Cube factorizations of complete multipartite graphs

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Abstract

Let λK_{h^u} denote the λ-fold complete multipartite graph with u parts of size h. A cube factorization of λK_{h^u} is a uniform 3-factorization of λK_{h^u} in which the components of each factor are cubes. We show that there exists a cube factorization of λK_{h^u} if and only if $uh \equiv 0 \pmod 8), \lambda(u-1)h \equiv 0 \pmod 3$ and $u \geq 2$. It gives a new family of uniform 3-factorizations of λK_{h^u}. We also establish the necessary and sufficient conditions for the existence of cube frames of λK_{h^u}.

Keywords : decomposition; factorization; cube; frame; uniform 3-factorization

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1 Introduction

Let $\mathcal{G}$ and $\mathcal{H}$ be two graphs. An $\mathcal{H}$-decomposition of $\mathcal{G}$ is a decomposition of the edges of $\mathcal{G}$ into isomorphic copies of $\mathcal{H}$, the copies of $\mathcal{H}$ are called *blocks*. Such a decomposition is called *resolvable* if it is to partition the blocks into *classes* $\mathcal{P}_i$ (called *parallel classes or resolution classes*) such that every vertex of $\mathcal{G}$ appears exactly once in each $\mathcal{P}_i$. A resolvable $\mathcal{H}$-decomposition of $\mathcal{G}$ is sometimes also referred to as an $\mathcal{H}$-factorization of $\mathcal{G}$, a class can be called an $\mathcal{H}$-factor of $\mathcal{G}$.

Necessary and sufficient conditions for a (resolvable) $\mathcal{H}$-decomposition of $\mathcal{G}$ have been established for various $\mathcal{H}$ and $\mathcal{G}$. The most common problem considered is that given a graph $\mathcal{H}$, for which u does there exist a (resolvable) $\mathcal{H}$-decomposition of K_u, the complete graph on u vertices. Other common choices for $\mathcal{G}$ include the λ-fold complete graph λK_u, and the λ-fold complete multipartite graph λK_{h^u} with u parts each of size

h. $\mathcal{H}$-factorizations (decompositions) of the above graphs have been considered for many different graphs $\mathcal{H}$. An $\mathcal{H}$-factorization of λK_u is also known as a resolvable design. An $\mathcal{H}$-factorization of λK_{h^u} is known as a resolvable group divisible design (RGDD) with index λ, the parts of size h are called the groups of the design. We also for groups of differing sizes use an exponential notation $h_1^{u_1} h_2^{u_2} \cdots h_n^{u_n}$ to specify that there are u_i groups of size h_i, this is the factorization's or the RGDD's *type*. When $\mathcal{H} = K_k$ we call it a (k, λ)-RGDD. For $k = 3, 4$, the existence for a (k, λ)-RGDD of type h^u has been extensively studied by several authors in [4, 22, 23, 11, 12, 13, 14, 15, 16, 17, 24].

In this paper, we consider $\mathcal{H}$-factorization of λK_{h^u} where $\mathcal{H}$ is a 3-cube. The d-cube is a graph whose vertex set can be labelled with the set of all binary d-tuples, so that its edge set consists of all pairs of vertices which differ in exactly one coordinate. It is clear that d-cube has 2^d vertices, $d \cdot 2^{d-1}$ edges, and is d-regular and bipartite. So for the existence of a d-cube factorization of λK_{h^u}, we have the following lemma.

Lemma 1.1 *Necessary conditions for the existence of a d-cube factorization of λK_{h^u} are that $uh \equiv 0 \pmod{2^d}$ and $\lambda(u-1)h \equiv 0 \pmod{d}$.*

In 1979, Kotzig [19] posed two problems of d-cube decompositions and factorizations of K_u. Since these problems were introduced, the cube decomposition problem and its variations have been investigated by many people and several results have been obtained, although the cube decomposition problem itself is far from being completely solved (see [20, 2, 3, 5, 7, 8, 9]). Progress on the cube factorization problem of K_u has been restricted to sporadic values of u or the case where u is a power of 2 (see [7, 8]). In 2004, Adams et al.[1] settled the cube factorization problem for $d = 3$ by showing that these necessary conditions in Lemma 1.1 with $h = 1, \lambda = 1$ are also sufficient. Namely

Theorem 1.2 ([1]) *There exists a cube factorization of K_u if and only if $u \equiv 16 \pmod{24}$.*

Note that $d = 3$ is the first non-trivial value of d. A 1-cube factorization is simply a 1-factorization. In this paper, we will settle the cube factorization problem of λK_{h^u} for $d = 3$ by showing that the necessary conditions in Lemma 1.1 are also sufficient (see Theorem 3.21).

