Chromaticity of Bipartite Graphs With Seven Edges Deleted

H. Roslan[1] and Y.H. Peng[2]

[1]*School of Mathematical Sciences*
Universiti Sains Malaysia, 11800 Penang, Malaysia

[2]*Department of Mathematics, and*
Institute for Mathematical Research
University Putra Malaysia
43400UPM Serdang, Malaysia

[1]*E-mail: hroslan@cs.usm.my*
[2]*E-mail: yhpeng@fsas.upm.edu.my*

ABSTRACT

For integers p, q, s with $p \geq q \geq 2$ and $s \geq 0$, let $\mathcal{K}_2^{-s}(p,q)$ denote the set of 2−connected bipartite graphs which can be obtained from the complete bipartite graph $K_{p,q}$ by deleting a set of s edges. F.M.Dong et al. (Discrete Math. vol.224 (2000) 107–124) proved that for any graph $G \in \mathcal{K}_2^{-s}(p,q)$ with $p \geq q \geq 3$ and $0 \leq s \leq \min\{4, q-1\}$, then G is chromatically unique. In [13], we extended this result to $s = 5$ and $s = 6$. In this paper, we consider the case when $s = 7$.

Keywords: Chromatic Polynomial; Chromatically unique; Chromatically equivalent.

1 Introduction

All graphs considered here are simple graphs. For a graph G, let $V(G)$, $\Delta(G)$ and $P(G,\lambda)$ be the vertex set, maximum degree and the chromatic polynomial of G, respectively.

Two graphs G and H are said to be *chromatically equivalent* (or simply χ−equivalent), symbolically $G \sim H$, if $P(G,\lambda) = P(H,\lambda)$. The equivalence class determined by G under $\sim$ is denoted by $[G]$. A graph G is *chromatically unique* (or simply χ−unique) if $H \cong G$ whenever $H \sim G$, i.e, $[G] = \{G\}$ up to isomorphism. For a set $\mathcal{G}$ of graphs, if $[G] \subseteq \mathcal{G}$ for every $G \in \mathcal{G}$, then $\mathcal{G}$ is said to be χ−*closed*. For two sets $\mathcal{G}_1$ and $\mathcal{G}_2$ of graphs, if

$P(G_1, \lambda) \neq P(G_2, \lambda)$ for every $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}_2$, then $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be chromatically disjoint, or simply $\chi-$*disjoint.*

For integers p, q, s with $p \geq q \geq 2$ and $s \geq 0$, let $\mathcal{K}^{-s}(p,q)$ (resp. $\mathcal{K}_2^{-s}(p,q)$) denote the set of connected (resp. 2−connected) bipartite graphs which can be obtained from the complete bipartite graph $K_{p,q}$ by deleting a set of s edges.

In [5], Dong et al. proved the following result.

Theorem 1.1 *For integers p, q, s with $p \geq q \geq 2$ and $0 \leq s \leq q-1$, $K_2^{-s}(p,q)$ is χ-closed.*

Teo and Koh [14] showed that every graph in $\mathcal{K}(p,q) \cup \mathcal{K}^{-1}(p,q)$ is $\chi-$unique. The case when $s \geq 2$ has been studied by Giudici and Lima de Sa [6], Peng [7], Borowiecki and Drgas-Burchardt [1]. Their typical results are of the following:

(i) If $2 \leq s \leq 4$ and $p-q$ is *small enough*, then each graph in $\mathcal{K}^{-s}(p,q)$ is $\chi-$unique;

(ii) If $G \in \mathcal{K}^{-s}(p,q)$, where $0 \leq p-q \leq 1$, such that the set of s edges deleted forms a matching, then G is $\chi-$unique.

Chen [2] showed that if $G \in \mathcal{K}^{-s}(p,q)$, where $3 \leq s \leq p-q$ and

$$q \geq \max\left\{\frac{1}{2}(p-q)(s-1)+\frac{3}{2},\ \frac{8}{27}(p-q)^2+\frac{1}{3}(p-q)+5s+6\right\},$$

and the set of s edges deleted forms a matching or a star, then G is $\chi-$unique. In [5], Dong et al. proved that any 2−connected graph obtained from $K_{p,q}$ by deleting a set of edges that forms a matching of size at most $q-1$ or that induces a star is chromatically unique.

Dong et al. [4] showed that any graph in $K_2^{-s}(p,q)$ is χ-unique if $p \geq q \geq 3$ and $1 \leq s \leq \min\{4, q-1\}$. Very recently, we [13] extend this result to $p \geq q \geq 6$ and $s=5$ or 6. The case when $s=7$ shall be presented in this paper. For the sake of completeness, we state again in the next section, all known results listed in [13] which will be used to prove this case.

2 Preliminary results and notation

Throughout this paper, we fix the following conditions for p, q and s:

$$p \geq q \geq 3 \quad \text{and} \quad 1 \leq s \leq q-1.$$

For a bipartite graph $G = (A, B; E)$ with bipartition A and B and edge set E, let $G' = (A', B'; E')$ be the bipartite graph induced by the edge set $E' = \{xy \mid xy \notin E, x \in A, y \in B\}$, where $A' \subseteq A$ and $B' \subseteq B$. We write $G' = K_{p,q} - G$, where $p = |A|$ and $q = |B|$. Let $\Delta(G')$ denote the maximum degree of G'. Partition $\mathcal{K}^{-s}(p,q)$ into the following subsets:

$$\mathcal{D}_i(p,q,s) = \left\{ G \in \mathcal{K}^{-s}(p,q) \;\middle|\; \Delta(G') = i \right\}, \quad i = 1, 2, \ldots, s.$$

The following two results were obtained in [3].

