ON GRACEFUL AND CORDIAL LABELING OF SHELL GRAPHS

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Abstract

We recall from [13] a *shell* graph of size n, denoted $C(n, n\ \ 3)$, is the graph obtained from the cycle $C_n\langle v_0, v_1, v_2, \ldots, v_{n\ 1}\rangle$ by adding $n\ \ 3$ consecutive chords incident at a common vertex, say v_0. The vertex v_0 of $C(n, n\ \ 3)$ is called *apex* of the shell $C(n, n\ \ 3)$. The vertex v_l of $C(n, n\ \ 3)$ is said to be at *level* l.

A graph $C(2n, n\ \ 2)$ is called an *alternate shell*, if $C(2n, n\ \ 2)$ is obtained from the cycle $C_{2n}\langle v_0, v_1, v_2, \ldots, v_{2n\ 1}\rangle$ by adding $n\ \ 2$ chords between the vertex v_0 and the vertices $v_{2i\ 1}$, for $1\ \ i\,\delta\, n\ \ \mathbf{2}$. If the vertex v_l of $C(2n, n\ \ 2)$ at level l and is adjacent with v_0 then v_l is said to be at *level* l *with a chord,* otherwise the vertex v_l is said to be at *level* l *without a chord.*

A graph, denoted $G\ \ 2n_i, n_i\ \ 2, k, l$ is called *one vertex union of alternate shells with a path at any common level l (with or without chords),* if it is obtained from k alternate shells $C(2n_i, n_i\ \ 2)$'s, $1\ \ i\,\delta\, k$, by merging them together at their apex and joining k vertices each chosen from a distinct alternate shell in a particular level l (with or without chords) by a path $P_{2k\ 1}$ such that the chosen vertex of the i th alternate shell $C(2n_i, n_i\ \ 2)$ is at the $(2i\ \ 1)$ th vertex of the $P_{2k\ 1}$, for $1\ \ i\,\delta\, k$. We denote the graph $G\ \ 2n_i, n_i\ \ 2, k, l$ as $G\ \ 2n_i, n_i\ \ 2, k, l_c$ if the path $P_{2k\ 1}$ joins the vertices only at the common *level l with chords.*

In this paper, we show that $G\ 2n_l, n_l\ \ 2, k, l_c$ is graceful and admits an Δ-labeling, for $k\ \tau 1,\ n_l\ \ 3, 1 \tau i$ &δand $G\ 2n_l, n_l\ \ 2, k, l$ is cordial, for $n_l\ \ n\ \ 3,\ k\ \ 1,\ 1 \tau i$ &i.

Keywords: *Graph labeling, graceful labeling, cordial labeling, one vertex union of alternate shells with a path at any common level.*

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1. Introduction

At the Smolenic symposium in 1963, Ringel conjectured that the complete graph $K_{2m\ 1}$ can be decomposed into $2m\ \ 1$ isomorphic copies of a given tree with m edges. In an attempt to solve Ringel's conjecture, Rosa [12] introduced graceful labeling, an Δ-labeling and other labeling as a tool to attack the Ringel's conjecture.

A function f is called a *graceful labeling* of G with m edges, if f is an injection from the vertices of G to the set $\{0,1,2,\ldots,m\}$ such that when each edge uv is assigned the label $|f(u)\ \ f(v)|$, then the resulting edge labels are distinct. A graceful labeling f is called an *Δ-labeling* of G if there exists an integer C such that $f(u)$ Œ $f(v)$ (or) $f(v)$ Œ $f(u)$, for every edge $uv\ \ E(G)$. A graph G admitting an Δ-labeling is necessarily bipartite.

Harmonious labeling was introduced by Graham and Sloane [9] in connection with their study on error correcting codes. A function f is called *harmonious labeling* of a graph G with m edges, if f is an injection from the vertices of G to the group of integers modulo m, such that when each edge uv is assigned the label $f(u)\ \ f(v)\ (\mathrm{mod}\ m)$, then the resulting edge labels are distinct.

Over the period, variations of these two labelings were studied with different motivations. Chang [4] introduced elegant labeling as a variation of harmonious labeling. A graph G with m edges is called *elegant*, if there is an injection $f : V(G)$ o $\{0,1,2,...,m$ such that when each edge uv is assigned the label $f(u) \quad f(v)$ (mod $m \quad 1$), then the resulting edge labels are distinct and non-zero.

In [3], cordial labeling was introduced by Cahit as a variation of both graceful and harmonious. A function f from the set of vertices of the graph G to the set $\{0,1$ and for each edge uv assign the label $|f(u) \quad f(v)|$ is called *cordial* labeling, if the number of vertices labeled 0's and the number of vertices labeled 1's differ by at most 1, and the number of edges labeled 0's and the number of edges labeled 1's differ by at most 1. Though the cordial labeling looks much simpler to graceful labeling, but it is surprising to note that there are families of graphs which are graceful but not cordial, for example, K_4 is graceful but not cordial.

The above labelings are not only useful in theoretical studies, but also play an important role in applications, (refer [2]).

It is extremely hard to characterize graceful / harmonious / elegant / cordial graphs. Also determining a graph is harmonious / cordial are NP-complete. Due to the inherent difficulties of these labelings, researchers have investigated these labelings on various specific families of graphs.

