Nearly antipodal chromatic number of even paths*

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Abstract

For paths P_n, Chartrand, Nebeský and Zhang gave the exact value of $\text{ac}'(P_n)$ for $n \le 8$, and showed that $\text{ac}'(P_n) \le \binom{n-2}{2} + 2$ for every positive integer n, where $\text{ac}'(P_n)$ denotes the nearly antipodal chromatic number of P_n. In this paper, we determine the exact values of $\text{ac}'(P_n)$ for all even integers $n \ge 8$.

Keywords: Radio colorings; Nearly antipodal chromatic number; Paths.

1 Introduction

Radio k-colorings were introduced by Chartrand, Erwan, Harary and Zhang [1], which were inspired by (FM Radio) Channel Assignments Problem (see [6]). For a connected graph G of order n and diameter d and a integer k with $1 \le k \le d$, a radio k-coloring of G is a function c: $V(G) \to \mathbf{N}$, such that $d(u,v) + |c(u) - c(v)| \ge k+1$ for every pair u and v of distinct vertices of G, where $d(u,v)$ denotes the distance between u and v (the length of a shortest $u-v$ path) in G. The *value* $\text{rc}_k(c)$ of a radio k-coloring c of G is

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the maximum color assigned to a vertex of G; while the *radio k-chromatic number* $\mathrm{rc}_k(G)$ of G is $\min\{\mathrm{rc}_k(c)\}$ taken over all k-coloring c of G.

It is easy to see that the radio 1-coloring and ordinary colorings are synonymous, and the radio 2-coloring problem corresponds to the well studied $L(2,1)$ (see [5] and references therein). Consequently, radio k-colorings generalize many graph colorings. Moreover, radio d-colorings are referred to as *radio labelings* and the *radio d-chromatic number* is called the *radio number*, denoted by $\mathrm{rn}(G)$. Radio $(d-1)$-colorings are referred to as *antipodal coloring* and the *radio $(d-1)$-chromatic number* is called the *antipodal chromatic number*, denoted by $\mathrm{ac}(G)$. Radio $(d-2)$-colorings are referred to as *nearly antipodal coloring* and the *radio $(d-2)$-chromatic number* is called the *nearly antipodal chromatic number*, denoted by $\mathrm{ac}'(G)$. Some results of radio k-coloring, radio labeling, antipodal coloring and nearly antipodal coloring of some graphs can be found in [1, 2, 3, 4, 7, 8, 9, 10].

For paths $P(_n)$, the exact values of $\mathrm{rn}(P_n)$ and $\mathrm{ac}(P_n)$ were determined in [9] and [7], respectively. Note that if G is a connected graph of diameter 1 or 2, then $\mathrm{ac}'(G) = 1$; while if $\mathrm{diam}(G) = 3$, then $\mathrm{ac}'(G)$ is the chromatic number of G. Thus nearly antipodal colorings are most interesting for connected graphs of diameter 4 or more. For this reason, the nearly antipodal chromatic number of paths P_n were investigated in [4] by Chartrand, Nebeský and Zhang. They showed that $\mathrm{ac}'(P_5) = 5$, $\mathrm{ac}'(P_6) = 7$, $\mathrm{ac}'(P_7) = 11$ and $\mathrm{ac}'(P_8) = 16$, but the exact values of $\mathrm{ac}'(P_n)$ for all integers $n \geq 9$ were not determined.

In [4], Chartrand, Nebeský and Zhang presented an upper bound of $\mathrm{ac}'(P_n)$ for every positive integer n as follows.

Theorem 1.1 ([4]). *If n is a path of order $n \geq 1$, $\mathrm{ac}'(P_n) \leq \binom{n-2}{2} + 2$.*

In [10], Shen et al. provided an improved version for Theorem 1.1, they showed that

Theorem 1.2 ([10]). *If n is even and $n \geq 10$, then $\mathrm{ac}'(P_n) \leq \binom{n-2}{2} - \frac{n}{2} - \lfloor \frac{10}{n} \rfloor + 7$; if n is odd and $n \geq 13$, then $\mathrm{ac}'(P_n) \leq \binom{n-2}{2} - \frac{n-1}{2} - \lfloor \frac{13}{n} \rfloor + 8$.*

In this paper, we will present the exact values of $\mathrm{ac}'(P_n)$ for all even integers $n \geq 8$. We will show that $\mathrm{ac}'(P_{2p}) = 2p^2 - 6p + 8$ for every integer $p \geq 4$.