Theorem 2.1 *Let p, q, s be integers with $p \geq q \geq 3$ and $1 \leq s \leq q-1$. The following sets are pairwise χ-disjoint:*

$$\mathcal{D}_1(p,q,s), \cup_{2 \leq i < t} \mathcal{D}_i(p,q,s), \mathcal{D}_t(p,q,s), \mathcal{D}_{t+1}(p,q,s), \ldots, \mathcal{D}_s(p,q,s),$$

where $t = \lceil (s+3)/2 \rceil$.

Theorem 2.2 *Let $p \geq q \geq 3$ and $1 \leq s \leq q-1$.*

(i) $\mathcal{D}_1(p,q,s)$ is χ-closed.

(ii) $\cup_{2 \leq i < (s+3)/2} \mathcal{D}_i(p,q,s)$ is χ-closed for $s \geq 2$.

(iii) $\mathcal{D}_i(p,q,s)$ is χ-closed for each i with $\lceil (s+3)/2 \rceil \leq i \leq \min\{s, q-2\}$.

(iv) $\mathcal{D}_{q-1}(p,q,s) \cap \mathcal{K}_2^{-s}(p,q)$ is χ-closed for $s = q-1$.

For a graph G and a positive integer k, a partition $\{A_1, A_2, \ldots, A_k\}$ of $V(G)$ is called a *k-independent partition* in G if each A_i is a non-empty independent set of G. Let $\alpha(G,k)$ denote the number of k-independent partitions in G. For any graph G of order n, we have (see [8]):

$$P(G,\lambda) = \sum_{k=1}^{n} \alpha(G,k)\lambda(\lambda-1)\cdots(\lambda-k+1).$$

Thus, we have

Lemma 2.1 *If $G \sim H$, then $\alpha(G,k) = \alpha(H,k)$ for $k = 1, 2, \ldots$.*

For any bipartite graph $G = (A, B; E)$ with bipartition A and B and edge set E, let

$$\alpha'(G,3) = \alpha(G,3) - (2^{|A|-1} + 2^{|B|-1} - 2). \tag{1}$$

In [5], the authors found the following sharp bounds for $\alpha'(G,3)$:

Theorem 2.3 *For $G \in \mathcal{K}_2^{-s}(p,q)$ with $p \geq q \geq 3$ and $0 \leq s \leq q-1$,*

$$s \leq \alpha'(G,3) \leq 2^s - 1,$$

where $\alpha'(G,3) = s$ iff $\Delta(G') = 1$ and $\alpha'(G,3) = 2^s - 1$ iff $\Delta(G') = s$.

For $t = 0, 1, 2, \ldots$, let $\mathcal{B}(p,q,s,t)$ denote the set of graphs $G \in \mathcal{K}^{-s}(p,q)$ with $\alpha'(G,3) = s + t$. Thus, $\mathcal{K}^{-s}(p,q)$ is partitioned into the following subsets:

$$\mathcal{B}(p,q,s,0), \mathcal{B}(p,q,s,1), \ldots, \mathcal{B}(p,q,s,2^s - s - 1).$$

Assume that $\mathcal{B}(p,q,s,t) = \emptyset$ for $t > 2^s - s - 1$.

Lemma 2.2 *(Dong et al. [4]) For $p \geq q \geq 3$ and $0 \leq s \leq q-1$, if $0 \leq t \leq 2^{q-1} - q - 1$, then*

$$\mathcal{B}(p,q,s,t) \subseteq \mathcal{K}_2^{-s}(p,q).$$

Dong et al. [5] have shown that any graph G in $\mathcal{B}(p,q,s,0) \cup \mathcal{B}(p,q,s,2^s - s - 1)$, if G is 2-connected, is χ-unique. In [4], Dong et al. proved that every 2-connected graph in $\mathcal{B}(p,q,s,t)$ is χ-unique for $1 \leq t \leq 4$. Roslan and Peng in [10,11,12] proved that every 2-connected graph in $\mathcal{B}(p,q,s,t)$ is also χ-unique for $5 \leq t \leq 7$.

For a bipartite graph $G = (A, B; E)$, let

$$\Omega(G) = \{\, Q \mid Q \text{ is an independent sets in } G \text{ with } Q \cap A \neq \emptyset, Q \cap B \neq \emptyset \,\}.$$

Lemma 2.3 *(Dong et al. [5]) For $G \in \mathcal{K}^{-s}(p,q)$,*

$$\alpha'(G,3) = |\Omega(G)| \geq 2^{\Delta(G')} + s - 1 - \Delta(G').$$

For a bipartite graph $G = (A, B; E)$, the number of 4−independent partitions $\{A_1, A_2, A_3, A_4\}$ in G with $A_i \subseteq A$ or $A_i \subseteq B$ for all $i = 1, 2, 3, 4$ is

$$(2^{|A|-1} - 1)(2^{|B|-1} - 1) + \frac{1}{3!}(3^{|A|} - 3 \cdot 2^{|A|} + 3) + \frac{1}{3!}(3^{|B|} - 3 \cdot 2^{|B|} + 3)$$
$$= (2^{|A|-1} - 2)(2^{|B|-1} - 2) + \frac{1}{2}(3^{|A|-1} + 3^{|B|-1}) - 2.$$

Define

$$\alpha'(G,4) = \alpha(G,4) - \{\, (2^{|A|-1} - 2)(2^{|B|-1} - 2) + \frac{1}{2}(3^{|A|-1} + 3^{|B|-1}) - 2 \,\}.$$

Observe that for G, $H \in \mathcal{K}^{-s}(p,q)$,

$$\alpha(G,4) = \alpha(H,4) \quad \text{iff} \quad \alpha'(G,4) = \alpha'(H,4).$$

The following results will be used to prove our main theorems.