A line of work on graceful graphs has concentrated on graphs related to the cycles stemming from Rosa's result [12] that a cycle C_n is graceful if and only if n {0,3 (mod 4). In [5], Delmore *et al.* proved that every cycle with a chord is graceful. In this direction, Koh *et al.* [10] have shown that every cycle with t consecutive chords is graceful for $t \quad 2,3$ and $n \quad 3$, and this result had been extended to all t, where $4 \quad t \, \delta \, n \quad 3$ by Goh and Lim [8].

We recall from [13], a *shell graph* of size n, denoted $C(n, n \quad 3)$, is the graph obtained from the cycle $C_n \langle v_0, v_1, v_2, ... v_{n \quad 1} \rangle$ by adding $n \quad 3$

consecutive chords incident with a common vertex, say v_0. The common vertex v_0 of $C(n,n\ \ 3)$ is called *apex* of the shell $C(n,n\ \ 3)$. The vertex v_l of $C(n,n\ \ 3)$ is said to be at the *level* l.

In [13], Sethuraman and Dhavamani have shown that one edge union of shell graphs is graceful. In [14], Sethuraman and Selvaraju have shown that the one edge union of shell graphs is cordial.

A graph $C(2n,n\ \ 2)$ is called an *alternate shell,* if $C(2n,n\ \ 2)$ is obtained from the cycle $C_{2n}\langle v_0, v_1, v_2, \ldots, v_{2n\ 1}\rangle$ by adding $n\ \ 2$ chords between the vertex v_0 and the vertices $v_{2i\ 1}$, for $1\ \ i\,\delta\, n\ \ $ã. If the vertex v_l of $C(2n,n\ \ 2)$ at level l and is adjacent with v_0 then v_l is said to be at *level* l *with chord,* otherwise the vertex v_l is said to be at *level* l *without chord.*

Let $C(2n_1,n_1\ \ 2)$, $C(2n_2,n_2\ \ 2)$, $\ldots, C(2n_k,n_k\ \ 2)$ be any k alternate shells, without loss of generality, we assume that $n_1\ \ n_2\,\delta \ldots$ δæß. The alternate shell $C(2n_i,n_i\ \ 2)$ is called i th alternate shell. Let $\bigcup_{i\ 1}^{k} C(2n_i,n_i\ \ 2)$ denote the *one vertex union of* k *alternate shells* $C(2n_i,n_i\ \ 2)$'s, for $1\ \ i\,\delta\, k$, obtained by merging the apex of each of these k alternate shells together. Let v_0 denotes the common vertex of $\bigcup_{i\ 1}^{k} C(2n_i,n_i\ \ 2)$, and let $v_{i,j}$ be the vertex of the i th alternate shell $C(2n_i,n_i\ \ 2)$ at the j th level, where $1\ \ j\,\delta\, 2n_i\ \ $ß. The $(i\ \ 1)$ th alternate shell $C(2n_{i\ 1},n_{i\ 1}\ \ 2)$ and the i th alternate shell $C(2n_i,n_i\ \ 2)$ are said to be *adjacent* alternate shells in $\bigcup_{i\ 1}^{k} C(2n_i,n_i\ \ 2)$, for $2\ \ i\,\delta\, k$.ì

We arrange the vertices of each alternate shell $C(2n_i,n_i\ \ 2)$'s excluding the apex v_0 in the one vertex union of alternate shells $\bigcup_{i\ 1}^{k} C(2n_i,n_i\ \ 2)$ with certain hierarchy, so that k suitable common level vertices (with chords and without chords) can be identified for joining them by a path $P_{2k\ 1}$.

Level arrangement of first and second alternate shells

First arrange the vertices of the first alternate shell $C(2n_1,n_1\ \ 2)$ as a chain $v_{1,1}, v_{1,2},..., v_{1,2n_1\ 1}$ such that $v_{1,1}$ is in the bottom level vertex and $v_{1,2n_1\ 1}$ is in the top level vertex of $C(2n_1,n_1\ \ 2)$. Now we arrange the vertices of $C(2n_2,n_2\ \ 2)$ as a chain $v_{2,1},v_{2,2},...v_{2,2n_2\ 1}$ such that the top level vertex $v_{1,2n_1\ 1}$ of the first alternate shell $C(2n_1,n_1\ \ 2)$ and the top level vertex $v_{2,2n_2\ 1}$ of the second alternate shell $C(2n_2,n_2\ \ 2)$ are in the same level, so that the bottom level vertex $v_{1,1}$ of the first alternate shell $C(2n_1,n_1\ \ 2)$ and the vertex $v_{2,2n_2\ 2n_1\ 1}$ of the second alternate shell $C(2n_2,n_2\ \ 2)$ are in the same level.

Observe that there are n_1 pairs of vertices $\langle v_{1,2n_1\ 1}, v_{2,2n_2\ 1}\rangle$, $\langle v_{1,2n_1\ 3}, v_{2,2n_2\ 3}\rangle$, $...,\langle v_{1,1}, v_{2,2n_2\ 2n_1\ 1}\rangle$ are in the same levels with chords, and $n_1\ \ 1$ pairs of vertices $\langle v_{1,2n_1\ 2}, v_{2,2n_2\ 2}\rangle$, $\langle v_{1,2n_1\ 4}, v_{2,2n_2\ 4}\rangle$, $...,\langle v_{1,2}, v_{2,2n_2\ 2n_1\ 2}\rangle$ are in the same level without chords, see Figure1.