In order to settle the 3-cube factorization problem, we want to apply a construction technique that uses frames. In the section 2, we will give a complete solution to the existence of 3-cube frames of λK_{h^u}.

In remainder of this paper deals exclusively with 3-cubes, so the term cube will be used for 3-cube. We will use the notation $[v_1, v_2, v_3, v_4; v_5,$

$v_6, v_7, v_8]$ to denote the cube with vertex set $\{v_1, v_2, \ldots, v_8\}$ and edge set

$$\{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_5v_6, v_6v_7, v_7v_8, v_8v_5, v_1v_5, v_2v_6, v_3v_7, v_4v_8\}.$$

2 Cube frames of λK_{h^u}'s

A useful tool in construction of factorizations is a frame. A λ-fold $\mathcal{H}$-frame of type h^u (or an $\mathcal{H}$-frame of λK_{h^u}) is a decomposition of λK_{h^u} into edge-disjoint copies of $\mathcal{H}$, called blocks, such that the set of blocks can be partitioned into subsets, called *partial factors* (or *holey parallel classes*), which satisfy the following conditions:

(1) each partial factor is a set of blocks in which each vertex of λK_{h^u} occurs either one or zero times;

(2) in each partial factor, the vertices that don't occur are precisely the vertices in a part (of size h) of λK_{h^u}.

An $\mathcal{H}$-frame of λK_u is usually called a almost resolvable $\mathcal{H}$ design of order u. An $\mathcal{H}$-frame is called a k-frame if $\mathcal{H} = K_k$. Similarly, an $\mathcal{H}$-frame is called a cube frame if the graph $\mathcal{H}$ is a cube. For more information about frames and almost resolvable designs, we can refer the interested readers to [10] or [6].

In this section, we will establish the spectrum of cube frames of type h^u. To do this, we give some recursive constructions for cube frames.

Let G is a graph, we denote by $G \bigotimes I_m$ the graph whose vertex-set is formed by replacing each vertex x of G by m vertices $(x, 1), (x, 2), \ldots, (x, m)$, with (x, i) adjacent to (y, j) if and only if x adjacent to y in G.

Lemma 2.1 *Let m be a positive integer and C be a cube. Then there exists a cube factorization of $C \bigotimes I_m$.*

Proof. Let $C = [v_1, w_2, v_4, w_3; w_4, v_3, w_1, v_2]$, and let $\mathcal{F}$ be a 1-factorization of K_{m^2} over the vertex set $X_v \cup X_w$, which exists by [6], where $X_v = \{(v, j) | j = 1, 2, \ldots, m\}$, $X_w = \{(w, j) | j = 1, 2, \ldots, m\}$. Furthermore, we let $\mathcal{F} = \{F_1, F_2, \ldots, F_m\}$, and $F_i = \{(v, k)\ (w, j_k^{(i)})\ | k = 1, 2, \ldots, m\}$ where $\{j_k^{(i)} | k = 1, 2, \ldots, m\} = \{1, 2, \ldots, m\}$. Now we use F_i to construct a cube factor of $C \bigotimes I_m$. For each edge $(v, k)(w, j_k^{(i)}) \in F_i$, we use $C_k^{(i)}$ to denote the cube $[(v_1, k), (w_2, j_k^{(i)}), (v_4, k), (w_3, j_k^{(i)}); (w_4, j_k^{(i)}), (v_3, k), (w_1, j_k^{(i)}), (v_2, k)]$ in $C \bigotimes I_m$. Let $C^i = \{C_1^{(i)}, C_2^{(i)}, \ldots, C_m^{(i)}\}$. Then it is not difficult to verify that C^i is a cube factor of $C \bigotimes I_m$ and $\mathcal{C} = \{C^1, C^2, \ldots, C^m\}$ is a cube factorization of $C \bigotimes I_m$.

Lemma 2.2 *Let m be a positive integer. Suppose there is a cube frame of λK_{h^u}, then there is a cube frame of $\lambda K_{(mh)^u}$.*

Proof. Let C be a cube frame of λK_{h^u}. Replace each vertex x of λK_{h^u} by a set of m vertices $(x,1),(x,2),\cdots,(x,m)$ and each edge of λK_{h^u} by a copy of K_{m^2}. Replace each cube C in $\mathcal{C}$ by a cube factorization of $C \bigotimes I_m$, which exists by Lemma 2.1. This gives a cube frame of $\lambda K_{(mh)^u}$.