Lemma 2.4 *(Dong et al. [3]) For $G = (A, B; E) \in \mathcal{K}^{-s}(p, q)$ with $|A| = p$ and $|B| = q$,*

$$\alpha'(G, 4) = \sum_{Q \in \Omega(G)} (2^{p-1-|Q\cap A|} + 2^{q-1-|Q\cap B|} - 2) + \left| \{ \{Q_1, Q_2\} \mid Q_1, Q_2 \in \Omega(G), Q_1 \cap Q_2 = \emptyset \} \right|.$$

Lemma 2.5 *(Dong et al. [4]) For a bipartite graph $G = (A, B; E)$, if uvw is a path in G' with $d_{G'}(u) = 1$ and $d_{G'}(v) = 2$, then for any $k \geq 2$,*

$$\alpha(G, k) = \alpha(G + uv, k) + \alpha(G - \{u, v\}, k - 1) + \alpha(G - \{u, v, w\}, k - 1).$$

Lemma 2.6 *(Roslan and Peng [9]) For a bipartite graph $G = (A, B; E)$, if uvw, uvy and wvy are three paths in G' with $d_{G'}(u) = 1$ and $d_{G'}(v) = 3$, then for any $k \geq 2$,*

$$\alpha(G, k) = \alpha(G + uv, k) + \alpha(G - \{u, v\}, k - 1) + \alpha(G - \{u, v, w\}, k - 1) + \alpha(G - \{u, v, y\}, k - 1) + \alpha(G - \{u, v, w, y\}, k - 1).$$

Theorem 2.4 *(Dong et al.[3]) For any $G \in \mathcal{K}_2^{-s}(p, q)$, with $p \geq q \geq s + 1 \geq 6$, if $\Delta(G') = s - 1$, then G is χ-unique.*

Theorem 2.5 *(Dong et al.[3]) For any $G \in \mathcal{K}_2^{-s}(p, q)$, with $p \geq q \geq s + 1 \geq 8$, if $\Delta(G') = s - 2$, then G is χ-unique.*

Theorem 2.6 *(Dong et al.[5]) For any $G \in \mathcal{B}(p, q, s, 0) \cup \mathcal{B}(p, q, s, 2^s - s - 1)$, if G is 2-connected, then G is χ-unique.*

Theorem 2.7 *(Dong et al.[4]) For any $G \in \cup_{t=1}^{4} \mathcal{B}(p, q, s, t)$, if G is 2-connected, then G is χ-unique.*

Theorem 2.8 *(Roslan and Peng [10,11,12]) For any $G \in \cup_{t=5}^{7} \mathcal{B}(p, q, s, t)$, if G is 2-connected, then G is χ-unique.*

3 Main result

Dong et al. in [4] proved that every graph in $\mathcal{K}_2^{-s}(p, q)$ is χ-unique if $p \geq q \geq 3$ and $1 \leq s \leq \min\{4, q - 1\}$. In this section, we shall show that every graph in $\mathcal{K}_2^{-s}(p, q)$ is χ-unique if $p \geq q \geq 8$ and $s = 7$.

Our main result is the following theorem.

Theorem 3.1 *Every graph in $\mathcal{K}_2^{-7}(p, q)$ with $p \geq q \geq 8$ is χ-unique.*

Proof. Let G be a graph in $\mathcal{K}_2^{-s}(p,q)$ with $p \geq q \geq 8$ and $s = 7$. If $\Delta(G') \in \{1,7\}$, then $\alpha'(G,3) = s$ or $\alpha'(G,3) = 2^{s-1}$ and thus G is χ−unique by Theorem 2.6. If $\Delta(G') = 5$, then G is χ−unique by Theorem 2.5. If $\Delta(G') = 6$, then G is χ−unique by Theorem 2.4. If $\Delta(G') = 2$ and $G' \ncong C_4 \cup 3K_2$ or $G' \ncong C_4 \cup P_3 \cup K_2$ or $G' \ncong C_4 \cup P_4$ or $G' \ncong C_6 \cup K_2$, then $\alpha'(G,3) \leq s+6$ and thus G is χ−unique by Theorems 2.7 and 2.8. If $G' \cong C_4 \cup 3K_2$, then $\alpha'(G,3) = 12 = s+5$ and thus G is χ−unique by Theorem 2.8. If $G' \cong C_4 \cup P_3 \cup K_2$ or $G' \cong C_6 \cup K_2$, then $\alpha'(G,3) = 13 = s+6$ and thus G is χ−unique by Theorem 2.8. If $G' \cong C_4 \cup P_4$, then $\alpha'(G,3) = 14 = s+7$ and thus G is χ−unique by Theorem 2.8.

We now consider $G \in \mathcal{K}_2^{-s}(p,q)$ with $\Delta(G') = 3$. Let Y_n and Z_i be the graphs shown in Figure 1.

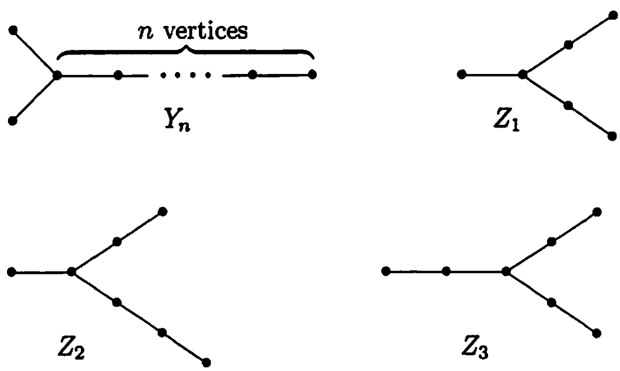


Figure 1: The graphs Y_n and Z_i

If $G' \cong K_{1,3} \cup 4K_2$, then $\alpha'(G,3) = 11 = s+4$ and thus G is χ−unique by Theorem 2.7. If $G' \cong K_{1,3} \cup P_3 \cup 2K_2$ or $G' \cong Y_3 \cup 3K_2$, then $\alpha'(G,3) = 12 = s+5$ and thus G is χ−unique by Theorem 2.8. If $G' \cong Y_4 \cup 2K_2$ or $G' \cong Y_3 \cup P_3 \cup K_2$ or $G' \cong Z_1 \cup 2K_2$ or $G' \cong K_{1,3} \cup P_4 \cup K_2$ or $G' \cong K_{1,3} \cup 2P_3$, then $\alpha'(G,3) = 13 = s+6$ and thus G is χ−unique by Theorem 2.8. If $G' \cong Z_3 \cup K_2$ or $G' \cong Z_2 \cup K_2$ or $G' \cong Z_1 \cup P_3$ or $G' \cong Y_4 \cup P_3$ or $G' \cong Y_3 \cup P_4$ or $G' \cong Y_5 \cup K_2$ or $G' \cong K_{1,3} \cup P_5$, then $\alpha'(G,3) = 14 = s+7$ and thus G is χ−unique by Theorem 2.8.