2 The exact values of $\mathrm{ac}'(P_n)$ for $p \geq 4$

Theorem 2.1. *For every integer $p \geq 1$, $\mathrm{ac}'(P_{2p}) \leq 2p^2 - 6p + 8$.*

Proof. Let $P_{2p} = (u_1, u_2, \ldots, u_{2p})$. Define a coloring c of path P_{2p}:

$$\begin{cases} f(u_1) = p - 1, \\ f(u_i) = (p-i)(2p-3) + 2, \quad 2 \le i \le p-1, \\ f(u_p) = 2p^2 - 6p + 8, \\ f(u_{p+i}) = 2p^2 - 6p + 9 - f(u_{p-i+1}), \quad 1 \le i \le p. \end{cases}$$

It is easy to see that the vertex u_p has the maximum color $c(u_p) = 2p^2 - 6p + 8$. It suffices to show the distance condition:

$$d(u_i, u_j) + |c(u_i) - c(u_j)| \ge (d-2) + 1 = 2p - 2 \tag{1}$$

for every pair u_i, u_j of distinct vertices of P_{2p}.

We only need to verify that (1) holds for two vertices u_i and u_{p+j} for $2 \le i \le p-1$ and $1 \le j \le p$ (the other cases can be checked easily). In fact,

$$\begin{aligned} & d(u_i, u_{p+j}) + |c(u_i) - c(u_{p+j})| \\ &= (p+j-i) + |c(u_i) - [(2p^2 - 6p + 9) - c(u_{p-j+1})]| \\ &= (p+j-i) + |(p+j-i-1)(2p-3) + 4 - (2p^2 - 6p + 9)| \\ &= (p+j-i) + |(j-i)(2p-3) - (p-2)| \\ &= \begin{cases} (j-i)(2p-2) + 2 \ge 2p-2, & \text{if } i \le j-1; \\ 2p - 2 + (i-j)(2p-4) \ge 2p-2, & \text{if } i \ge j. \end{cases} \end{aligned}$$

□

In order to show that $2p^2 - 6p + 8$ is also a lower bound of $\mathrm{ac}'(P_{2p})$ for every integer $p \ge 4$, we establish a lemma as follows.

Lemma 2.1. *Let c be a nearly antipodal coloring of path P_n, order the vertices of P_n as $x_1, x_2, \ldots, x_n$ such that $c(x_{i-1}) \le c(x_i)$ for $i = 2, 3, \ldots, n$. Denote $c(x_i) - c(x_{i-1}) = (n-2) - d(x_{i-1}, x_i) + \varepsilon_i$, $i = 2, 3, \ldots, n$, where $\varepsilon_i \ge 0$. If there exists x_i $(2 \le i \le n-1)$ such that $\min\{d(x_{i-1}, x_i), d(x_i, x_{i+1})\} \ge \frac{n}{2}$, then $\varepsilon_i + \varepsilon_{i+1} \ge 2$.*

Proof. Assume, to the contrary, that $\varepsilon_i + \varepsilon_{i+1} < 2$. Without loss of generality, we assume that $d(x_i, x_{i+1}) \ge d(x_{i-1}, x_i) \ge \frac{n}{2}$. Then $d(x_i, x_{i+1}) = d(x_{i-1}, x_i) + d(x_{i-1}, x_{i+1})$, and

$$\begin{aligned} & c(x_{i+1}) - c(x_{i-1}) \\ &= c(x_{i+1}) - c(x_i) + c(x_i) - c(x_{i-1}) \\ &= (n-2) - d(x_i, x_{i+1}) + \varepsilon_{i+1} + (n-2) - d(x_{i-1}, x_i) + \varepsilon_i \\ &< (n-2) - d(x_{i-1}, x_{i+1}) + n - 2d(x_{i-1}, x_i) \\ &\le (n-2) - d(x_{i-1}, x_{i+1}), \end{aligned}$$

contrary to that c is a nearly antipodal coloring of P_n. □

Theorem 2.2. *For every integer $p \geq 4$, $\mathrm{ac}'(P_{2p}) \geq 2p^2 - 6p + 8$.*