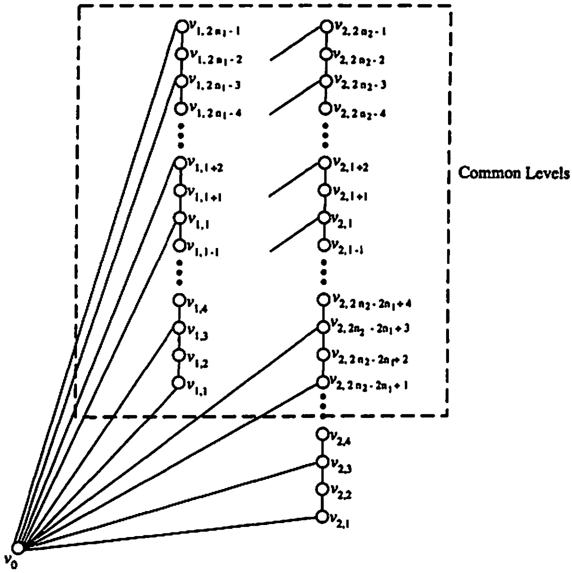

Figure 1 : Level arrangement of $C(2n_1, n_1 - 2)$ and $C(2n_2, n_2 - 2)$

Level arrangement of *i* th alternate shell

When 3 i δ k *and* i *odd,* arrange the i th alternate shell $C(2n_i, n_i \;\; 2)$ as a chain $v_{i,1}, v_{i,2}, \ldots, v_{i,2n_i \;\; 1}$ such that the bottom level vertex $v_{i \;\; 1,1}$ of the $(i \;\; 1)$ th alternate shell $C(2n_{i \;\; 1}, n_{i \;\; 1} \;\; 2)$ and the bottom level vertex $v_{i,1}$ of the i th alternate shell $C(2n_i, n_i \;\; 2)$ are in the same level, so that the top level vertex $v_{i \;\; 1, 2n_{i \;\; 1} \;\; 1}$ of the $(i \;\; 1)$ th alternate shell $C(2n_{i \;\; 1}, n_{i \;\; 1} \;\; 2)$ and the vertex $v_{i, 2n_{i \;\; 1} \;\; 1}$ of the i th alternate shell $C(2n_i, n_i \;\; 2)$ are in the same level.

Thus, there are $n_{i \;\; 1}$ pairs of vertices $\langle v_{i \;\; 1,1}, v_{i,1} \rangle$, $\langle v_{i \;\; 1,3}, v_{i,3} \rangle, \ldots, \langle v_{i \;\; 1, 2n_{i \;\; 1} \;\; 1}, v_{i, 2n_{i \;\; 1} \;\; 1} \rangle$ are in the same levels with chords, and there are $n_{i \;\; 1} \;\; 1$ pairs of vertices $\langle v_{i \;\; 1,2}, v_{i,2} \rangle$,

$\left\langle v_{i\ 1,4},\ v_{i,4}\right\rangle,\ldots,\left\langle v_{i\ 1,2n_{i\ 1}\ 2},\ v_{i,2n_{i\ 1}\ 2}\right\rangle$ are in the same levels without chords.

When $3\ \ i\,\delta\, k$ *and* i *even*, arrange the vertices of the i th alternate shell $C(2n_i,n_i\ \ 2)$ as a chain $v_{i,1},\ v_{i,2},\ldots,v_{i,2n_i\ 1}$ such that the top level vertex $v_{i\ 1,\ 2n_{i\ 1}\ 1}$ of the $(i\ \ 1)$ th alternate shell $C(2n_{i\ 1},\ n_{i\ 1}\ \ 2)$ and the top level vertex $v_{i,\ 2n_i\ 1}$ of the i th alternate shell $C(2n_i,n_i\ \ 2)$ are in the same level, so that the bottom level vertex $v_{i\ 1,1}$ of $(i\ \ 1)$ th alternate shell $C(2n_{i\ 1},\ n_{i\ 1}\ \ 2)$ and the vertex $v_{i,2n_i\ 2n_{i\ 1}\ 1}$ of the i th alternate shell $C(2n_i,n_i\ \ 2)$ are in the same level.

Thus, there are $n_{i\ 1}$ pairs of vertices $\left\langle v_{i\ 1,\ 2n_{i\ 1}\ 1},\ v_{i,\ 2n_i\ 1}\right\rangle$, $\left\langle v_{i\ 1,\ 2n_{i\ 1}\ 3},\ v_{i,\ 2n_i\ 3}\right\rangle,\ \ldots,\left\langle v_{i\ 1,1},\ v_{i,2n_i\ 2n_{i\ 1}\ 1}\right\rangle$ are in the same levels with chords, and $n_{i\ 1}\ \ 1$ pairs of vertices $\left\langle v_{i\ 1,\ 2n_{i\ 1}\ 2},\ v_{i,\ 2n_i\ 2}\right\rangle$, $\left\langle v_{i\ 1,\ 2n_{i\ 1}\ 4},\ v_{i,\ 2n_i\ 4}\right\rangle,\ \ldots,\left\langle v_{i\ 1,2},\ v_{i,2n_i\ 2n_{i\ 1}\ 2}\right\rangle$ in the same levels without chords.