Otherwise, there are 109 possible structures for G' with $\Delta(G') \in \{3,4\}$ and they are named as $G'_1, G'_2, \ldots, G'_{109}$ which are listed the table in [16]. We group them into 13 families T_i $(1 \leq i \leq 13)$ according to their values of $\alpha'(G_i,3)$ which can be calculated by using Lemma 2.3 and these values are in column three of the table. Let

$$
\begin{aligned}
\mathcal{T}_1 &= \{ G_1 \} \\
\mathcal{T}_2 &= \{ G_2, G_3 \} \\
\mathcal{T}_3 &= \{ G_4, G_5, G_6, G_7 \} \\
\mathcal{T}_4 &= \{ G_8, G_9 \} \\
\mathcal{T}_5 &= \{ G_{10}, G_{11}, \ldots, G_{17} \} \\
\mathcal{T}_6 &= \{ G_{18}, G_{19}, \ldots, G_{25} \} \\
\mathcal{T}_7 &= \{ G_{26}, G_{27}, \ldots, G_{38} \} \\
\mathcal{T}_8 &= \{ G_{39}, G_{40}, \ldots, G_{48} \} \\
\mathcal{T}_9 &= \{ G_{49}, G_{50}, \ldots, G_{62} \} \\
\mathcal{T}_{10} &= \{ G_{63}, G_{64}, \ldots, G_{67} \} \\
\mathcal{T}_{11} &= \{ G_{68}, G_{69}, \ldots, G_{77} \} \\
\mathcal{T}_{12} &= \{ G_{78}, G_{79}, \ldots, G_{95} \} \\
\mathcal{T}_{13} &= \{ G_{96}, G_{97}, \ldots, G_{109} \}
\end{aligned}
$$

Observe that for any i, j with $1 \leq i < j \leq 13$, $\alpha'(G,3) > \alpha'(H,3)$ if $G \in \mathcal{T}_i$ and $H \in \mathcal{T}_j$. Thus by Lemma 2.1 and Equation (1), $\mathcal{T}_i$ and $\mathcal{T}_j$ $(1 \leq i < j \leq 13)$ are χ-disjoint and since $\cup_{i=2}^{4}\mathcal{D}_i(p,q,s)$ is χ-closed (see Theorem 2.2), each $\mathcal{T}_i$ $(1 \leq i \leq 13)$ is χ-closed. Hence, for each i, to show that all graphs in $\mathcal{T}_i$ are χ-unique, it suffices to show that for any two graphs $G, H \in \mathcal{T}_i$, if $G \not\cong H$, then either $\alpha'(G,4) \neq \alpha'(H,4)$ or $\alpha(G,5) \neq \alpha(H,5)$. Note that $\mathcal{T}_1$ contains only one graph G_1, and hence G_1 is χ-unique. The remaining works is to compare every two graphs in each $\mathcal{T}_i$ for $2 \leq i \leq 13$.

We shall establish several inequalities of the form $\alpha'(G_i,4) < \alpha'(G_j,4)$ for some i,j. Since the methods used to obtain inequalities is standard, long and rather repetitive, we shall not discuss all of them here. In the following we shall only show the detail camparisons of every two graphs in $\mathcal{T}_i$ for $i = 2,3,4,5$ and 6. The reader may refer to [15] for other detail comparisons. For convenient, we show all the graphs in $\mathcal{T}_i$ $(2 \leq i \leq 6)$ together with their values of $\alpha'(G_j,3)$ and $\alpha'(G_j,4)$ in Table 1.

(1) $\mathcal{T}_2$.
(1.1) When $p = q$, $G_2 \cong G_3$.
(1.2) When $p > q$,

$$\alpha'(G_2,4) - \alpha'(G_3,4)$$

$$= \left[\left\{ \sum_{i=1}^{4} \binom{4}{i} (2^{p-i-1} + 2^{q-2} - 2) \right\} + 2^{p+1} + 5 \cdot 2^{p-4} + 5 \cdot 2^{q-1} + 5 \right] -$$

$$\left[\left\{\sum_{i=1}^{4}\binom{4}{i}(2^{q-i-1}+2^{p-2}-2)\right\}+2^{q+1}+5\cdot 2^{q-4}+5\cdot 2^{p-1}+5\right]$$

$$=\left\{\sum_{i=1}^{4}\binom{4}{i}(2^{p-i-1}-2^{q-i-1})(1-2^{i-1}\right\}+(2^{p+1}-2^{q+1})+$$
$$5(2^{p-4}-2^{q-4})-5(2^{p-1}-2^{q-1})$$
$$<-12(2^{p-4}-2^{q-4})+2^5(2^{p-4}-2^{q-4})+5(2^{p-4}-2^{q-4})-5\cdot 2^3(2^{p-4}-2^{q-4})$$
$$=-15(2^{p-4}-2^{q-4})<0.$$