Proof. Let $P_{2p} = (u_1, u_2, \ldots, u_{2p})$ and let c be a nearly antipodal coloring of P_{2p}. Reorder the vertices of P_{2p} as $x_1, x_2, \ldots, x_{2p}$ such that $c(x_{i-1}) \leq c(x_i)$ for $i = 2, 3, \ldots, 2p$. Denote $c(x_i) - c(x_{i-1}) = (2p-2) - d(x_{i-1}, x_i) + \varepsilon_i, i = 2, 3, \ldots, 2p$, where $\varepsilon_i \geq 0$. Let $x_i = u_{\sigma(i)}$, where σ is a permutation of $\{1, 2, \ldots, 2p\}$.

By the definition of c, $c(x_1) \geq 1$ and $c(x_i) \geq c(x_{i-1}) + (2p-2) - d(x_{i-1}, x_i)$ for $i = 2, 3, \ldots, 2p$. Thus, we have that

$$\begin{aligned} & c(x_{2p}) \\ & \geq 1 + \textstyle\sum_{i=2}^{2p}[(2p-2) - d(x_{i-1}, x_i) + \varepsilon_i] \\ & = (4p^2 - 6p + 3) - \textstyle\sum_{i=2}^{2p} d(x_{i-1}, x_i) + \sum_{i=2}^{2p} \varepsilon_i. \end{aligned} \qquad (2)$$

If $\sum_{i=2}^{2p} d(x_{i-1}, x_i) \leq 2p^2 - 5$, then $c(x_{2p}) \geq 2p^2 - 6p + 8$ by (2), and we are done. Hence, assume $\sum_{i=2}^{2p} d(x_{i-1}, x_i) > 2p^2 - 5$.

Claim 2.1. *If $\sum_{i=2}^{2p} d(x_{i-1}, x_i) > 2p^2 - 5$, then $2p^2 - 4 \leq \sum_{i=2}^{2p} d(x_{i-1}, x_i) \leq 2p^2 - 1$.*

In fact, note that $d(x_{i-1}, x_i)$ is equal to either $\sigma(i) - \sigma(i-1)$ or $\sigma(i-1) - \sigma(i)$, whichever is positive. By replacing each term $d(x_{i-1}, x_i)$ with the corresponding $\sigma(i) - \sigma(i-1)$ or $\sigma(i-1) - \sigma(i)$, whichever is positive, we obtain a summation whose entries are $\pm j$ for $j \in \{1, 2, \ldots, 2p\}$.

All together, there are $4p - 2$ terms in the summation $\sum_{i=2}^{2p} d(x_{i-1}, x_i)$, half of them are positive and half of them are negative. Each $j \in \{1, 2, \ldots, 2p\}$ occurs as $\pm j$ exactly twice in the summation, except for two j's where each occurs as j or $-j$ only once.

To maximize the summation $\sum_{i=2}^{2p} d(x_{i-1}, x_i)$, one needs to minimize the absolute values for the negative terms while maximize the values for the positive terms. It is easy to verify that there is only one combination achieving the maximum summation: each of the number in $\{p+2, p+3, \ldots, 2p\}$ occurs twice as a positive, each of $\{1, 2, \ldots, p-1\}$ occurs twice as a negative, while p and $p+1$ occurs once, respectively, as a negative term and a positive term. Thus, the maximum summation is equal to $2[(p+2)+(p+3)+\ldots+2p] - 2[1+2+\ldots+(p-1)] - p + (p+1) = 2p^2 - 1$. Therefore, it holds that if $\sum_{i=2}^{2p} d(x_{i-1}, x_i) > 2p^2 - 5$, then $2p^2 - 4 \leq \sum_{i=2}^{2p} d(x_{i-1}, x_i) \leq 2p^2 - 1$.

In the following, according to the values of $\sum_{i=2}^{2p} d(x_{i-1}, x_i)$, we consider four cases to prove that $c(x_{2p}) \geq 2p^2 - 6p + 8$ respectively.

Case 1. $\sum_{i=2}^{2p} d(x_{i-1}, x_i) = 2p^2 - 1$.

