the diagonal procedure, picking a maximal independent set at each stage.

Klavzar [4] gives examples showing that each approach may yield the exact value for $\alpha(G\Box H)$.

In the sequel we are partially motivated by the following open problem [4]: characterize the graphs or bipartite graphs G and H for which $\alpha(G\Box H) = \lambda(G\Box H)$.

2 Binary trees

As is usual, a *tree* is a graph which has a unique path from a designated vertex (called the *root)* to any vertex. We assume the reader is familiar with the standard definitions of sibling, parent, children, level, and leaves. (See, for example, [6].) We follow [6] in the following definition of a binary tree. (Notice that some authors define binary trees differently.)

Definition 3 *A tree is binary if each vertex has zero or exactly two children. The height of a tree is the length of the longest path starting from the root. A tree of height h is called full if all the leaves are at level h.*

Proposition 4 *Let G and H be full binary trees.*

1. If G and H both have odd height, then $\alpha(G\Box H) = 5/4\alpha(G)\alpha(H)$
2. If both G and H have even height then $\alpha(G\Box H) = 5/4\alpha(G)\alpha(H) - 1/4\alpha(G) - 1/4\alpha(H) + 1/4$
3. If G has even height and H odd, then $\alpha(G\Box H) = 5/4\alpha(G)\alpha(H) - 1/4\alpha(G)$

Proof. These results follow from Remark 2 (4) above, the fact that for full binary trees, $\alpha(G) = V_1$ and $\alpha(H) = W_1$, and also from the easily proved result that if G is full and has odd height then $\overline{\alpha}(G) = 1/2\alpha(G)$, while if G has even height then $\overline{\alpha}(G) = 1/2(\alpha(G) - 1)$. ∎

The greedy algorithm below finds a maximum independent set in a (not necessarily full) binary tree. For a binary tree with root r and vertex set V, we shall use the following notation:

For $0 \leq k \leq h$, let $V_k = \{v \in V$: distance from v to r is exactly $k\}$, let $n_k = |V_k|$, and label $V_k = \{v_{(k,1)}, v_{(k,2)}, ...v_{(k,n_k)}\}$. If $v \in V$ and v is not a leaf, then v^1 and v^2 will denote the two children of v.

Algorithm 5

> **Input**: T a binary tree with root r, height h, vertex set V.
> **Begin**
> Set $S = \phi$
> For $j = 1$ to n_h, add $v_{(h,j)}$ to S. End for.
> For $count = h - 1$ downto 0 do
> For $j = 1$ to n_{count} do
> If $v_{(count,j)}$ is a leaf, then add $v_{(count,j)}$ to S
> Else
> If $v^1_{(count,j)}$ is not in S and $v^2_{(count,j)}$ is not in S,
> then add $v_{(count,j)}$ to S.
> End if
> End for
> End for
> Return S
> **End**

Theorem 6 *S is a maximum independent set.*

Proof. Clearly, S is an independent set of vertices. Assume there is a set S' which is of maximum size. Then $|S| \leq |S'|$. Using the notation of the algorithm, for each $j = 1...h$, label $S_j = V_j \cap S$ and $S'_j = S' \cap V_j$. We prove that for every j, $S_j = S'_j$, from which it will follow that $S = S'$.

For $j = h$, consider a pair of siblings, $x = v^1$ and $y = v^2$, where $v \in V_{h-1}$. Then both x and y are in S_h by the algorithm. We claim both x and y are in S'_h. If one of them, say y, is not in S'_h, then if $v \notin S'_{h-1}$, then $\{y\} \cup S'$ is an independent set and $|\{y\} \cup S'| > |S'|$ Thus, if $y \notin S'_h$, then $v \in S'_{h-1}$ and so by the independence of S', $x \notin S'_h$. That is, if one of x or y is not in S'_h, then neither is the other and $v \in S'_{h-1}$. But if $v \in S'_{h-1}$, consider $S'' = (S' \backslash \{v\}) \cup \{x, y\}$. If v is the root, then S'' is an independent set and $|S''| > |S'|$, while if v is not the root, v's parent cannot be in S'_{h-2} so that again S'' is an independent set and $|S''| > |S'|$. So, whether or not v is the root, we reach a contradiction, and therefore both x and y must be in S'_h . Since every pair of vertices with distance h to the root must both be in S_h and in S'_h, $S_h = S'_h$.