(2) $\mathcal{T}_3$.
Note that $\alpha'(G_i,4)$ is odd when $i=4,5$ and even when $i=6,7$. Thus $\alpha'(G_i,4)\neq\alpha'(G_j,4)$ if $i=4,5$ and $j=6,7$.
(2.1) The graphs G_i when $i=4,5$.
(2.1.1) When $p=q$, $G_4\cong G_5$.
(2.1.2) When $p>q$,

$$\alpha'(G_4,4)-\alpha'(G_5,4)$$
$$=\left[\left\{\sum_{i=1}^{4}\binom{4}{i}(2^{p-i-1}+2^{q-2}-2)\right\}+7\cdot 2^{p-2}+2^{p-4}+2^{q+1}+2^{q-3}+29\right]-$$
$$\left[\left\{\sum_{i=1}^{4}\binom{4}{i}(2^{q-i-1}+2^{p-2}-2)\right\}+7\cdot 2^{q-2}+2^{q-4}+2^{p+1}+2^{p-3}+29\right]$$
$$=\left\{\sum_{i=1}^{4}\binom{4}{i}(2^{p-i-1}-2^{q-i-1})(1-2^{i-1})\right\}+7(2^{p-2}-2^{q-2})+$$
$$(2^{p-4}-2^{q-4})-(2^{p+1}-2^{q+1})-(2^{p-3}-2^{q-3})$$
$$<-12(2^{p-4}-2^{q-4})+7\cdot 2^2(2^{p-4}-2^{q-4})+(2^{p-4}-2^{q-4})-$$
$$2^5(2^{p-4}-2^{q-4})-2(2^{p-4}-2^{q-4})$$
$$=-17(2^{p-4}-2^{q-4})<0.$$

(2.2) The graphs G_i when $i=6,7$.
(2.2.1) When $p=q$, $G_6\cong G_7$.
(2.2.2) When $p>q$,

$$\alpha'(G_6,4)-\alpha'(G_7,4)$$
$$=\left[\left\{\sum_{i=1}^{4}\binom{4}{i}(2^{p-i-1}+2^{q-2}-2)\right\}+2^{p+1}+2^{p-2}+3\cdot 2^{q-1}+3\cdot 2^{q-4}+14\right]-$$

$$\left[\left\{\sum_{i=1}^{4}\binom{4}{i}(2^{q-i-1}+2^{p-2}-2)\right\}+2^{q+1}+2^{q-2}+3\cdot 2^{p-1}+3\cdot 2^{p-4}+14\right]$$

$$=\left\{\sum_{i=1}^{4}\binom{4}{i}(2^{p-i-1}-2^{q-i-1})(1-2^{i-1})\right\}+(2^{p+1}-2^{q+1})+$$

$$(2^{p-2}-2^{q-2})-3(2^{p-1}-2^{q-1})-3(2^{p-4}-2^{q-4})$$

$$< -12(2^{p-4}-2^{q-4})+2^5(2^{p-4}-2^{q-4})+2^2(2^{p-4}-2^{q-4})-$$

$$3\cdot 2^3(2^{p-4}-2^{q-4})-3(2^{p-4}-2^{q-4})$$

$$= -3(2^{p-4}-2^{q-4})<0.$$

(3) $\mathcal{T}_4$.

(3.1) When $p=q$, $G_8 \cong G_9$.

(3.2) When $p>q$,

$$\alpha'(G_8,4)-\alpha'(G_9,4)$$

$$=\left[\left\{\sum_{i=1}^{3}\binom{3}{i}(2^{p-i-1}+2^{q-2}-2)\right\}+13\cdot 2^{p-2}+2^{p-4}+12\cdot 2^{q-2}+2^{q-4}-7\right]-$$

$$\left[\left\{\sum_{i=1}^{3}\binom{3}{i}(2^{q-i-1}+2^{p-2}-2)\right\}+13\cdot 2^{q-2}+2^{q-4}+12\cdot 2^{p-2}+2^{p-4}-7\right]$$

$$=\left\{\sum_{i=1}^{3}\binom{3}{i}(2^{p-i-1}-2^{q-i-1})(1-2^{i-1})\right\}+(2^{p-2}-2^{q-2})$$

$$= -5(2^{p-4}-2^{q-4})<0.$$

(4) $\mathcal{T}_5$.

(4.1) When $p=q$, $G_{10}\cong G_{11}$, $G_{12}\cong G_{13}$, $G_{14}\cong G_{15}$, $G_{16}\cong G_{17}$ and

$$\alpha'(G_{16},4)-\alpha'(G_{14},4)$$

$$=\left[\left\{\sum_{i=1}^{4}\binom{4}{i}(2^{p-i-1}+2^{q-2}-2)\right\}+7\cdot 2^{p-2}+2^{q}+4\cdot 2^{q-3}+17\right]-$$

$$\left[\left\{\sum_{i=1}^{4}\binom{4}{i}(2^{p-i-1}+2^{q-2}-2)\right\}+2^{p+1}+5\cdot 2^{q-2}+2^{q-4}+21\right]$$