Lemma 2.3 *Let $h, t_1, \ldots, t_n$ be positive integers and $h|t_i, 1 \le i \le n$. Suppose there is a cube frame of $\lambda K_{t_1,t_2,\ldots,t_n}$, and for $1 \le i \le n$ there is a cube frame of $\lambda K_{h^{(t_i/h+1)}}$. Then there is a cube frame of λK_{h^u} where $u = \sum_{1\le i\le n}(t_i/h)+1$.*

Proof. Let $X_0 = \{1,2,\ldots,h\}$, $X_{ij} = \{(i,j,1),(i,j,2),\ldots,(i,j,h)\}$, and $X_i = \cup_{1\le j\le t_i/h} X_{ij}$ for $1 \le i \le n$. Let the vertex set of $\lambda K_{t_1,t_2,\ldots,t_n}$ be $X = \cup_{1\le i\le n} X_i$, and let $\mathcal{P}$ be a cube frame of $\lambda K_{t_1,t_2,\ldots,t_n}$ with parts $X_i, 1 \le i \le n$. It is not difficult to see that there are $\lambda t_i/3$ partial factors missing the vertices in X_i, call these $\mathcal{P}_{ij1}, \mathcal{P}_{ij2}, \ldots, \mathcal{P}_{ij(\lambda h/3)}, 1 \le j \le t_i/h$.

For $1 \le i \le n$, let $\mathcal{Q}$ be a cube frame of $\lambda K_{h^{(t_i/h+1)}}$ on the vertex set $X_i \cup X_0$ with parts $X_{ij}, 1 \le j \le t_i/h$ and X_0. For each part X_{ij}, let $\mathcal{Q}_{ij1}, \mathcal{Q}_{ij2}, \ldots, \mathcal{Q}_{ij(\lambda h/3)}$ be $\lambda h/3$ partial factors missing the vertices in X_{ij}, and for part X_0, let $\mathcal{Q}_{i1}, \mathcal{Q}_{i2}, \ldots, \mathcal{Q}_{i(\lambda h/3)}$ be $\lambda h/3$ partial factors missing the vertices in X_0.

For $1 \le i \le n$, $1 \le j \le t_i/h$ and $1 \le k \le \lambda h/3$, let

$$\mathcal{R}_{ijk} = \mathcal{P}_{ijk} \bigcup \mathcal{Q}_{ijk}$$

and let

$$\mathcal{R}_k = \bigcup_{1\le i\le n} \mathcal{Q}_{ik}.$$

Then it is straightforward to check that

$$\mathcal{R} = \bigcup_{1\le k\le \lambda h/3} (\mathcal{R}_k \bigcup (\bigcup_{1\le i\le n, 1\le j\le t_i/h} \mathcal{R}_{ijk}))$$

forms a cube frame of λK_{h^u} on the vertex set $X \cup X_0$ with parts X_0 and $X_{ij}, 1 \le i \le n, 1 \le j \le t_i/h$. The partial factors missing the vertices in X_{ij} are $\mathcal{R}_{ij1}, \mathcal{R}_{ij2}, \ldots, \mathcal{R}_{ij(\lambda h/3)}$ and the partial factors missing the vertices in X_0 are $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_{\lambda h/3}$. This completes the proof.

To apply the above recursive constructions, we need the following designs constructed by using difference technique.

Lemma 2.4 *There exists a cube frame of K_{3^9}.*

Proof. Label the vertices of K_{3^9} with the elements of $Z_9 \times Z_3$, with the part set $\{\{i\} \times Z_3 : i \in Z_9\}$. A cube frame is given by developing the following starter partial cube factor missing the vertices in $\{0\} \times Z_3$ modulo (9,-):

$$[(1,0),(2,0),(4,0),(2,1);(5,0),(6,2),(7,2),(4,2)],$$
$$[(3,0),(6,0),(3,1),(8,1);(7,1),(5,1),(6,1),(2,2)],$$
$$[(7,0),(1,1),(8,0),(5,2);(3,2),(8,2),(1,2),(4,1)].$$

Lemma 2.5 *There exists a cube frame of $K_{3^{17}}$.*

Proof. Label the vertices of $K_{3^{17}}$ with the elements of $Z_{17} \times Z_3$, with the part set $\{\{i\} \times Z_3 : i \in Z_{17}\}$. The following two starter cubes generate a partial cube factor missing the vertices in $\{0\} \times Z_3$ by the operation modulo(-,3).

$$[(1,0),(2,0),(4,0),(7,0);(10,0),(14,0),(16,1),(6,1)],$$
$$[(3,0),(11,1),(5,0),(12,1);(13,1),(8,0),(9,1),(15,0)].$$

Then, we can get the desired cube frame by developing the above partial cube factor by using the operation modulo (17,-).