Assume that for all levels j from h down to $h - k$, S_j and S'_j are not only the same size, but are the same sets. We show $S_{h-k-1} = S'_{h-k-1}$.

If $h-k-1 = 0$, then if at least one child of r, say r^1, is in $S'_{h-k} = S_{h-k}$, then by the algorithm, $r \notin S_{h-k-1}$ and since S' is an independent set, $r \notin S'_{h-k-1}$, so $S_{h-k-1} = S'_{h-k-1} = \phi$. If neither r^1 nor r^2 is in $S_{h-k} = S'_{h-k}$, then by the algorithm, $r \in S_{h-k-1}$, and since $S'$is a maximum independent set, $r \in S'_{h-k-1}$. Thus, $S'_{h-k-1} = S_{h-k-1} = \{r\}$ and again the sets agree.

Now assume $h - k - 1 > 0$ and let $x = v^1$ and $y = v^2$ be siblings at the $(h - k - 1)$ level. We consider three possibilities: that both x and y are leaves; that exactly one is a leaf; and that neither is a leaf. In each case we show that

$x \in S_{h-k-1}$ iff $x \in S'_{h-k-1}$ and $y \in S_{h-k-1}$ iff $y \in S'_{h-k-1}$. Since we do this for every pair of vertices, it will follow that $S_{h-k-1} = S'_{h-k-1}$.

Case 1. If both x and y are both leaves, then using the same argument as for S_h, x and y are both in S'_{h-k-1} and also in S_{h-k-1}.

Case 2. If x is a leaf and y is not a leaf, then $x \in S_{h-k-1}$ by the algorithm and we label the two children of y, y^1 and y^2. We first consider the subcase that at least one of y's children, say y^1, is in $S_{h-k} = S'_{h-k}$. Then, since S and S' are independent sets of vertices, $y \notin S_{h-k-1}$ and $y \notin S'_{h-k-1}$. Now, if $x \notin S'_{h-k-1}$, then since x is a leaf and $x = v^1$ and $y = v^2$, it follows that $v \in S'_{h-k-2}$. (Otherwise $S'' = S' \cup \{x\}$ would be an independent set larger than S'.) Consider $S'' = (S' \backslash \{v\}) \cup \{x\}$. If v is the root then S'' is independent and $|S''| = |S'|$. If v is not the root, then since $v \in S'_{h-k-2}$, the parent of v is not in S'_{h-k-3} and so, again, S'' is independent and $|S''| = |S'|$. Without loss of generality, we replace S' by S''. Then x is in S'_{h-k-1} and in S_{h-k-1} ,while y is in neither S_{h-k-1} nor S'_{h-k-1}

For the remaining subcase of 2, assume x is a leaf, y is not a leaf, and neither child of y is in S_{h-k} (and so not in S'_{h-k} by the induction hypothesis). Then by the algorithm, $y \in S_{h-k-1}$ and since x is a leaf, we also have $x \in S_{h-k-1}$. We claim both x and y are in S'_{h-k-1}. The proof follows exactly as in Case1.

Case 3. Now, if neither x nor y is a leaf, then each has two children. We consider the subcases that x and y each has a child in S_{h-k}, that neither has a child in S_{h-k}, and that exactly one has a child in S_{h-k}. If x and y each has a child in S_{h-k}, then that child is in S'_{h-k} by induction, and so by independence, x is neither in S_{h-k-1} nor in S'_{h-k-1} and y is neither in S_{h-k-1} nor in S'_{h-k-1} (so that $x \in S_{h-k-1}$ iff $x \in S'_{h-k-1}$ and $y \in S_{h-k-1}$ iff $y \in S'_{h-k-1}$). If neither x nor y has a child in S_{h-k}, then by the assumption, neither has a child in S'_{h-k}. By the algorithm, it follows that $x \in S_{h-k-1}$ and $y \in S_{h-k-1}$. We claim both x and y are in S'_{h-k-1}. Again, this proof follows exactly as in the Case 1.