$$=(7\cdot 2^{p-2}-2^{p+1})+(2^{q}+4\cdot 2^{q-3}-5\cdot 2^{q-2}-2^{q-4})-4$$

$$= -2^{q-4}-4<0. \quad \text{(since } p=q\text{)}$$

$$\alpha'(G_{14},4)-\alpha'(G_{10},4)$$
$$= \left[\left\{\sum_{i=1}^{4}\binom{4}{i}(2^{p-i-1}+2^{q-2}-2)\right\}+5\cdot 2^{q-2}+2^{p+1}+2^{q-4}+21\right]-$$
$$\left[\left\{\sum_{i=1}^{4}\binom{4}{i}(2^{p-i-1}+2^{q-2}-2)\right\}+2^{p-3}+7\cdot 2^{p-2}+2^{q}+7\cdot 2^{q-4}+29\right]$$
$$= (2^{p+1}-7\cdot 2^{p-2}-2^{p-3})+(5\cdot 2^{q-2}+2^{q}-6\cdot 2^{q-4})-8$$
$$= 2^{p-3}-2^{q-3}-8 \qquad (2)$$
$$= -8 < 0.$$
$$\alpha'(G_{10},4)-\alpha'(G_{12},4)$$
$$= \left[\left\{\sum_{i=1}^{4}\binom{4}{i}(2^{p-i-1}+2^{q-2}-2)\right\}+7\cdot 2^{p-2}+2^{p-3}+2^{q}+7\cdot 2^{q-4}+29\right]-$$
$$\left[\left\{\sum_{i=1}^{4}\binom{4}{i}(2^{p-i-1}+2^{q-2}-2)\right\}+2^{p}+7\cdot 2^{p-4}+7\cdot 2^{q-2}+2^{q-3}+57\right]$$
$$= (7\cdot 2^{p-2}+2^{p-3}-2^{p}-7\cdot 2^{p-4})+(2^{q}+7\cdot 2^{q-4}-7\cdot 2^{q-2}-2^{q-3})-28$$
$$= 7\cdot 2^{p-4}-7\cdot 2^{q-4}-28 < 0.$$

Therefore, we have

$$\alpha'(G_{16},4) < \alpha'(G_{14},4) < \alpha'(G_{10},4) < \alpha'(G_{12},4).$$

(4.2) When $p > q$,

$$\alpha'(G_{12},4)-\alpha'(G_{16},4)$$
$$= \left[\left\{\sum_{i=1}^{4}\binom{4}{i}(2^{p-i-1}+2^{q-2}-2)\right\}+2^{p}+7\cdot 2^{p-4}+7\cdot 2^{q-2}+2^{q-3}+57\right]-$$
$$\left[\left\{\sum_{i=1}^{4}\binom{4}{i}(2^{p-i-1}+2^{q-2}-2)\right\}+7\cdot 2^{p-2}+2^{q}+4\cdot 2^{q-3}+17\right]$$
$$= -5\cdot 2^{p-4}+6\cdot 2^{q-4}+40$$
$$= -4\cdot 2^{q-4}+40 < 0. \quad (\text{since } q \geq 8)$$
$$\alpha'(G_{16},4)-\alpha'(G_{10},4)$$
$$= \left[\left\{\sum_{i=1}^{4}\binom{4}{i}(2^{p-i-1}+2^{q-2}-2)\right\}+2^{p}+7\cdot 2^{p-2}+2^{q}+4\cdot 2^{q-3}+17\right]-$$

$$\left[\left\{\sum_{i=1}^{4}\binom{4}{i}(2^{p-i-1}+2^{q-2}-2)\right\}+7\cdot 2^{p-2}+2^{p-3}+2^{q}+7\cdot 2^{q-4}+29\right]$$
$$= -2^{p-3}+2^{q-4}-12<0.$$

From Equation (2), we have

$$\alpha'(G_{10},4)-\alpha'(G_{14},4)$$
$$= -(2^{p-3}-2^{q-3}-8)<0.$$

$$\alpha'(G_{14},4)-\alpha'(G_{15},4)$$
$$= \left[\left\{\sum_{i=1}^{4}\binom{4}{i}(2^{p-i-1}+2^{q-2}-2)\right\}+2^{p+1}+5\cdot 2^{q-2}+2^{q-4}+21\right]-$$
$$\left[\left\{\sum_{i=1}^{4}\binom{4}{i}(2^{p-i-1}+2^{p-2}-2)\right\}+5\cdot 2^{p-2}+2^{q+1}+2^{p-4}+21\right]$$
$$= \left\{\sum_{i=1}^{4}\binom{4}{i}(2^{p-i-1}-2^{q-i-1})(1-2^{i-1})\right\}+(2^{p+1}-2^{q+1})-$$
$$5(2^{p-2}-2^{q-2})-(2^{p-4}-2^{q-4})$$
$$< -12(2^{p-4}-2^{q-4})+2^{5}(2^{p-4}-2^{q-4})-5\cdot 2^{2}(2^{p-4}-2^{q-4})-(2^{p-4}-2^{q-4})$$
$$= -(2^{p-4}-2^{q-4})<0.$$

$$\alpha'(G_{15},4)-\alpha'(G_{17},4)$$
$$= \left[\left\{\sum_{i=1}^{4}\binom{4}{i}(2^{q-i-1}+2^{p-2}-2)\right\}+2^{q+1}+5\cdot 2^{p-2}+2^{p-4}+21\right]-$$
$$\left[\left\{\sum_{i=1}^{4}\binom{4}{i}(2^{q-i-1}+2^{p-2}-2)\right\}+7\cdot 2^{q-2}+2^{p}+4\cdot 2^{p-3}+17\right]$$
$$= -3\cdot 2^{p-4}+4\cdot 2^{q-4}+4<0.$$

$$\alpha'(G_{15},4)-\alpha'(G_{11},4)$$
$$= \left[\left\{\sum_{i=1}^{4}\binom{4}{i}(2^{q-i-1}+2^{p-2}-2)\right\}+2^{q+1}+5\cdot 2^{p-2}+2^{p-4}+21\right]-$$
$$\left[\left\{\sum_{i=1}^{4}\binom{4}{i}(2^{q-i-1}+2^{p-2}-2)\right\}+7\cdot 2^{q-2}+2^{q-3}+2^{p}+7\cdot 2^{p-4}+29\right]$$

$$= -2^{p-3} + 2^{q-3} - 8 < 0.$$

$$\alpha'(G_{17}, 4) - \alpha'(G_{13}, 4)$$

$$= \left[\left\{ \sum_{i=1}^{4} \binom{4}{i} (2^{q-i-1} + 2^{p-2} - 2) \right\} + 7 \cdot 2^{q-2} + 2^p + 4 \cdot 2^{p-3} + 17 \right] -$$

$$\left[\left\{ \sum_{i=1}^{4} \binom{4}{i} (2^{q-i-1} + 2^{p-2} - 2) \right\} + 2^q + 7 \cdot 2^{q-4} + 7 \cdot 2^{p-2} + 2^{p-3} + 57 \right]$$