In this case, by the discussion above only one combination achieving $\sum_{i=2}^{2p} d(x_{i-1}, x_i) = 2p^2 - 1$, we must have $\{\sigma(1), \sigma(2p)\} = \{p+1, p\}$. And in the only one combination, $\sigma(i) \geq p+1$ if and only if $\sigma(i+1) \leq p$. Denote $\sigma(i_0) = 1$ and $\sigma(j_0) = 2p$, where $i_0 \neq j_0$.

Subcase 1.1. $\sigma(1) = p+1, \sigma(2p) = p$.

We have $\sigma(i_0-1) \geq p+1, \sigma(i_0+1) \geq p+2; \sigma(j_0-1) \leq p-1, \sigma(j_0+1) \leq p$. Thus x_{i_0} and x_{j_0} satisfy the condition of Lemma 2.1, respectively. Therefore we have that $\varepsilon_{i_0} + \varepsilon_{i_0+1} \geq 2$ and $\varepsilon_{j_0} + \varepsilon_{j_0+1} \geq 2$.

Subcase 1.1.1. $|j_0 - i_0| \geq 2$.

It is easy to see that $\sum_{i=2}^{2p} \varepsilon_i \geq 4$. Thus, by (2), we have $c(x_{2p}) \geq (4p^2 - 6p + 3) - (2p^2 - 1) + 4 = 2p^2 - 6p + 8$.

Subcase 1.1.2. $j_0 = i_0 + 1$.

Note that $\sigma(1) = p+1$, $\sigma(2p) = p$, $c(x_{i_0-1}) \leq c(x_{i_0})(= c(x_{j_0-1})) \leq c(x_{i_0+1})(= c(x_{j_0})) \leq c(x_{j_0+1})$ and $p \geq 4$, we have that $\sigma(i_0 - 1) > \sigma(1) = p+1$ or $\sigma(j_0 + 1) < \sigma(2p) = p$. If $\sigma(i_0 - 1) > \sigma(1) = p+1$, we consider three vertices x_{i_0-1}, x_{i_0} and x_{j_0} (If $\sigma(j_0 + 1) < \sigma(2p) = p$, consider three vertices x_{i_0}, x_{j_0} and x_{j_0+1}, and the proof is similar). Since $d(x_{i_0-1}, x_{j_0}) = \sigma(j_0) - \sigma(i_0 - 1) \leq 2p - (p+2) = p-2$, $d(x_{i_0}, x_{j_0}) = \sigma(j_0) - \sigma(i_0) = 2p-1$, $d(x_{i_0-1}, x_{i_0}) = \sigma(i_0 - 1) - \sigma(i_0) \geq (p+2) - 1 = p+1$, we have $d(x_{i_0-1}, x_{i_0}) \geq d(x_{i_0-1}, x_{j_0}) + 3$. We claim that $\varepsilon_{i_0} + \varepsilon_{j_0} \geq 4$, for otherwise, $c(x_{j_0}) - c(x_{i_0-1}) = c(x_{j_0}) - c(x_{i_0}) + c(x_{i_0}) - c(x_{i_0-1}) = (2p-2) - d(x_{i_0}, x_{j_0}) + \varepsilon_{j_0} + (2p-2) - d(x_{i_0-1}, x_{i_0}) + \varepsilon_{i_0} < (2p-2) - d(x_{i_0-1}, x_{j_0})$, contrary to that c is a nearly antipodal coloring of P_n. Thus, we have $\sum_{i=2}^{2p} \varepsilon_i \geq 4$, and by (2), we also have $c(x_{2p}) \geq 2p^2 - 6p + 8$.

Subcase 1.1.3. $i_0 = j_0 + 1$.

In this case, by the discussion above only one combination achieving $\sum_{i=2}^{2p} d(x_{i-1}, x_i) = 2p^2 - 1$ and $p \geq 4$, we must have $\sigma(i_0+1) > \sigma(1) = p+1$ and $\sigma(j_0 - 1) < \sigma(2p) = p$. So we can show $c(x_{2p}) \geq 2p^2 - 6p + 8$ by an argument similar to that used in Subcase 1.1.2.