For the final subcase of 3, assume x and y each has exactly two children, and say x has a child in S_{h-k} (so in S'_{h-k}) while y has no children in S_{h-k} (so none in S'_{h-k}). Then by the algorithm, $x \notin S_{h-k-1}$ and $y \in S_{h-k-1}$. We claim $x \notin S'_{h-k-1}$ and $y \in S'_{h-k-1}$. Since x has a child in S'_{h-k}, $x \notin S'_{h-k-1}$ by the independence of S'. Recall that v is the common parent of x and y. If $y \notin S'_{h-k-1}$, then $v \in S'_{h-k-2}$. (Otherwise $S'' = S \cup \{y\}$ is an independent which is larger than S.) Thus, if $y \notin S'_{h-k-1}$, $v \in S'_{h-k-2}$. Consider $S'' = (S' \backslash \{v\}) \cup \{y\}$. If v is the root then S'' is independent and $|S''| = |S'|$. If v is not the root, then since $v \in S'_{h-k-2}$, the parent of v is not in S'_{h-k-3} and so, again, S'' is independent and $|S''| = |S'|$. Without loss of generality, we replace S' by S''. Then we have $y \in S'_{h-k-1}$ and $y \in S_{h-k}$ and $x \notin S'_{h-k-1}$ and $x \notin S_{h-k-1}$.

We've shown that for each pair $x, y \in V_j$, $x \in S'_j$ iff $x \in S_j$ and similarly for y. Thus, $S_j = S'_j$ so that $S = S'$ and S is a maximum independent set. ■

Corollary 7 *If G is a full binary tree, then a maximum independent set is obtained by starting at the bottom, and including all vertices at every other level. If the height, h, of the tree is odd, then* $\alpha(G) = \dfrac{2(2^{h+1}-1)}{3}$, *while if h is* *even,* $\alpha(G) = \dfrac{2^{h+2}-1}{3}$.

There are several known upper and lower bounds for $\alpha(G\square H)$. Notice that $\lambda^*(G\square H)$ defined below satisfies $\lambda^*(G\square H) \le \lambda(G\square H) \le \alpha(G\square H)$ and that $\tau(H)\,|G| + \alpha(G)(|H| - 2\tau(H))$ is the upper bound in Remark 2 (2).

Theorem 8 *Let* $H = W_1 + W_2$ *be a bipartite graph satisfying* $\alpha(H) = w_1$ *and let* $G = V_1 + V_2$ *be a bipartite graph. Define* $\lambda^*(G\square H) = \max\{v_1w_1 + v_2w_2,\ \alpha(G)w_1 + \overline{\alpha}(G)w_2\}$. *Then* $\lambda^*(G\square H) = \tau(H)\,|G| + \alpha(G)(|H| - 2\tau(H))$ *if and only if* $v_1 = \alpha(G)$ *or* $w_1 = |H|\ /\ 2$.

Proof. We will denote $\tau(H)\,|G| + \alpha(G)(|H| - 2\tau(H))$ by (ub). Since for a biparitite graph H, $\alpha(H) + \tau(H) = |H|$, [4] , (ub) easily reduces to $w_2v_1 + w_2v_2 + \alpha(G)w_1 - \alpha(G)w_2$. Then $\lambda^*(G\square H) \le \alpha(G\square H) \le (ub)$.

Necessity. Since $\lambda^*(G\square H) = (ub)$, we have either
(1) $\lambda^*(G\square H) = v_1w_1 + v_2w_2$ or
(2) $\lambda^*(G\square H) = \alpha(G)w_1 + \overline{\alpha}(G)w_2$

If (1) holds, then $v_1w_1 + v_2w_2 = \lambda^*(G\square H) = w_2v_1 + w_2v_2 + \alpha(G)w_1 - \alpha(G)w_2$, so that $w_1(v_1 - \alpha(G)) = w_2(v_1 - \alpha(G))$. Equivalently, $0 = (w_1 - w_2)(v_1 - \alpha(G))$. Thus, $w_1 = w_2$ or $v_1 = \alpha(G)$, from which it follows that $w_1 = |H|/2$ or $v_1 = \alpha(G)$.