$$= -6 \cdot 2^{p-4} + 5 \cdot 2^{q-4} - 40 < 0.$$

$$\alpha'(G_{11}, 4) - \alpha'(G_{13}, 4)$$

$$= \left[\left\{ \sum_{i=1}^{4} \binom{4}{i} (2^{q-i-1} + 2^{p-2} - 2) \right\} + 7 \cdot 2^{q-2} + 2^{q-3} + 2^p + 7 \cdot 2^{p-4} + 29 \right] -$$

$$\left[\left\{ \sum_{i=1}^{4} \binom{4}{i} (2^{q-i-1} + 2^{p-2} - 2) \right\} + 2^q + 7 \cdot 2^{q-4} + 7 \cdot 2^{p-2} + 2^{p-3} + 57 \right]$$

$$= (2^p + 7 \cdot 2^{p-4} - 7 \cdot 2^{p-2} - 2^{p-3}) + (7 \cdot 2^{q-2} + 2^{q-3} - 2^q - 7 \cdot 2^{q-4}) - 28$$

$$= -7 \cdot 2^{p-4} + 7 \cdot 2^{q-4} - 28 < 0.$$

$$\alpha'(G_{17}, 4) - \alpha'(G_{11}, 4)$$

$$= \left[\left\{ \sum_{i=1}^{4} \binom{4}{i} (2^{q-i-1} + 2^{p-2} - 2) \right\} + 7 \cdot 2^{q-2} + 2^p + 4 \cdot 2^{p-3} + 17 \right] -$$

$$\left[\left\{ \sum_{i=1}^{4} \binom{4}{i} (2^{q-i-1} + 2^{p-2} - 2) \right\} + 7 \cdot 2^{q-2} + 2^{q-3} + 2^p + 7 \cdot 2^{p-4} + 29 \right]$$

$$= 2^{p-4} - 2 \cdot 2^{q-4} - 12$$

$$\begin{cases} < 0 & \text{if} \quad p - q = 1; \\ > 0 & \text{if} \quad p - q \geq 2, \end{cases}$$

Thus, if $p - q = 1$, we have

$$\alpha'(G_{12}, 4) < \alpha'(G_{16}, 4) < \alpha'(G_{10}, 4) < \alpha'(G_{14}, 4) < \alpha'(G_{15}, 4) <$$
$$\alpha'(G_{17}, 4) < \alpha'(G_{11}, 4) < \alpha'(G_{13}, 4);$$

and if $p - q \geq 2$, we have

$$\alpha'(G_{12}, 4) < \alpha'(G_{16}, 4) < \alpha'(G_{10}, 4) < \alpha'(G_{14}, 4) < \alpha'(G_{15}, 4) <$$
$$\alpha'(G_{11}, 4) < \alpha'(G_{17}, 4) < \alpha'(G_{13}, 4)$$

Therefore,

$$\alpha'(G_i,4) \neq \alpha'(G_j,4) \quad \text{for} \quad 10 \leq i < j \leq 17.$$

(5) $\mathcal{T}_6$.
Note that $\alpha'(G_i,4)$ is odd when $22 \leq i \leq 25$ and even when $18 \leq i \leq 21$.
Thus $\alpha'(G_i,4) \neq \alpha'(G_j,4)$ if $22 \leq i \leq 25$ and $18 \leq i \leq 21$.
(5.1) The graphs G_i when $18 \leq i \leq 21$.
(5.1.1) When $p = q$, $G_{18} \cong G_{19}$, $G_{20} \cong G_{21}$ and

$$\begin{aligned}
&\alpha'(G_{18},4) - \alpha'(G_{20},4) \\
&= \left[\left\{\sum_{i=1}^{4}\binom{4}{i}(2^{p-i-1}+2^{q-2}-2)\right\} + 6\cdot 2^{p-2} + 2^q + 3\cdot 2^{q-3} + 24\right] - \\
&\quad \left[\left\{\sum_{i=1}^{4}\binom{4}{i}(2^{p-i-1}+2^{q-2}-2)\right\} + 7\cdot 2^{p-2} + 2^q + 3\cdot 2^{q-4} + 28\right] \\
&= -4\cdot 2^{p-4} + 3\cdot 2^{q-4} - 4 \qquad (3) \\
&= -2^{q-4} - 4 < 0. \quad (\text{since } p = q)
\end{aligned}$$

(5.1.2) When $p > q$, from Equation (3), we have

$$\alpha'(G_{18},4) - \alpha'(G_{20},4) = -4\cdot 2^{p-4} + 3\cdot 2^{q-4} - 4 < 0.$$

$$\begin{aligned}
&\alpha'(G_{20},4) - \alpha'(G_{21},4) \\
&= \left[\left\{\sum_{i=1}^{4}\binom{4}{i}(2^{p-i-1}+2^{q-2}-2)\right\} + 7\cdot 2^{p-2} + 2^q + 3\cdot 2^{q-4} + 28\right] - \\
&\quad \left[\left\{\sum_{i=1}^{4}\binom{4}{i}(2^{q-i-1}+2^{p-2}-2)\right\} + 7\cdot 2^{q-2} + 2^p + 3\cdot 2^{p-4} + 28\right] \\
&= \left\{\sum_{i=1}^{4}\binom{4}{i}(2^{p-i-1}-2^{q-i-1})(1-2^{i-1})\right\} + 7(2^{p-2}-2^{q-2}) - \\
&\quad (2^p - 2^q) - 3(2^{p-4}-2^{q-4}) \\
&< -12(2^{p-4}-2^{q-4}) + 7\cdot 2^2(2^{p-4}-2^{q-4}) - 2^4(2^{p-4}-2^{q-4}) - 3(2^{p-4}-2^{q-4}) \\
&= -3(2^{p-4}-2^{q-4}) < 0.
\end{aligned}$$

$$\begin{aligned}
&\alpha'(G_{21},4) - \alpha'(G_{19},4) \\
&= \left[\left\{\sum_{i=1}^{4}\binom{4}{i}(2^{q-i-1}+2^{p-2}-2)\right\} + 7\cdot 2^{q-2} + 2^p + 3\cdot 2^{p-4} + 28\right] -
\end{aligned}$$

$$\left[\left\{ \sum_{i=1}^{4} \binom{4}{i} (2^{q-i-1} + 2^{p-2} - 2) \right\} + 6 \cdot 2^{q-2} + 2^p + 3 \cdot 2^{p-3} + 24 \right]$$
$$= -3 \cdot 2^{p-4} + 4 \cdot 2^{q-4} + 4 < 0.$$