Subcase 1.2. $\sigma(1) = p, \sigma(2p) = p+1$.

We have $\sigma(i_0-1) \geq p+2, \sigma(i_0+1) \geq p+1; \sigma(j_0-1) \leq p, \sigma(j_0+1) \leq p-1$. And we can show $c(x_{2p}) \geq 2p^2 - 6p + 8$ by an argument similar to that used in Subcase 1.1.

Case 2. $\sum_{i=2}^{2p} d(x_{i-1}, x_i) = 2p^2 - 2$.

In this case, it is easy to verify that there are two combinations of the terms in the summation $\sum_{i=2}^{2p} d(x_{i-1}, x_i)$ achieving $2p^2 - 2$, and we consider two subcases according to the two combinations as follows.

Subcase 2.1. $2p^2 - 2 = \sum_{i=2}^{2p} d(x_{i-1}, x_i) = 2[(p+2) + (p+3) + \ldots + 2p] - 2[1 + 2 + \ldots + (p-2) + p] - (p-1) + (p+1)$.

In this case, we must have $\{\sigma(1), \sigma(2p)\} = \{p+1, p-1\}$. And it is easy to see that $\sigma(i) \geq p+1$ if and only if $\sigma(i+1) \leq p$. Denote $\sigma(i_0) = 1$ and $\sigma(j_0) = 2p$.

Subcase 2.1.1. $\sigma(1) = p+1, \sigma(2p) = p-1$.

Note that $\sigma(i) \geq p+1$ if and only if $\sigma(i+1) \leq p$, we have $\sigma(i_0 - 1) \geq$

$p+1, \sigma(i_0+1) \geq p+2; \sigma(j_0-1) \leq p, \sigma(j_0+1) \leq p$. Thus x_{i_0} and x_{j_0} satisfy the condition of Lemma 2.1, respectively. Therefore we have that $\varepsilon_{i_0} + \varepsilon_{i_0+1} \geq 2$ and $\varepsilon_{j_0} + \varepsilon_{j_0+1} \geq 2$.

Subcase 2.1.1.1. $|j_0 - i_0| \geq 2$.

It is easy to see that $\sum_{i=2}^{2p} \varepsilon_i \geq 4$. Thus, by (2), we have $c(x_{2p}) \geq (4p^2 - 6p + 3) - (2p^2 - 2) + 4 = 2p^2 - 6p + 9$.

Subcase 2.1.1.2. $j_0 = i_0 + 1$.

Note that $\sigma(1) = p+1$, $\sigma(2p) = p-1$ (that is $x_1 = u_{p+1}$, $x_{2p} = u_{p-1}$), $c(x_{i_0-1}) \leq c(x_{i_0})(= c(x_{j_0-1})) \leq c(x_{i_0+1})(= c(x_{j_0})) \leq c(x_{j_0+1})$ and $p \geq 4$. If $\sigma(i_0 - 1) > p+1$ or $\sigma(j_0+1) < p$, we can show that $\varepsilon_{i_0} + \varepsilon_{j_0} \geq 4$ or $\varepsilon_{j_0} + \varepsilon_{j_0+1} \geq 4$ by an argument similar to that used in Subcase 1.1.2, and hence $\sum_{i=2}^{2p} \varepsilon_i \geq 4$. Thus, by (2), we have $c(x_{2p}) \geq (4p^2 - 6p + 3) - (2p^2 - 2) + 4 = 2p^2 - 6p + 9$.

If $\sigma(i_0 - 1) = p + 1$ and $\sigma(j_0 + 1) = p$, recall that $\sigma(1) = p + 1$, $\sigma(2p) = p - 1$ and $p \geq 4$, there must exist $s_0 > j_0 + 1 > i_0 + 1$, such that $\sigma(s_0) = 2$, $\sigma(s_0 - 1) \geq p+2$ and $\sigma(s_0+1) \geq p+2$. Then it is easy to see that x_{i_0} and x_{s_0} satisfy the condition of Lemma 2.1, respectively. Therefore we have that $\varepsilon_{i_0} + \varepsilon_{i_0+1} \geq 2$ and $\varepsilon_{s_0} + \varepsilon_{s_0+1} \geq 2$. Thus, we have $\sum_{i=2}^{2p} \varepsilon_i \geq 4$, and by (2), we also have $c(x_{2p}) \geq 2p^2 - 6p + 9$.