Lemma 2.6 *There exists a cube frame of K_{6^5}.*

Proof. Label the vertices of K_{6^5} with the elements of Z_{30}, with the part set $\{\{i, i+5, \cdots, i+25\} : 0 \le i \le 4\}$. A cube frame is given by developing the following starter partial cube factor missing the vertices in $\{0, 5, \cdots, 25\}$ +3modulo 30:

$$[1,3,4,7;13,21,27,24],[2,16,12,26;6,23,14,22],[8,9,17,19;11,18,29,28].$$

Now we can establish the existence of cube frames with index $\lambda = 1$. The following known result was obtained by Adams et al.

Lemma 2.7 ([1, Lemma 4.1]) *There exists a cube frame of K_{24^u} for $u \ge 3$.*

Lemma 2.8 *There exists a cube frame of K_{3^u} for $u \equiv 1 \pmod 8$.*

Proof. Cube frames of K_{3^9} and $K_{3^{17}}$ are given by Lemmas 2.4 and 2.5. Thus it remains to consider $u = 8n + 1$ and $n \ge 3$. These designs can be obtained by applying Lemma 2.3 with $h = 3$ to cube frames of K_{24^n} for $n \ge 3$ which come from Lemma 2.7.

Lemma 2.9 *There exists a cube frame of K_{6^u} for $u \equiv 1 \pmod 4$.*

Proof. Cube frame of K_{6^5} is given by Lemma 2.6. Cube frame of K_{6^9} is given by applying Lemma 2.2 with $m = 2$ to a cube frame of K_{3^9} which comes from Lemma 2.4. Now applying Lemma 2.3 with $h = 6$ to cube frames of K_{24^n} for $n \geq 3$ which come from Lemma 2.7 gives the desired designs.

Lemma 2.10 *There exists a cube frame of K_{12^u} for $u \equiv 1 \pmod 2$.*

Proof. Cube frame of K_{12^3} can be obtained from a cube factorization of K_{12^2}, which exists by [1, Lemma 3.2]. Cube frame of K_{12^5} is given by applying Lemma 2.2 with $m = 2$ to a cube frame of K_{6^5} which comes from Lemma 2.6. Now applying Lemma 2.3 with $h = 12$ to cube frames of K_{24^n} for $n \geq 3$ which come from Lemma 2.7 gives the desired designs.

Applying Lemmas 2.2 and 2.7-2.10, we have

Theorem 2.11 *There exists a cube frame of K_{h^u} for $(u-1)h \equiv 0 \pmod 8$, $h \equiv 0 \pmod 3$ and $u \geq 3$.*

Next we will focus our attention on constructing the cube frames with index $\lambda = 3$.

Lemma 2.12 *There exists a 3-fold cube factorization of type 4^2. Hence a 3-fold cube frame of type 4^3 exists.*

Proof. Label the vertices of $3K_{4^2}$ with the elements of Z_8, with the part set $\{\{i, i+2, i+4, i+6\} : i = 0, 1\}$. A cube factorization is given by the following cube factors.

Factor 1: $[0, 1, 2, 3; 5, 4, 7, 6]$. Factor 2: $[0, 1, 2, 3; 7, 6, 5, 4]$.
Factor 3: $[0, 1, 4, 5; 7, 6, 3, 2]$. Factor 4: $[0, 3, 4, 7; 5, 6, 1, 2]$.

Lemma 2.13 *There exists a 3-fold cube frame of type 8^u for $u \geq 3$.*

Proof. A 2-frame of type 2^u exists for all $u \geq 3$ [6]. Let $\mathcal{M}$ be a 2-frame of type 2^u with the underlying graph K_{2^u}. Replace each vertex of K_{2^u} by a set of 4 vertices, and each edge of K_{2^u} by a copy of $3K_{4^2}$. Replace each block in $\mathcal{M}$ by a cube factorization of $3K_{4^2}$, which exists by Lemma 2.12. This gives a 3-fold cube frame of type 8^u.

Lemma 2.14 *There exists a 3-fold cube frame of type 1^u for $u \equiv 1 \pmod 8$.*

Proof. For $u = 9$ and 17. Label the vertices of $3K_u$ with the elements of Z_u. The two designs can be obtained by developing the following starter blocks modulo u.

(1) $u = 9$: $[1, 2, 3, 6; 8, 4, 7, 5]$.

(2) $u = 17$: $[1, 2, 3, 4; 10, 16, 9, 14], [5, 7, 11, 13; 8, 12, 6, 15]$.

For $u = 8n + 1, n \geq 3$, applying Lemma 2.3 with $h = 1$ to 3-fold cube frames of type 8^n for $n \geq 3$ coming from Lemma 2.13 gives the desired designs.