If (2) holds, then $\alpha(G)w_1 + \overline{\alpha}(G)w_2 = w_2v_1 + w_2v_2 + \alpha(G)w_1 - \alpha(G)w_2$, so $w_2(\alpha(G) + \overline{\alpha}(G)) = v_1w_2 + v_2w_2 = w_2(v_1 + v_2)$. Since $w_2 \neq 0$, $\alpha(G) + \overline{\alpha}(G) = v_1 + v_2 = |G|$. Since the size of a bipartition of a bipartitie graph is unique, $\alpha(G) = v_1$, and the conclusion follows.

For the sufficiency, if $\alpha(G) = V_1$, then $v_1w_1 + v_2w_2 = \lambda^*(G\square H) \le (\text{ub}) = w_2v_1 + w_2v_2 + \alpha(G)w_1 - \alpha(G)w_2 = w_2v_1 + w_2v_2 + v_1w_1 - v_1w_2 = v_1w_1 + v_2w_2$. Thus, $\lambda^*(G\square H) = (\text{ub})$

If $w_1 = |H|\ /\ 2$, then $w_1 - w_2 = 0$, so $v_1w_1 + v_2w_2 \le \lambda^*(G\square H) \le (\text{ub}) = w_2v_1 + w_2v_2 + \alpha(G)w_1 - \alpha(G)w_2 = w_1v_1 + w_2v_2 + 0 = v_1w_1 + v_2w_2$. Again, $\lambda^*(G\square H) = (\text{ub})$ ∎

We will prove in Corollary 11 that if $G = V_1 + V_2$ is biparitite and H is a full binary tree of odd height, then $\lambda^*(G\square H) = (ub)$ if and only if $v_1 = \alpha(G)$.

In [2], Hagauer and Klavzar characterized $\alpha(G\square P_{2n+1})$, where P_{2n+1} is a path of odd length. The lemma below shows that full binary trees act similarly to paths in products with bipartite graphs. A word on notation: for $h \in H$, we have defined $G^h = \{(g,h) : g \in G\}$ and represented a maximum independent set S in $G\square H$ as $S = < S^1, S^2, ..., S^k >$, where $S^k = S \cap G^{h_k}$. We slightly expand this notation for full binary trees. Say the vertices of the full binary tree H are listed from the root down, left to right, as $x_{01}, x_{11}, x_{12}, x_{21}, x_{22}, x_{23}, x_{24}, ..., x_{h1}, x_{h2}, ..., x_{h2^h}$. Let $S_{ij} = S \cap G^{x_{ij}}$. Then

$$S = < S_{01},\ S_{11}S_{12},\ S_{21}S_{22}S_{23}S_{24}, .., S_{h1}S_{h2}...S_{h2^h} > .$$

Notice that we will use subscripts and omit commas between sets at the same level.

Lemma 9 *Let H be a full binary tree of height h. For any graph G, there exists a maximum independent set in $G\Box H$ of the form $< B,\ AA,\ BBBB, ..., AA..A >$ if h is odd and $< A,\ BB,\ AAAA, ..., \ AA...A >$ if h is even, where $|A| \geq |B|$.*

Proof. Let S be a maximum independent set in $G\Box H$. We list the vertices of H from the root down, left to right, as $x_{01}, x_{11}, x_{12}, x_{21}, x_{22}, x_{23}, x_{24}, ..., x_{h1}, x_{h2}, ..., x_{h2^h}$. Let $S_{ij} = S \cap G^{x_{ij}}$. Then

$$S =< S_{01},\ \ S_{11}S_{12}, ..., S_{h1}S_{h2}...S_{h2^h} > .$$

We show, starting at level h, that for every full subtree T of H, $S \cap (G\Box T)$ has constant values on every level of T. Consider the full subtree with root $x_{(h-1)j}$. Since $S_{(h-1)j}$ is independent of both $S_{h(2j-1)}$ and S_{h2j}, then if $|S_{h(2j-1)}| \neq |S_{h2j}|$, we could replace the smaller value with the larger one and have an independent set in $G\Box H$ larger than S, which is not possible. Thus, we can assume $S_{h(2j-1)} = S_{h2j}$.