Subcase 2.1.1.3. $i_0 = j_0 + 1$.

Note that $\sigma(i_0 + 1) \geq p + 2$, we consider three vertices x_{j_0}, x_{i_0} and x_{i_0+1}. We can show that $\varepsilon_{i_0} + \varepsilon_{i_0+1} \geq 4$ by an argument similar to that used in Subcase 1.1.2, and hence $\sum_{i=2}^{2p} \varepsilon_i \geq 4$. Thus, by (2), we have $c(x_{2p}) \geq (4p^2 - 6p + 3) - (2p^2 - 2) + 4 = 2p^2 - 6p + 9$.

Subcase 2.1.2. $\sigma(1) = p - 1, \sigma(2p) = p + 1$.

Note that $\sigma(i) \geq p+1$ if and only if $\sigma(i+1) \leq p$, we have $\sigma(i_0 - 1) \geq p+2, \sigma(i_0+1) \geq p+1; \sigma(j_0-1) \leq p, \sigma(j_0+1) \leq p$. Thus x_{i_0} and x_{j_0} satisfy the condition of Lemma 2.1, respectively. Therefore we have that $\varepsilon_{i_0} + \varepsilon_{i_0+1} \geq 2$ and $\varepsilon_{j_0} + \varepsilon_{j_0+1} \geq 2$.

Subcase 2.1.2.1. $|j_0 - i_0| \geq 2$.

Similar to Subcase 2.1.1.1, we have $c(x_{2p}) \geq 2p^2 - 6p + 9$.

Subcase 2.1.2.2. $j_0 = i_0 + 1$.

By $\sigma(i_0 - 1) \geq p+2$, we can show $c(x_{2p}) \geq 2p^2 - 6p + 9$ by an argument similar to that used in Subcase 1.1.2.

Subcase 2.1.2.3. $i_0 = j_0 + 1$.

Note that $\sigma(1) = p - 1$, $\sigma(2p) = p + 1$, $c(x_{j_0-1}) \leq c(x_{j_0})(= c(x_{i_0-1})) \leq c(x_{i_0})(= c(x_{j_0+1})) \leq c(x_{i_0+1})$ and $p \geq 4$. Then, by the symmetry of P_{2n}, we can show that $c(x_{2p}) \geq 2p^2 - 6p + 9$ by an argument similar to that used in Subcase 2.1.1.2.

Subcase 2.2. $2p^2 - 2 = \sum_{i=2}^{2p} d(x_{i-1}, x_i) = 2[(p+1) + (p+3) + \ldots + 2p] - 2[1 + 2 + \ldots + (p-1)] - p + (p+2)$.

In this case, we must have $\{\sigma(1), \sigma(2p)\} = \{p, p+2\}$. And it is easy to see that $\sigma(i) \geq p+1$ if and only if $\sigma(i+1) \leq p$. Denote $\sigma(i_0) = 1$ and $\sigma(j_0) = 2p$.

Subcase 2.2.1. $\sigma(1) = p, \sigma(2p) = p+2$.

By the symmetry of P_{2n}, we can show $c(x_{2p}) \geq 2p^2 - 6p + 9$ by an argument similar to that used in Subcase 2.1.1.

Subcase 2.2.2. $\sigma(1) = p+2, \sigma(2p) = p$.

By the symmetry of P_{2n}, we can show $c(x_{2p}) \geq 2p^2 - 6p + 9$ by an argument similar to that used in Subcase 2.1.2.

Case 3. $\sum_{i=2}^{2p} d(x_{i-1}, x_i) = 2p^2 - 3$.

In this case, it is easy to verify that there are three combinations of the terms in the summation $\sum_{i=2}^{2p} d(x_{i-1}, x_i)$ achieving $2p^2 - 3$, and we consider three subcases according to the three combinations as follows.

Subcase 3.1. $2p^2 - 3 = \sum_{i=2}^{2p} d(x_{i-1}, x_i) = 2[(p+2)+(p+3)+\ldots+2p] - 2[1+2+\ldots+(p-3)+(p-1)+p] - (p-2)+(p+1)$.