We refer the above hierarchical level arrangement of vertices of the alternate shell $C(2n_i,\ n_i\ \ 2)$ in $\bigcup_{i\ 1}^{k} C(2n_i,n_i\ \ 2)$ is called *Top-Bottom -Level arrangement* (*TBL- arrangement*). In the TBL-arrangement of $\bigcup_{i\ 1}^{k} C(2n_i,n_i\ \ 2)$ there are n_1 levels of vertices in each alternate shell which are in the common level with chords and $n_1\ \ 1$ levels of vertices in each alternate shell which are in the common level without chords, the vertices in the common levels with chords are called *common level vertices with chords,* and the vertices in the common level without chords are called *common level vertices without chords*. Observe that, there exists n_1 different sets of common level vertices with chords where each set containing a vertex from a distinct alternate shell $C(2n_i,\ n_i\quad 2)$, for $1\quad i \,\delta\, k$, and there exists $n_1\quad 1$ different sets of common level vertices without chords where each set containing a vertex from a distinct alternate shell $C(2n_i,\ n_i\quad 2)$, for $1\quad i\,\delta\, k$.

Let $\bigcup_{i\ 1}^{k} C(2n_i, n_i\quad 2)$ be one vertex union of k alternate shell $C(2n_i, n_i\quad 2)$'s, $1\quad i\,\delta\, k$, with TBL- arrangement, choose k vertices, each from a distinct alternate shell $C(2n_i,\ n_i\quad 2)$, such that they are in any fixed common level l (with or without chords). Join these k vertices by a path $P_{2k\ 1}$ such that the ith alternate shell $C(2n_i,\ n_i\quad 2)$ at the (2*i*-1)th vertex of $P_{2k\ 1}$. The vertex w_i of the path $P_{2k\ 1}$ is called *middle vertex* between the ith alternate shell $C(2n_i,\ n_i\quad 2)$ and the $(i\quad 1)$th alternate shell $C(2n_{i\ 1},\ n_{i\ 1}\quad 2)$, for $1\quad i\,\delta\, k\quad \delta$. The graph thus obtained is denoted by $G(2n_i, n_i\quad 2, k, l)$ and is called *one vertex union of* k *alternate shells with a path* $P_{2k\ 1}$ *at any common level* l *(with or without chords).*

In particular, we denote the graph $G(2n_i, n_i\quad 2, k, l)$ as $G(2n_i, n_i\quad 2, k, l_c)$ if the path $P_{2k\ 1}$ joins the vertices only at the common level l with chords, see Figure 2.

We observe the following two observations from the graph $G(2n_i, n_i\quad 2, k, l)$.

Observation 1

In each of the alternate shell $C(2n_i, n_i\quad 2)$ excluding the apex v_0 there is a path $P_{2n_i\ 1}$ of length $2n_i\quad 1$ in $G(2n_i, n_i\quad 2, k, l)$, for $1\quad i\,\delta\, k$.

In the path $P_{2k\ 1}$ of $G(2n_i, n_i\quad 2, k, l)$ joining the k alternate shell with alternate vertices, there are k vertices which are common to the path $P_{2k\ 1}$ and the k alternate shell, called *shared vertices* of the alternating path $P_{2k\ 1}$ in $G(2n_i, n_i\quad 2, k, l)$, and the $(k\quad 1)$ middle vertices w_i's $(1\quad i\,\delta\, k\quad \delta)$ which lies only on the path $P_{2k\ 1}$.

By the construction of the graph $G(2n_l, n_i \;\; 2, k, l)$, there is no odd cycle. Therefore, $G(2n_l, n_i \;\; 2, k, l)$ is a bipartite graph.

Rosa [12] has proved the following significant theorem.

Theorem [Rosa].

If a graph G with m edges has an Δ-labeling, then there exists a cyclic decomposition of the edges of the complete graph $K_{2pm \;\; 1}$ into sub graphs isomorphic to G, where p is an arbitrary natural number.

In 1996, El-Zanati and Vanden [6] have proved the following theorem.

Theorem [El-Zanati]

If G has m edges and admits an Δ-labeling then $K_{pm, qm}$ can be partitioned into sub graphs isomorphic to G for all positive integers p and q.

Observation 2

When $n_i \;\; n$, for $1 \;\; i \, \delta \, k$, then we denote the graph $G(2n_l, n_i \;\; 2, k, l)$ as $G(2n, n \;\; 2, k, l)$. We arrange the k copies of the alternate shells $C(2n, \; n \;\; 2)$'s in $G(2n, n \;\; 2, k, l)$ as first, second,..., k th copy. Observe that there are $2n \;\; 2$ pairs of vertices which are in the common levels between any two adjacent copies were not joined in the path $P_{2k \;\; 1}$. Join $2t \; P_3$ paths between $2t$ pairs of vertices where each pair of vertex is selected from any adjacent alternate shells in the same common level. The graph thus obtained is denoted by $G(2n, n \;\; 2, k, l, 2t)$, where $1 \;\; t \, \delta \, (k \;\; 1)(n \, \delta \, 1)$.