Lemma 2.15 *There exists a 3-fold cube frame of type 2^u for $u \equiv 1 \pmod 4$.*

Proof. For $u = 5$, we give its direct construction as follows. Label the vertices of $3K_{2^5}$ with the elements of Z_{10}, with the part set $\{\{i, i+5\} : 0 \leq i \leq 4\}$. Developing the following starter cube $[1, 2, 3, 7; 8, 4, 6, 9]$ modulo 10 gives the desired designs. For $u = 9$, applying Lemma 2.2 to a 3-fold cube frame of type 1^u which comes from Lemma 2.14 gives the desired designs. For $u = 4n + 1, n \geq 3$, applying Lemma 2.3 with $h = 2$ to 3-fold cube frames of type 8^n for $n \geq 3$ coming from Lemma 2.13 gives the desired designs.

Lemma 2.16 *There exists a 3-fold cube frame of type 4^u for $u \equiv 1 \pmod 2$.*

Proof. For $u = 3$, the desired design comes from Lemma 2.12. For $u = 5$, applying Lemma 2.2 to a 3-fold cube frame of type 2^5 which comes from Lemma 2.15 gives the desired designs. For $u = 2n + 1, n \geq 3$, applying Lemma 2.3 with $h = 4$ to 3-fold cube frames of type 8^n for $n \geq 3$ coming from Lemma 2.13 gives the desired designs.

Applying Lemmas 2.2 and 2.13-2.16, we have

Theorem 2.17 *There exists a 3-fold cube frame of type h^u for $(u-1)h \equiv 0 \pmod 8$ and $u \geq 3$.*

We are now in a position to give our main result in this section.

Theorem 2.18 *There exists a cube frame of λK_{h^u} if and only if $(u-1)h \equiv 0 \pmod 8$, $\lambda h \equiv 0 \pmod 3$ and $u \geq 3$.*

Proof. By simple counting argument, necessity is clear. Sufficiency can be divided into the following two cases.

Case I: $\lambda \equiv 1, 2 \pmod 3$, $(u-1)h \equiv 0 \pmod 8$, $h \equiv 0 \pmod 3$, and $u \geq 3$.

Case II: $\lambda \equiv 0 \pmod 3$, $(u-1)h \equiv 0 \pmod 8$, and $u \geq 3$.

The conclusion of Case I follows from Theorem 2.11 by repeating blocks λ times. The conclusion of Case II follows from Theorem 2.17 by repeating blocks $\lambda/3$ times. This completes the proof.

3 Cube factorizations of λK_{h^u}'s

In this section, we will establish the existence of cube factorizations of λK_{h^u}. For cube factorizations, we first present the following recursive constructions which are similar to Lemmas 2.2 and 2.3.

Lemma 3.1 *Let m be a positive integer. Suppose there is a cube factorization of λK_{h^u}, then there is a cube factorization of $\lambda K_{(mh)^u}$.*

Proof. Let $\mathcal{C}$ be a cube factorization of λK_{h^u}. Replace each vertex x of λK_{h^u} by a set of m vertices $(x,1),(x,2),\cdots,(x,m)$ and each edge of λK_{h^u} by a copy of K_{m^2}. Replace each cube C in $\mathcal{C}$ by a cube factorization of $C\bigotimes I_m$, which exists by Lemma 2.1. This gives a cube factorization of $\lambda K_{(mh)^u}$.

Lemma 3.2 *Let h,t be positive integers and $h|t$. Suppose there is a cube factorization of λK_{t^n}, and there is a cube factorization of $\lambda K_{h^{(t/h)}}$. Then there is a cube factorization of λK_{h^u} where $u = nt/h$.*

Proof. Let $X_{ij} = \{(i,j,1),(i,j,2),\ldots,(i,j,h)\}$, and $X_i = \cup_{1\le j\le t/h}X_{ij}$ for $1\le i\le n$. Let the vertex set of λK_{t^n} be $X = \cup_{1\le i\le n}X_i$, and let $\mathcal{P}$ be a cube factorization of λK_{t^n} with parts $X_i, 1\le i\le n$. It is not difficult to see that there are $\lambda(n-1)t/3$ factors, call these $\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_{\lambda(n-1)t/3}$. Now, replacing each part X_i of λK_{t^n} by the graph $\lambda K_{h^{t/h}}$ with parts X_{ij} gives a graph λK_{h^u} with parts X_{ij}, $1\le i\le n$ and $1\le j\le t/h$. Let $\mathcal{Q}_i$ be a cube factorization of $\lambda K_{h^{(t/h)}}$ on the vertex set X_i with parts $X_{ij}, 1\le j\le t/h$. And let $\mathcal{Q}_{i1},\mathcal{Q}_{i2},\ldots,\mathcal{Q}_{i(\lambda(t-h)/3)}$ be $\lambda(t-h)/3$ factors of $\mathcal{Q}_i$.