Subcase 3.2. $2p^2 - 3 = \sum_{i=2}^{2p} d(x_{i-1}, x_i) = 2[(p+1)+(p+3)+\ldots+2p] - 2[1+2+\ldots+(p-2)+p] - (p-1)+(p+2)$.

Subcase 3.3. $2p^2 - 3 = \sum_{i=2}^{2p} d(x_{i-1}, x_i) = 2[(p+1)+(p+2)+(p+4)+\ldots+2p] - 2[1+2+\ldots+(p-1)] - p+(p+3)$.

In the three cases above, we must have that $\{\sigma(1), \sigma(2p)\}$ is equal to $\{p+1, p-2\}$, $\{p+2, p-1\}$ or $\{p+3, p-p\}$, respectively. And it is easy to see that $\sigma(i) \geq p+1$ if and only if $\sigma(i+1) \leq p$ in all the three cases above. If denote $\sigma(i_0) = 1$, then $\sigma(i_0 - 1) \geq p+1$ and $\sigma(i_0+1) \geq p+1$. Thus x_{i_0} satisfy the condition of Lemma 2.1, therefore we have that $\varepsilon_{i_0} + \varepsilon_{i_0+1} \geq 2$. This implies that $\sum_{i=2}^{2p} \varepsilon_i \geq 2$. Thus, by (2), we have $c(x_{2p}) \geq (4p^2 - 6p + 3) - (2p^2 - 3) + 2 = 2p^2 - 6p + 8$.

Case 4. $\sum_{i=2}^{2p} d(x_{i-1}, x_i) = 2p^2 - 4$.

In this case, it is easy to verify that there are four combinations of the terms in the summation $\sum_{i=2}^{2p} d(x_{i-1}, x_i)$ achieving $2p^2 - 4$, and we consider four subcases according to the four combinations as follows.

Subcase 4.1. $2p^2 - 4 = \sum_{i=2}^{2p} d(x_{i-1}, x_i) = 2[(p+2)+(p+3)+\ldots+2p] - 2[1+2+\ldots+(p-4)+(p-2)+(p-1)+p] - (p-3)+(p+1)$.

Subcase 4.2. $2p^2 - 4 = \sum_{i=2}^{2p} d(x_{i-1}, x_i) = 2[(p+1)+(p+3)+\ldots+2p] - 2[1+2+\ldots+(p-3)+(p-1)] - (p-2)+(p+2)$.

Subcase 4.3. $2p^2 - 4 = \sum_{i=2}^{2p} d(x_{i-1}, x_i) = 2[(p+1)+(p+2)+(p+4)+\ldots+2p] - 2[1+2+\ldots+(p-2)+p] - (p-1)+(p+3)$.

Subcase 4.4. $2p^2 - 4 = \sum_{i=2}^{2p} d(x_{i-1}, x_i) = 2[(p+1)+(p+2)+(p+3)+(p+5)+\ldots+2p] - 2[1+2+\ldots+(p-1)] - p+(p+4)$.

In the four cases above, we must have that $\{\sigma(1), \sigma(2p)\}$ is equal to $\{p+1, p-3\}$, $\{p+2, p-2\}$, $\{p+3, p-1\}$ or $\{p+4, p\}$, respectively. And it is easy to see that $\sigma(i) \geq p+1$ if and only if $\sigma(i+1) \leq p$ in all the four cases above. If denote $\sigma(i_0) = 1$, then $\sigma(i_0 - 1) \geq p+1$ and $\sigma(i_0+1) \geq p+1$. Therefore x_{i_0} satisfies the condition of Lemma 2.1, and we have that $\varepsilon_{i_0} + \varepsilon_{i_0+1} \geq 2$. This implies that $\sum_{i=2}^{2p} \varepsilon_i \geq 2$. Thus, by (2), we have $c(x_{2p}) \geq (4p^2 - 6p + 3) - (2p^2 - 4) + 2 = 2p^2 - 6p + 9$. □

Theorem 2.3. *For every integer $p \geq 4$, $\mathrm{ac}'(P_{2p}) = 2p^2 - 6p + 8$.*

Proof. Combining Theorem 2.1 and Theorem 2.2, Theorem 2.3 holds. □

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