In this paper, we show that the graph $G(2n_l, n_i \;\; 2, k, l_c)$ is graceful and admits an Δ-labeling, for $n_i \;\; 3, 1 \, \tau \, i \;\; k, k \, \delta \, б$. Thus it follows from Rosa's [12] and El-Zanati's [6] theorems. We show that the graph $G(2n, n \;\; 2, k, l)$ is cordial, for $n \, \tau 3$, $k \, \tau 1$, and the graph $G(2n, n \;\; 2, k, l, 2t)$ is cordial, for $n \;\; 3, k \, \tau \, 1$. Finally, we discuss related open problems.

2. One vertex union of alternate shells with a path at any common level with chords is graceful

In this section, we show that the graph $G(2n_i, n_i \quad 2, k, l_c)$ one vertex union of k alternate shells $C(2n_i,\ n_i \quad 2)$'s with a path $P_{2k \quad 1}$ at any common level l with chords is graceful and admits an Δ-labeling, for $n_i \quad 3,\ 1 \tau i \quad k,\ k \delta$ ß.

Theorem 1. *For* $k \quad 1, n_i \tau 3, 1 \quad i \tau$ ß, *the graph* $G(2n_i, n_i \quad 2, k, l_c)$ *is graceful.*

Proof. Let $G(2n_i, n_i \quad 2, k, l_c)$ be the one vertex union of k alternate shells $C(2n_i,\ n_i \quad 2)$'s with a path $P_{2k \quad 1}$ at any common level l with chords, for $n_i \quad 3,\ 1 \tau i \quad k,\ k \delta$ ß.

Observe that $|V(G(2n_i, n_i \quad 2, k, l_c)| \quad \underset{i \quad 1}{\overset{k}{f}}(2n_i \quad 1) \quad k \quad = 2 \underset{i \quad 1}{\overset{k}{f}} n_i$, and

$$|E(G(2n_i, n_i \quad 2, k, l_c)| \quad \underset{i \quad 1}{\overset{k}{f}}(3n_i \quad 2) \quad 2(k \quad 1) \quad = 3(\underset{i \quad 1}{\overset{k}{f}} n_i) \quad 2.$$

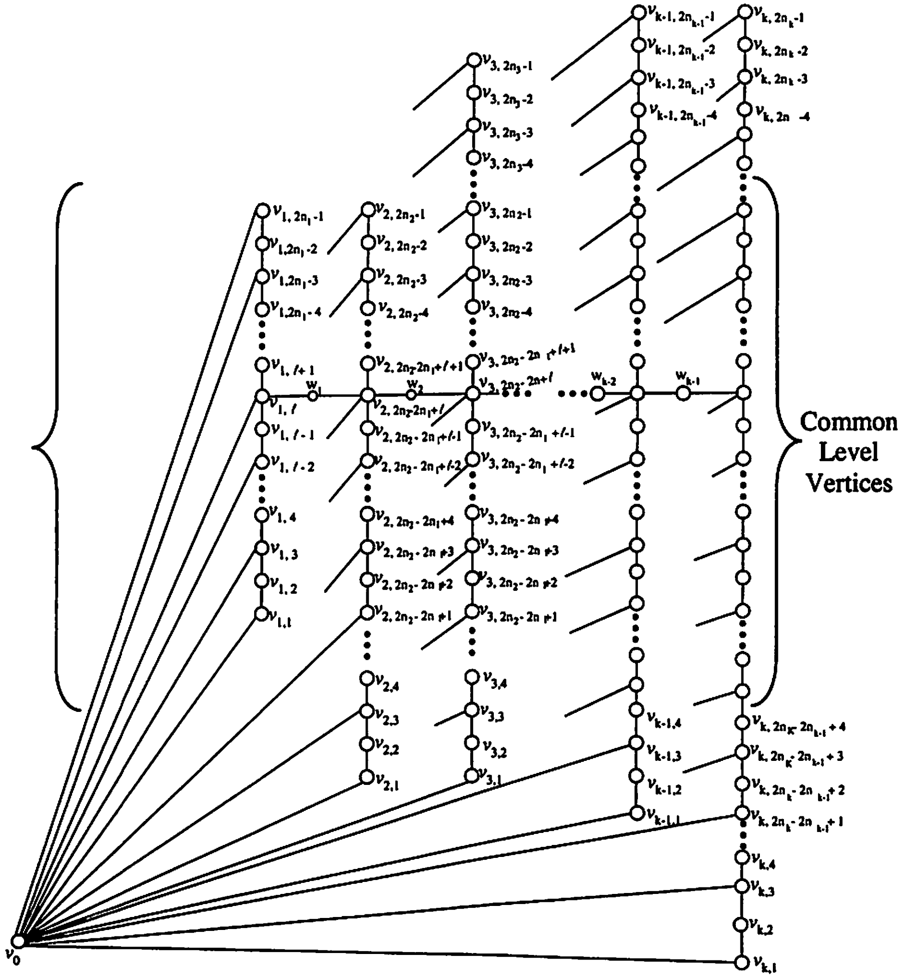

Figure 2 : The graph G($2n_l$, n_l-2, k, l_c) with k even

Let $m \quad |E(G(2n_l, n_l \quad 2, k, l_c)|$.