For $1\le j\le\lambda(t-h)/3$, Let

$$\mathcal{R}_j = \bigcup_{1\le i\le n}\mathcal{Q}_{ij},$$

and let

$$\mathcal{R} = (\bigcup_{1\le j\le\lambda(n-1)t/3}\mathcal{P}_j)\bigcup(\bigcup_{1\le j\le\lambda(t-h)/3}\mathcal{R}_j)).$$

Then it is straightforward to check that $\mathcal{R}$ forms a cube factorization of λK_{h^u} on the vertex set X with parts and $X_{ij}, 1\le i\le n, 1\le j\le t/h$. And $\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_{\lambda(n-1)t/3}$ and $\mathcal{R}_1,\mathcal{R}_2,\ldots,\mathcal{R}_{\lambda(t-h)/3}$ are its all factors. This completes the proof.

Lemma 3.3 *Let h,t be positive integers and $h|t$. Suppose there is a cube frame of λK_{t^n}, and there is a cube factorization of $\lambda K_{h^{(t/h+1)}}$. Then there is a cube factorization of λK_{h^u} where $u = nt/h+1$.*

Proof. Let $X_0 = \{1,2,\ldots,h\}$, $X_{ij} = \{(i,j,1),(i,j,2),\ldots,(i,j,h)\}$, and $X_i = \cup_{1\le j\le t/h} X_{ij}$ for $1 \le i \le n$. Let the vertex set of λK_{t^n} be $X = \cup_{1\le i\le n} X_i$, and let $\mathcal{P}$ be a cube frame of λK_{t^n} with parts $X_i, 1 \le i \le n$. For each part X_i, there are exactly $\lambda t/3$ partial cube factors missing the vertices in X_i, call these $\mathcal{P}_{i1}, \mathcal{P}_{i2}, \ldots, \mathcal{P}_{i(\lambda t/3)}$.

Now, replacing each part X_i of λK_{t^n} by the graph $\lambda K_{h^{(t/h+1)}}$ with parts X_{ij} and X_0 gives a graph λK_{h^u} with parts X_0 and X_{ij}, $1 \le i \le n$ and $1 \le j \le t/h$. For $1 \le i \le n$, let $\mathcal{Q}_i$ be a cube factorization of $\lambda K_{h^{(t/h+1)}}$ on the vertex set $X_i \cup X_0$ with parts X_0 and $X_{ij}, 1 \le j \le t/h$, and call the $\lambda t/3$ cube factors $\mathcal{Q}_{i1}, \mathcal{Q}_{i2}, \ldots, \mathcal{Q}_{i(\lambda t/3)}$.

For $1 \le i \le n$, $1 \le j \le \lambda t/3$, let

$$\mathcal{R}_{ij} = \mathcal{P}_{ij} \bigcup \mathcal{Q}_{ij}$$

and let

$$\mathcal{R} = \bigcup_{1\le i\le n, 1\le j\le \lambda t/3} \mathcal{R}_{ij}.$$

Then $\mathcal{R}$ forms a cube factorization of λK_{t^u} on the vertex set $X \cup X_0$ with parts X_0 and $X_{ij}, 1 \le i \le n, 1 \le j \le t/h$. Where $\mathcal{R}_{ij}, 1 \le i \le n, 1 \le j \le \lambda t/3$ are its all cube factors. This completes the proof.

Now we consider the case with index $\lambda = 1$.

Lemma 3.4 [1, Lemma 2.1] *There exists a cube factorization of K_{2^4}.*

By Lemmas 3.1 and 3.4, we have

Lemma 3.5 *For any positive integer m, there exists a cube factorization of $K_{(2m)^4}$.*

Lemma 3.6 *There exists a cube factorization of K_{h^u} for $h \equiv 1, 5, 7, 11, 13, 17, 19, 23 \pmod{24}$ and $u \equiv 16 \pmod{24}$.*

Proof. Applying Lemma 3.1 to a cube factorization of K_u , which exists by Theorem 1.2, gives the desired design.

Lemma 3.7 *There exists a cube factorization of K_{2^u} for $u \equiv 4 \pmod{12}$.*

Proof. For $u \equiv 4 \pmod{12}$, a (4,1)-RGDD of type 1^u exists [6]. Let $\mathcal{R}$ be a (4,1)-RGDD of type 1^u with the underlying graph K_u. Replace each vertex of K_u by a set of 2 vertices, and each edge of K_u by a copy of K_{2^2}. Replace each block in $\mathcal{R}$ by a cube factorization of K_{2^4}, which exists by Lemma 3.4. This gives a cube factorization of K_{2^u}.