Now, working our way up, assume that each full binary subtree has constant values at each level and consider the left subtree, call it T_1, with x_{11} as its root and the right subtree, T_2, with x_{12} as its root. That is, we assume that for each i $(i = 2...h)$, $S_{i1} = S_{ij}$ for all $j = 2...2^{i-1}$ and also for each i $(i = 2...h)$, S_{ij} $=S_{i2^i}$ for all $j = 2^{i-1}+1, ..., 2^i$. Let $ST_1 = S \cap (G\Box T_1)$ and $ST_2 = S \cap (G\Box T_2)$. Now, since S_{01} must be independent of both S_{11} and S_{12}, then if $|ST_1| \neq |ST_2|$, then we could replace the subtree with smaller absolute value with the entire larger one and obtain an independent set in $G\Box H$ which is larger than S. Thus, we can assume that they have the same absolute value, and without loss of generality, we can assume they are the same; that is, $ST_1 = ST_2$. We conclude that the independent set which has for each i $(i = 2...h)$, $S_{ij} = S_{i1}$ for all $j = 1...2^i$ has value $|S|$

For each i, let S_i be the common set on the ith level. Then

$$S =< S_0,\ S_1S_1,\ S_2,S_2S_2S_2, ..., S_hS_h....S_h >$$

Now, pick i so that $|S_i| + |S_{i+1}| = \max\{|S_j| + |S_{j+1}|\}$. Pick $A, B \in \{S_i,\ S_{i+1}\}$ so that $|A| = \max\{|S_i|,\ |S_{i+1}|\}$ and $B = \min\{|S_i|, |S_{i+1}|\}$. Finally, let

$$S' =< A,\ BB,\ AAAA, ...,\ AA...A > \textit{if the height h is even}$$

$$\textit{and} \quad S' =< B,\ AA,\ BBBB, ...,\ AA...A > \textit{if h is odd}$$

Then S is independent and $|S| \geq |S|$, so by maximality, $|S'| = |S|$. ■

Theorem 10 *Let $H = W_1 + W_2$ be a full binary tree and let $G = V_1 + V_2$ be a bipartite graph. Then*

1. $\lambda(G\Box H) = \alpha(G\Box H) = \max\{|A|\,w_1 + |B|\,w_2 : A$ *independent in* G, B *independent in* $G\backslash A\}$

2. If H has odd height, then $\alpha(G\Box H) = v_1w_1 + v_2w_2 = \alpha(G)w_1 + \overline{\alpha}(G)w_2$ whenever $v_2 \leq 3$ or $\alpha(G) = v_1$

3. If H has odd height and $v_2 > 3$, then $\alpha(G\Box H) \leq v_1w_1 + v_2w_2 + w_2(v_2 - 3)$

4. If H has even height, then $\alpha(G\Box H) = v_1w_1 + v_2w_2$ whenever $v_2 \leq 2$

Proof. (1) By the preceeding lemma, there exists a maximum independent set of $G\Box H$, of the form $S = < A, BB, AAAA, ..., AAA..A >$ or
$< B, AA, BBBB, ..., AAA...A >$ depending on whether the height is even or odd. Let $a = |A|$ and $b = |B|$ and $w_i = |W_i|$. Then, in either case, we have $|S| = aw_1 + bw_2$, with $a \geq b$. We know that $\lambda(G\Box H) \leq \alpha(G\Box H) = |S| = aw_1 + bw_2$. By definition of $\lambda(G\Box H)$, $aw_1 + bw_2$ is less than $\lambda(G\Box H)$, so $\lambda(G\Box H) \leq \alpha(G\Box H) = \max\{aw_1 + bw_2 : A \text{ independent in } G, B \text{ independent in } G - A\} \leq \lambda(G\Box H)$ and the result follows.