For the convenience of graceful labeling, we rename the vertices of $G(2n_l, n_l \quad 2, k, l_c)$ as shown in Figure 3.

Define $l: V(G(2n_l, n_l \quad 2, k, l_c)) \quad \{0, 1, 2, \ldots, m$ by

$$I(v_0)=0,\quad I(v_j) = m-(j-1),\ \textit{for}\ 1 \le j \le \sum_{i=1}^{k} n_i, \qquad \text{and}$$

$$I(u_j) = m-1-j-2\sum_{i=1}^{k} n_i,\ \textit{for}\ 1 \le j \le \left(\sum_{i=1}^{k} n_i\right)-1 .$$

It is clear that the vertex labels $I(v_j)$'s, for $0 \le j \le \sum_{i=1}^{k} n_i$, and

$I(u_j)$'s, for $1 \le j \le \left(\sum_{i=1}^{k} n_i\right)-1$ are distinct.

Let A be the set of edges of $G(2n_i, n_i-2, k, l_c)$ which are adjacent to v_0, that is,

$$A = \{\langle v_0, v_j\rangle \in E\ G(2n_i, n_i-2, k, l_c),\ \textit{for}\ 1 \le j \le \sum_{i=1}^{k} n_i\}.$$

Let B be the set of edge of $G(2n_i, n_i-2, k, l_c)$ which are not in A.

Let A' and B' denote the sets of edge labels of the edges of the sets A and B respectively.

Observe that

$$A' = \{m, m-1, m-2, \ldots, m-1-\sum_{i=1}^{k} n_i\},$$

$$B' = \{m-(\sum_{i=1}^{k} n_i), m-(\sum_{i=1}^{k} n_i)-1, \ldots, 2, 1\}.$$

It is clear that the edge labels in the sets A' and B' are distinct, and

$$A' \cup B' = \{1, 2, 3, \ldots, m-1, m\}.$$

Hence, $G(2n_i, n_i-2, k, l_c)$ is graceful.

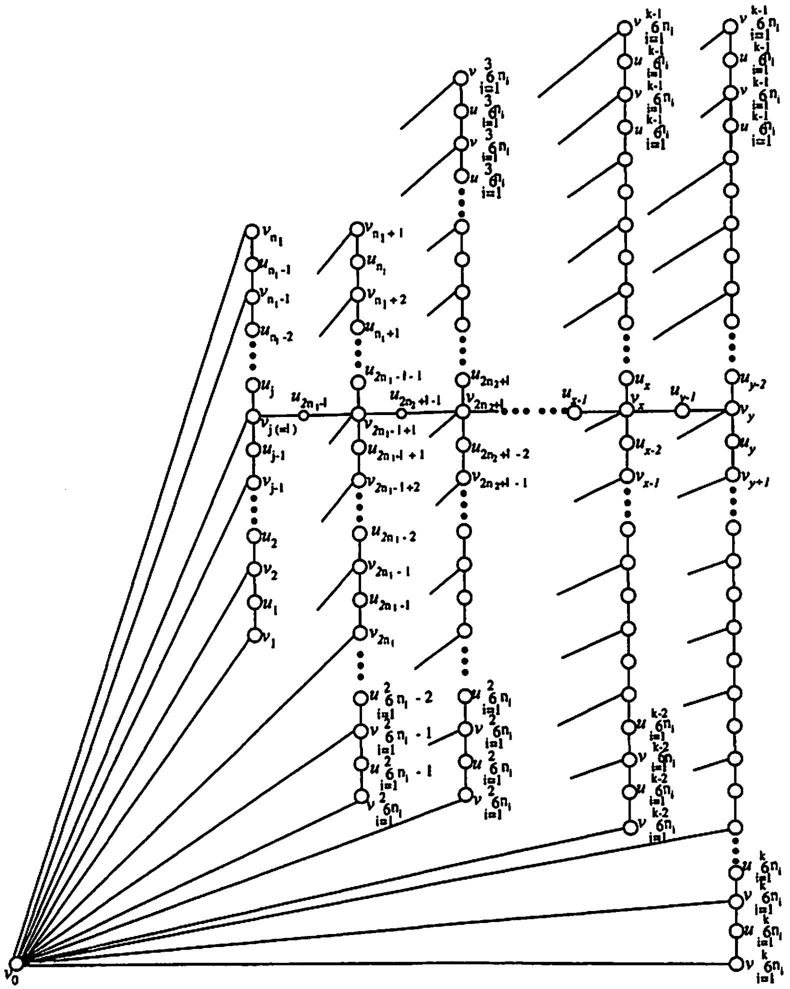

Figure 3 : The graph G $(2n_l, n_l\text{-}2, k, l_c)$ with k even

From Theorem 1, observe that for every edge uv $E(G(2n_l, n_l\ \ 2, k, l_c))$ such that $f(u)$ Œ $f(v)$ or $f(v)$ Œ $f(u)$, where $O\ m\ \ \overset{k}{\underset{l\ \ 1}{f}} n_l$, hence $G(2n_l, n_l\ \ 2, k, l_c)$ admits an Δ-labeling.