Lemma 3.8 *There exists a cube factorization of K_{24^u} for $u \ge 2$.*

Proof. There is a 1-factorization of K_{2^u} for $u \geq 2$ [6]. Let $\mathcal{P}$ be a 1-factorization of type 2^u with the underlying graph K_{2^u}. Replace each vertex of K_{2^u} by a set of 12 vertices, and each edge of K_{2^u} by a copy of K_{12^2}. Replace each block in $\mathcal{P}$ by a cube factorization of K_{12^2}, which exists by Lemma 3.2 in [1]. This gives a cube factorization of K_{24^u}.

Lemma 3.9 *There exists a cube factorization of* K_{12^u} *for* $u \equiv 0 \pmod 2$.

Proof. There is a 1-factorization of K_u for $u \equiv 0 \pmod 2$ [6]. Let $\mathcal{P}$ be a 1-factorization of type 1^u with the underlying graph K_u. Replace each vertex of K_u by a set of 12 vertices, and each edge of K_u by a copy of K_{12^2}. Replace each block in $\mathcal{P}$ by a cube factorization of K_{12^2}, which exists by Lemma 3.2 in [1]. This gives a cube factorization of K_{12^u}.

Lemma 3.10 *There exists a cube factorization of* K_{4^u} *for* $u \equiv 4 \pmod 6$.

Proof. By Lemma 3.5 there is a cube factorization of K_{4^4}. For $u = 6n+4, n \geq 1$, applying Lemma 3.3 with $h = 4$ to a cube frame of $K_{12^{2n+1}}$ coming from Lemma 2.10 gives the desired design.

Lemma 3.11 *There exists a cube factorization of* K_{8^u} *for* $u \equiv 1 \pmod 3$.

Proof. For $u \equiv 1 \pmod 3$, a (4,1)-RGDD of type 4^u (i.e., a resolvable $(4u, 4, 1)$-BIBD) exists [17]. Let $\mathcal{R}$ be a (4,1)-RGDD of type 4^u with the underlying graph K_{4^u}. Replace each vertex of K_{4^u} by a set of 2 vertices, and each edge of K_{4^u} by a copy of K_{2^2}. Replace each block in $\mathcal{R}$ by a cube factorization of K_{2^4}, which exists by Lemma 3.4. This gives a cube factorization of K_{8^u}.

Lemma 3.12 *There exists a cube factorization of* K_{6^u} *for* $u \equiv 0 \pmod 4$.

Proof. There is a cube factorization of K_{6^4} by Lemma 3.5. For $u = 4n, n \geq 2$, applying Lemma 3.2 with $h = 6$ to a cube factorization of K_{24^n} coming from Lemma 3.8 gives the desired design.

Lemma 3.13 *There exists a cube factorization of* K_{3^u} *for* $u \equiv 0 \pmod 8$.

Proof. For $u = 8$, label the vertices of K_{3^8} with the elements of $Z_{21} \cup \{\infty_1, \infty_2, \infty_3\}$, with the parts $\{\infty_1, \infty_2, \infty_3\}$ and $\{i, i+7, i+14\}, 0 \leq i \leq 6$. A cube factorization is given by developing the following starter cube factor +3 modulo 21.

$[\infty_1, 0, 1, 2; 4, 6, 9, 17], [\infty_2, 11, 7, 13; 12, 16, 19, 3], [\infty_3, 5, 8, 10; 15, 14, 18, 20]$.

For $u = 8n, n \geq 2$, applying Lemma 3.2 with $h = 3$ to a cube factorization of K_{24^n} coming from Lemma 3.8 gives the desired design.

Summing up, we have

Theorem 3.14 *There exists a cube factorization of K_{h^u} for $uh \equiv 0 \pmod 8$, $(u-1)h \equiv 0 \pmod 3$ and $u \geq 2$.*

Proof. Conclusion follows from Lemmas 3.1 and 3.6- 3.13.

Next we consider the case with index $\lambda = 3$.

Lemma 3.15 *There exists a 3-fold cube factorization of type 2^u for $u \equiv 0 \pmod 4$.*

Proof. For $u \equiv 0 \pmod 4$, a (4,3)-RGDD of type 1^u exists [6]. Let $\mathcal{R}$ be a (4,3)-RGDD of type 1^u with the underlying graph $3K_u$. Replace each vertex of $3K_u$ by a set of 2 vertices, and each edge of $3K_u$ by a copy of K_{2^2}. Replace each block in $\mathcal{R}$ by a cube factorization of K_{2^4}, which exists by Lemma 3.4. This gives a 3-fold cube factorization of type 2^u.