(2) and (3). We assume H has odd height. Then

$$S' =< V_2, V_1V_1, V_2V_2V_2V_2, ..., V_1V_1...V_1 >$$

is an independent set by the construction of $\alpha(H)$ in Algorithm 5. Let S be a maximum independent set in $G\Box H$. Then $|S'| \leq |S|$ and, by the preceeding lemma, we can assume $S =< B, AA, BBBB, ..., AAA...A >$, so that, using the same notation as above, $|S| = aw_1 + bw_2$ with $a \geq b$. Notice that $a + b \leq v_1 + v_2 = |G|$. We can easily verify that in a full binary tree of odd height, $w_2 = 1/2w_1$, so that $|S'| = w_1v_1 + w_2v_2 = 2w_2v_1 + w_2v_2 = w_2(2v_1 + v_2)$. Similarly $|S| = w_2(2a + b)$.

If $a + b = v_1 + v_2$, then since bipartitions are unique, $a = v_1$ and $b = v_2$. It follows that $|S'| = w_2(2v_1 + v_2) = w_2(2a + b) = |S|$.

If $a + b < v_1 + v_2$, then (following [2]), $a + b \leq v_1 + v_2 - 1$ and $a \leq v_1 + v_2 - 1 - b \leq v_1 + v_2 - 2$ since $b \geq 1$. Then $|S| = w_2(2a + b) = w_2(a + (a + b)) \leq w_2(v_1 + v_2 - 1 + v_1 + v_2 - 2) = w_2(2v_1 + v_2 + (v_2 - 3))$.

Whenever $v_2 \leq 3$, this last term is less than or equal to $w_2(2v_1 + v_2) = |S'|$. Thus, in this case $|S| = |S'| = v_1w_1 + v_2w_2$. Now, also consider

$$S'' =< \alpha(G),\ \overline{\alpha}(G)\overline{\alpha}(G), ..., \overline{\alpha}(G)\overline{\alpha}(G)...\overline{\alpha}(G) >$$

Then S" is an independent set in $G\Box H$,and $|S''| = \alpha(G)w_1 + \overline{\alpha}(G)w_2$, so that $|S''| \leq |S| = |S'|$. If $\alpha(G) + \overline{\alpha}(G) = v_1 + v_2$, then again using the fact that bipartitions are unique, we have $|S'| = |S''|$. If $\alpha(G) + \overline{\alpha}(G) < v_1 + v_2$, then $\alpha(G) > v_1$, so that $\overline{\alpha}(G) < v_2 \leq 3$. Thus $\alpha(G) \geq v_1 + 1$ and $3 \geq v_2 > \overline{\alpha}(G) \geq v_2 - 2$. Now we have

$$|S''| = w_2(2\alpha(G) + \overline{\alpha}(G)) \geq w_2(2(v_1 + 1) + v_2 - 2) = w_2(2v_1 + v_2) = |S'| = |S|.$$

This completes the case when $v_2 \leq 3$. When $v_2 > 3$, we have the inequality of the statement. When $\alpha(G) = v_1$, the result follows from [4].

(4) When H has even height, it is easily verified that $w_2 = 1/2(w_1 - 1)$ or equivalently, $w_1 = 2w_2 + 1$. Then using the same notation and inequalities as in part (2), we have:

$$\begin{aligned}|S| &= a(2w_2 + 1) + bw_2 = w_2(2a + b) + a \leq \\ &w_2(v_1 + v_2 - 1 + v_1 + v_2 - 2) + a \leq \\ &w_2(2v_1 + v_2 + v_2 - 3) + v_1 + v_2 - 2 = \\ &w_2(2v_1 + v_2) + v_1 + w_2(v_2 - 3 + v_2 - 2) = \\ &w_2(2v_1 + v_2) + v_1 + w_2(2v_2 - 5) = |S'| + w_2(2v_2 - 5)\end{aligned}$$