The following two corollaries are an immediate consequence of Rosa's Theorem and El-Zanati's Theorem.

Corollary 1.

The complete graphs $K_{2pm\ 1}$ *can be decomposed into sub graphs isomorphic to* $G(2n_l, n_l\ \ 2, k, l_c)$, *where* $m\ \ |E(G(2n_l, n_l\ \ 2, k, l_c))|$, *and* p *is an arbitrary positive integer.*

Corollary 2.

The edges of the complete graphs $K_{pm,\, qm}$ *can be partitioned into sub graphs isomorphic to* $G(2n_l, n_l\ \ 2, k, l_c)$, *where* $m\ \ |E(G(2n_l, n_l\ \ 2, k, l_c))|$, p *and* q *are arbitrary positive integer.*

3. One vertex union of copies of alternate shells with a path at any level is cordial

In this section, we show that the graph $G(2n, n\ \ 2, k, l)$ the one vertex union of k copies of alternate shell $C(2n, n\ \ 2)$ with a path $P_{2k\ 1}$ at any level l (with or without chords) is cordial, for $n\ \ 3$, $k \tau 1$. We also show that the graph $G(2n, n\ \ 2, k, l, 2t)$ is cordial, for $n\ \ 3$, $k \tau 1$, $1\ \ t \delta (k\ \ 1)(n \delta 1)$.

Theorem 2.

For $n\ \ 3, k \tau 1$, *the graph* $G(2n, n\ \ 2, k, l)$ *is cordial.*

Proof. Let $G(2n, n\ \ 2, k, l)$ one vertex union of k copies of the alternate shells $C(2n, n\ \ 2)$ with a path $P_{2k\ 1}$ at any level l (with or without chords), whose vertices are described as shown in Figure 2, where $n_1\ \ n_2\ \ \ldots\ \ n_k\ \ n$.

Observe that $|V(G(2n, n\ \ 2, k, l))|\ \ 2kn$,

$|E(G(2n, n\ \ 2, k, l))|\ \ 3kn\ \ 2$.

Define $I: V(G(2n, n\ \ 2, k, l))\ \ \rfloor 0, 1$ by $I(v_0) = 0$,

Labeling of the first copy in $G(2n, n \ \ 2, k, l)$

For $1 \ \ j \, \delta \, 2n \ \ ŝ$,

$$\text{Let } I(v_{1,j}) \ \ \begin{cases} 1, & if \ j \ \{1,2 \ \ \text{mod } 4 \\ 0, & if \ j \ \{0,3 \ \ \text{mod } 4 \end{cases}$$

Labeling of the i th copy in $G(2n, n \ \ 2, k, l)$

Let $I \, v_{i,j} \ = 1 \ \ I(v_{i \ \ 1,j})$, for $2 \ \ i \, \delta \, k, 1 \ \ j \, \delta \, 2n \ \ ŝ$.

Labeling of middle vertices in the path $P_{2k \ \ 1}$

$$\text{Let} \quad I \, w_i \ \ \begin{cases} 1, & for \ 1 \ \ i \, \delta \, k \ \ ŝ \ and \ i \ odd \\ 0, & for \ 1 \ \ i \, \delta \, k \ \ ŝ \ and \ i \ even \end{cases}$$

Let V_0 and V_1 denotes the set of vertices of $G(2n, n \ \ 2, k, l)$ were assigned the labels 0's and 1's respectively.

Let E_0 and E_1 denotes the set of edges of $G(2n, n \ \ 2, k, l)$ were having the labels 0's and 1's respectively. Let $A \ \ \bigcup_{i \ 1}^{k} A_i$, where A_i be the set of all edge labels of the edges of the i th copy of the alternate shell which are adjacent to v_0, that is,

$A_i \ \ \{ \, (v_0) \ \ I(v_{i,j}) \, |, \, for \ 1 \ \ j \ \ 2n \ \ ŝ \, ℓ$.

Let B be the set of all edge labels of the edges of the paths $P_{2n \ \ 1}$'s in each copy of the k alternate shells $C(2n, n \ \ 2)$. Let C be the set all edge labels of the edges of the path $P_{2k \ \ 1}$ at any level l, where $l \ \ \{1, 2, 3, ..., 2n \ \ 1$.

Table 1. Edge labeling of $G(2n, n \;\; 2, k, l)$.

Nature of k	$2n$	A_i		B	C		$\lvert E_0\rvert \cdot \lvert E_1\rvert$
		i {1,3 (mod4)	i {0,2 (mod4)	i {0,1,2, (mod 4)	l {1,2 (mod4	l {0,3 (mod4	
Odd	$4r$	$(10)^r$	$(01)^r$	$(01)^{(2r\;\;1)k}$	$(01)^{k\;\;1}$	$(10)^{k\;\;1}$	$\lvert E_0\rvert \;\; \lvert E_1\rvert$
	$4r \;\; 2$	$(10)^r 1$	$(01)^r 0$	$(01)^{(2r\;\;1)k}$	$(01)^{k\;\;1}$	$(10)^{k\;\;1}$	$\lvert E_0\rvert \;\; \lvert E_1\rvert$
Even	$4r$	$(10)^r$	$(01)^r$	$(01)^{(2r\;\;1)k}$	$(01)^{k\;\;1}$	$(10)^{k\;\;1}$	$\lvert E_0\rvert \;\; \lvert E_1\rvert$
	$4r \;\; 2$	$(10)^r 1$	$(01)^r 0$	$(01)^{(2r\;\;1)k}$	$(01)^{k\;\;1}$	$(10)^{k\;\;1}$	$\lvert E_0\rvert \;\; \lvert E_1\rvert$

From the Table 1, it is clear that the graph $G(2n, n \;\; 2, k, l)$ is cordial.