Lemma 3.16 *There exists a 3-fold cube factorization of type 4^u for $u \equiv 0 \pmod 2$.*

Proof. There is a 1-factorization of K_u for $u \equiv 0 \pmod 2$ [6]. Let $\mathcal{P}$ be a 1-factorization of type 1^u with the underlying graph K_u. Replace each vertex of K_u by a set of 4 vertices, and each edge of K_u by a copy of $3K_{4^2}$. Replace each block in $\mathcal{P}$ by a 3-fold cube factorization of type 4^2, which exists by Lemma 2.12. This gives a 3-fold cube factorization of type 4^u.

Lemma 3.17 *There exists a 3-fold cube factorization of type 8^u for $u \geq 2$.*

Proof. There is a 1-factorization of K_{2^u} for $u \geq 2$ [6]. Let $\mathcal{P}$ be a 1-factorization of type 2^u with the underlying graph K_{2^u}. Replace each vertex of K_{2^u} by a set of 4 vertices, and each edge of K_{2^u} by a copy of $3K_{4^2}$. Replace each block in $\mathcal{P}$ by a 3-fold cube factorization of type 4^2, which exists by Lemma 2.12. This gives a 3-fold cube factorization of type 8^u.

Lemma 3.18 *There exists a 3-fold cube factorization of type 1^8.*

Proof. Label the vertices of $3K_8$ with the elements of $Z_7 \cup \{\infty\}$, A cube factorization is given by developing the following cube factor modulo 7.

$$[\infty, 0, 1, 5; 6, 4, 2, 3].$$

Lemma 3.19 *There exists a 3-fold cube factorization of type 1^u for $u \equiv 0 \pmod 8$.*

Proof. There is a 3-fold cube factorization of type 1^8 by Lemma 3.18. For $u = 8n, n \geq 2$, applying Lemma 3.2 with $h = 1$ to a 3-fold cube factorization of 8^n coming from Lemma 3.17 gives the desired design.

Applying Lemmas 3.1, 3.15-3.17 and 3.19, we have

Theorem 3.20 *There exists a 3-fold cube factorization of type h^u for $uh \equiv 0 \pmod 8$ and $u \geq 2$.*

We are now in a position to give our main result in this paper.

Theorem 3.21 *There exists a cube factorization of λK_{h^u} if and only if $uh \equiv 0 \pmod 8$, $\lambda(u-1)h \equiv 0 \pmod 3$ and $u \geq 2$.*

Proof. Necessity is from Lemma 1.1. Sufficiency can be divided into the following two cases.
Case I: $\lambda \equiv 1, 2 \pmod 3$, $uh \equiv 0 \pmod 8$, $(u-1)h \equiv 0 \pmod 3$, and $u \geq 2$.
Case II: $\lambda \equiv 0 \pmod 3$, $uh \equiv 0 \pmod 8$, and $u \geq 2$.
The conclusion of Case I follows from Theorem 3.14 by repeating blocks λ times. The conclusion of Case II follows from Theorem 3.20 by repeating blocks $\lambda/3$ times. This completes the proof.

4 Concluding remarks

A *k-factor* of a graph G is a k-regular spanning subgraph of G. A *k-factorization* of G is a set of k-factors of G whose edge sets partition the edge set of G. A k-factorization in which all of the k-factors are isomorphic is called *uniform*. For uniform 3-factorizations of K_{h^u} in which the structure of the 3-factors is specified in advance, very little is known. For $u = 10$, the smallest non-trivial value of u for which there exists a 3-factorization of K_u, a complete enumeration of 3-factorizations of K_u is given in [18]. Resolvable group divisible designs (RGDDs) of type h^u with block size $k = 4$ and index $\lambda = 1$ or 3 are equivalent to 3-factorizations of λK_{h^u} in which each 3-factor consists of $hu/4$ vertex disjoint copies of K_4, and are known to exist if and only if $uh \equiv 0 \pmod 4$ and $\lambda(u-1)h \equiv 0 \pmod 3$ and $u \geq 4$ with 6 definite exceptions and a handful of possible exceptions of (h, u, λ) (see [11, 12, 13, 14, 15, 16, 17, 24]). This is a first family of uniform 3-factorization of λK_{h^u}. Adams et al. [1] gave a second complete family of uniform 3-factorizations of K_u (see Theorem 1.2). In this paper, we generalize Adams et al.'s result and give a second complete family of uniform 3-factorizations of λK_{h^u} (see Theorem 3.21). For 3-factorization problem of λK_{h^u}, there is still much work to do.

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