If $v_2 \leq 2$, this last term is less than $|S'|$. ■

Corollary 11 *Let $G = V_1 + V_2$ be bipartite and H a full binary tree of odd height. Then $\lambda(G\Box H) = (ub)$ if and only if $v_1 = \alpha(G)$.*

Proof. Since H has odd height, $w_1 = 2w_2$ and $\lambda(G\Box H) = \alpha(G\Box H) = \max\{Aw_1 + Bw_2 : A \text{ independent in } G,\ B \text{ independent in } G\setminus A\}$. If $\alpha(G) = v_1$, then $\alpha(G\Box H) = v_1w_1 + v_2w_2 = \lambda(G\Box H)$. It than follows from Theorem 10, that $\lambda^*(G\Box H) = \text{(ub)}$, and hence, $\lambda(G\Box H) = (ub)$.

Conversely, if $\lambda(G\Box H) = \text{(ub)}$, then we have $\lambda(G\Box H) = \alpha(G\Box H) = \max\{aW_1 + bW_2 : A \text{ independent in } G,\ B \text{ independent in } G \setminus A\} = (ub)$, so there are independent sets A and B so that

$$aw_1 + bw_2 = v_2w_2 + v_1w_2 + \alpha(G)(w_1 - w_2) = w_2\,|G| + \alpha(G)w_2$$

since $w_1 = 2w_2$. Then $aw_1 + bw_2 = w_2(|G| + \alpha(G))$, and hence $aw_1 = w_2(|G| + \alpha(G) - b)$. But since $B \subseteq G\backslash A$, $A \subseteq G\backslash B$, and we have $|G| - b \geq a$. It follows that

$$aw_1 = w_2(|G| + \alpha(G) - b) \geq w_2(a + \alpha(G)) \geq w_2(a + a) = 2w_2a = w_1a.$$

Therefore, equalities hold throughout, and so $|G| - b = a$. Thus, $a + b = |G| = v_1 + v_2$. Since bipartitions of bipartite graphs are unique, it must be that $A = V_1$ and $B = V_2$, and hence, $\lambda^*(G\Box H) \leq \lambda(G\Box H) = \alpha(G\Box H) = v_1w_1 + v_2w_2 \leq \lambda^*(G\Box H)$. Thus, $\lambda^*(G\Box H) = (ub)$ and so it follows from Theorem 8 that $\alpha(G) = v_1$. ∎

As mentioned in the introduction, one technique that can be used to find the independence number of a product is to find lower and upper bounds and then examine each possibility between these bounds. It would be useful to know when the upper bound is not too much larger than the lower bound. Unfortunately, the next example shows that the upper bound may be arbitrarily larger than the independence number, complicating the computational process, even for very simple graphs.

Example 12 *Let $H = K_{1,2}$, the complete bipartite graph on one and two vertices. Then for each positive integer k, there is a binary tree $G_k = V_1 + V_2$ satisfying:*

1. $\alpha(G_k) = v_1 + k$
2. $\lambda(G_k\Box H) = \alpha(G_k\Box H) = 2v_1 + v_2 = 2\alpha(G) + \overline{\alpha}(G) = 2 + 12k$
3. $(ub) = \alpha(G_k\Box H) + k$

Proof. Let G^* be the binary graph on nine vertices shown in Figure 1. Let $G_1 = G^*$. It is easily verified that $v_1 = 5$, $v_2 = 4$, $\alpha(G) = 6$, and $\overline{\alpha}(G) = 2$. Assume G_{k-1} has been defined. Identify the bottom right vertex of G_{k-1} with the root of a copy of G^*. The resulting graph is G_k.

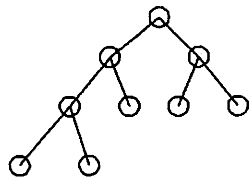

Figure 1: A tree on 9 vertices.