Corollary 3. *For* $k \,\tau 1$, $n \,\tau 3$, *the graph* $G(2n, n \;\; 2, k, l, 2t)$ *is cordial, for* $1 \;\; t\,\delta\,(k \;\; 1)(n\,\delta\,1)$.

Proof. The graph $G(2n, n \;\; 2, k, l)$ is cordial. Observe that, any two adjacent copies of the alternate shells are complement to each other in $G(2n, n \;\; 2, k, l)$, therefore joining any two adjacent copies of an alternate shell by a path P_3 with middle vertex having the label 0 or having the label 1 will give one of the edge label as 1 and the other edge label as 0.

From Observation 2, join t P_3 paths having middle vertex labeled 0's and t P_3 paths having middle vertex labeled 1's between any adjacent alternate shells in $G(2n, n \;\; 2, k, l)$.

Hence, $G(2n, n \;\; 2, k, l, 2t)$ is cordial.

4. Discussion

In Section 2 of this paper we have shown that the graph $G(2n_l, n_l \;\; 2, k, l_c)$ the one vertex union of k alternate shells with a path $P_{2k\;\;1}$ at any common level l with chords is graceful. We feel that tendency towards having the graceful labeling of $G(2n_l, n_l \;\; 2, k, l)$ the one vertex union of

k alternate shells with a path $P_{2k\ 1}$ at any common level l without chords seems to be negative. It prompts to ask the following question.

Is $G(2n_i, n_i\ \ 2, k, l)$ *the one vertex union of* k *alternate shells with a path* $P_{2k\ 1}$ *at any common level* l *without chords graceful, for* $k\ \tau 1, n_i\ \tau 3, 1\ \ i\, \delta\, k$*?*

In Section 3 of this paper, we have shown that $G(2n, n\ \ 2, k, l)$ is cordial. It appears that proving the cordialness of $G(n_i, n_i\ \ 3, k, l)$ for arbitrary n_i's, seem to be hard to establish. So we conclude this paper with the following question.

Is $G(n_i, n_i\ \ 3, k, l)$ *the one vertex union of* k *alternate shells with a path* $P_{2k\ 1}$ *at any common level* l *(with or without chord) is cordial, for all* $n_i\ \ 3,\ 1 \tau i\ \ k,\ k\, \delta$ ß?1

References:

[1] R. Balakrishnan, R. Samoathkumar, Decomposition of complete graphs into isomorphic bipartite subgraphs, Graphs Combin. 10 (1994), 19-25

[2] G. S. Bloom, S. W. Golomb, Applications of numbered undirected graphs. *Proc.*, IEEE, 65 (1977), 562-570.

[3] I. Cahit, ' Recent results and open problems on cordial graphs' *Contemporary methods in Graph theory*, R. Bodendiek (ed.), Wissenschaftsverlag, Mannheim, (1990), 209-230.

[4] G. J. Chang, D. F. Hsu, D. G. Rogers, Additive variations on a graceful theme: Some results on harmonious and other related graphs, *congress. Numer.* 32 (1981), 181-197.

[5] C. Delmore, K. M. Koh, M. Maheo, H. K. Teo, H. Thuillier, Cycles with a chord are graceful, *J. Graph Theory,* 4 (1980), 409-415.

[6] S. El-Zanati, C. Vanden Eynden, Decomposition of $K_{m,n}$ into cubes, J. Combin. Designs 4 (1996), 51-67.

[7] J. A. Gallian , A dynamic survey of graph labeling, *Electro. J. Combin.* 5 (2005).

[8] C. G. Goh, C. K. Lim, Graceful numberings of cycles with consecutive chords, Manuscript, 1992.

[9] R. L. Graham, N. J. A. Sloane, On additive bases and harmonious graphs *SIAM J. Alg. Discrete Math.*, 1 (1980), 382-404.

[10] K. M. Koh, N. Punnium, On graceful graphs: cycles with 3-consecutive chords, *Bull. Malaysian Math. Soc.,* 5 (1982), 49-64

[11] A. Kotzig, Decomposition of complete graphs into isomorphis cubes, J. Combin. Theory Ser. B 31 (1981), 292-296

[12] A. Rosa, On certain valuations of vertices of a graph, in; *Theory of graphs, International symposium, Rome, July* 1966 *Gordon and Breach*, New York, Dunod, Paris, (1967), 349-355.

[13] G. Sethuraman, R. Dhavamani, Graceful numbering of an edge- gluing of shell graphs, *Discrete Mathematics* 218 (2000), 283-287.

[14] G. Sethuraman, P.Selvaraju, One edge union of shell graphs and one vertex union of complete bipartite graphs are cordial, *Discrete Math.,* 259 (2002), 243-350.