For G_k, the following are easily proved by induction: $\alpha(G) = 1 + 5k$, $v_1 = 1 + 4k$, $v_2 = 4k$, $\overline{\alpha}(G) = 2k$, and $|G| = 1 + 8k$. It then follows immediately that $\alpha(G_k) = v_1 + k$ and then from these results and the previous theorem that

$$\begin{aligned} 2v_1 + v_2 &= 2\alpha(G) + \overline{\alpha}(G) = 2 + 12k \leq \lambda(G\Box H) = \alpha(G\Box H) \\ &= \max\{aw_1 + bw_2 : A \textit{ independent in } G,\ B \textit{ independent in } G\backslash A\} \\ &= \max\{(2a + b) : A \textit{ independent in } G, B \textit{ independent in } G\backslash A\}. \end{aligned}$$

Notice that $w_2 = 2w_1 = 2$, which explains the final equality. We claim that the inequality is actually an equality.

Assume there are sets A and B in G_k such that A is independent in G_k, B is independent in $G_k\backslash A$ and $2a + b$ is a maximum. In the top copy of G^*, the only choices for A and B force a and b to divide as 6 and 2 respectively or $5 - 4$ or $5 - 3$ and then the largest contribution to $2a + b$ from this top copy is 14. Notice that that the root and the bottom right vertex are both in A or both in B. As we move down through copies of G^*, then since the root of each copy is already labeled, the maximum contribution to $2a + b$ from each copy is 12. Thus the maximum value of $2a + b$ on G^* is $14 + 12(k - 1) = 12k + 2$, and so equality holds.

For (3), notice that $(ub) = w_2(|G|) + \alpha(G)(w_2 - w_1) = 1(1+8k) + (1+5k)(1) = 2 + 13k = \alpha(G_k\Box H) + k$. ∎

We've shown above that for a biparitie graph G and a full binary tree H, $\lambda(G\Box H) = \alpha(G\Box H)$. The following conjecture seems natural, yet the author has only been able to prove it in the special case that $|H| \leq 9$. G^* will denote the graph in the previous example.

Conjecture 13 *If G is bipartite and H is a binary tree, then $\lambda(G\Box H) = \alpha(G\Box H)$.*

Lemma 14 *Let $H = G^*$. If G is bipartite then $\lambda(G\Box H) = \alpha(G\Box H)$*

Proof. Let S be a maximum independent set in $G\Box H$. Then S must be of the form $< R, T, U, V, W, X, X, Y, Y >$, where we list the sets from the root down, left to right. Now say A, B, C are maximal independent sets in G with absolute values a, b, c respectively. If $|U| = |V|$, then assuming $a > b$, the largest possible value for $|S|$ is $5a+4b$. If $|U| \neq |V|$, then the largest possible value is $6a+5b+c$ or $5a+3b+c$. Thus $\lambda(G\Box H) \leq \alpha(G\Box H) = \max[5a+4b, 6a+2b+c, 5a+3b+c\}$, and this last set is in the form of a diagonal set, and so equality holds. ∎

Theorem 15 *Let H be a binary tree with $|H| \leq 9$ and let G be bipartite, then* $\lambda(G\Box H) = \alpha(G\Box H)$.

Proof. There are exactly seven binary trees with nine or fewer vertices, up to isomorphism. One of these seven is G^*, and so in that case, the result follows from the preceeding lemma. Two of the six are full binary trees, and then the result follows from Theorem 10 . There are four remaining cases. The first of these trees is shown below.

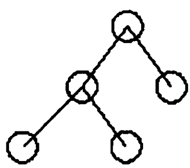

Figure 2: A tree on 5 vertices.

For this tree, $\alpha(H) = w_1 = 3$ and $\alpha(G) = w_2 = 2$. A maximum independent set S in $G\Box H$ is of the form $< R, T, U, V, V >$. If $|T| = |U|$, A is a maximal independent set in G, and B is a maximal independent sets in $G\backslash A$, then the largest possible value for $|S|$ is $3a + 2b$. If $|T| \neq |U|$, then the largest possible value for $|S|$ is still $3a + 2b$, and this value is $max\{aw_1 + bw_2 : A, B$ maximal independent in $G\} \leq \lambda(G\Box H)$, so that $\lambda(G\Box H) = \alpha(G\Box H)$.

The verifications for the other cases follow similarly. ∎

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