Therefore, we have

$$\alpha'(G_{18}, 4) < \alpha'(G_{20}, 4) < \alpha'(G_{21}, 4) < \alpha'(G_{19}, 4).$$

(5.2) The graphs G_i when $22 \le i \le 25$.
(5.2.1) When $p = q$, $G_{22} \cong G_{23}$, $G_{24} \cong G_{25}$ and

$$\begin{aligned} &\alpha'(G_{22}, 4) - \alpha'(G_{24}, 4) \\ &= \left[\left\{ \sum_{i=1}^{4} \binom{4}{i} (2^{p-i-1} + 2^{q-2} - 2) \right\} + 2^p + 3 \cdot 2^{p-4} + 7 \cdot 2^{q-2} + 91 \right] - \\ &= \left[\left\{ \sum_{i=1}^{4} \binom{4}{i} (2^{p-i-1} + 2^{q-2} - 2) \right\} + 7 \cdot 2^{p-2} + 2^q + 3 \cdot 2^{q-4} + 91 \right] \\ &= (2^p + 3 \cdot 2^{p-4} - 7 \cdot 2^{p-2}) + (7 \cdot 2^{q-2} - 2^q - 3 \cdot 2^{q-4}) \\ &= -9 \cdot 2^{p-4} + 9 \cdot 2^{q-4}. \qquad (4) \\ &= 0. \end{aligned}$$

Thus, we need to calculate $\alpha(G_{22}, 5) - \alpha(G_{24}, 5)$. By using Lemmas 2.6, we have

$$\begin{aligned} &\alpha(G_{22}, 5) - \alpha(G_{24}, 5) \\ &= \Big[\alpha(G_{22} + a_1b_1, 5) + \alpha(G_{22} - \{a_1, b_1\}, 4) + \alpha(G_{22} - \{a_1, b_1, c_1\}, 4) + \\ &\quad \alpha(G_{22} - \{a_1, b_1, d_1\}, 4) + \alpha(G_{22} - \{a_1, b_1, c_1, d_1\}, 4) \Big] - \\ &\quad \alpha(G_{24} + a_2b_2, 5) + \alpha(G_{24} - \{a_2, b_2\}, 4) + \alpha(G_{24} - \{a_2, b_2, c_2\}, 4) + \\ &\quad \alpha(G_{24} - \{a_2, b_2, d_2\}, 4) + \alpha(G_{24} - \{a_2, b_2, c_2, d_2\}, 4) \Big] \\ &= \Big[\alpha(G_{22} - \{a_1, b_1, c_1\}, 4) - \alpha(G_{24} - \{a_2, b_2, c_2\}, 4) \Big] + \\ &\quad \Big[\alpha(G_{22} - \{a_1, b_1, d_1\}, 4) - \alpha(G_{24} - \{a_2, b_2, d_2\}, 4) \Big] + \\ &\quad \Big[\alpha(G_{22} - \{a_1, b_1, c_1, d_1\}, 4) - \alpha(G_{24} - \{a_2, b_2, c_2, d_2\}, 4) \Big] \end{aligned}$$

since $G_{22} + a_1b_1 \cong G_{24} + a_2b_2, G_{22} - \{a_1, b_1\} \cong G_{24} - \{a_2, b_2\}$

$$= \left[\alpha'(G_{22} - \{a_1, b_1, c_1\}, 4) - \alpha'(G_{24} - \{a_2, b_2, c_2\}, 4) \right] +$$

$$\left[\alpha'(G_{22} - \{a_1, b_1, d_1\}, 4) - \alpha'(G_{24} - \{a_2, b_2, d_2\}, 4) \right] +$$

$$\left[\alpha'(G_{22} - \{a_1, b_1, c_1, d_1\}, 4) - \alpha'(G_{24} - \{a_2, b_2, c_2, d_2\}, 4) \right]$$

$$= \left[\sum_{i=1}^{4} \binom{4}{i} (2^{p-3-i} + 2^{q-3} - 2) - \sum_{i=1}^{4} \binom{4}{i} (2^{p-2-i} + 2^{q-4} - 2) \right] +$$

$$\left[\sum_{i=1}^{4} \binom{4}{i} (2^{p-3-i} + 2^{q-3} - 2) - \sum_{i=1}^{4} \binom{4}{i} (2^{p-2-i} + 2^{q-4} - 2) \right] +$$

$$\left[\sum_{i=1}^{4} \binom{4}{i} (2^{p-4-i} + 2^{q-3} - 2) - \sum_{i=1}^{4} \binom{4}{i} (2^{p-2-i} + 2^{q-5} - 2) \right]$$

$$= 2\sum_{i=1}^{4} \binom{4}{i} (2^{q-4} - 2^{p-3-i}) + \sum_{i=1}^{4} \binom{4}{i} (3 \cdot 2^{q-5} - 3 \cdot 2^{p-4-i})$$

$$= 7\sum_{i=1}^{4} \binom{4}{i} (2^{q-5} - 2^{p-4-i}) > 0.$$

(5.2.2) When $p > q$, from Equation (4), we have

$$\alpha'(G_{22}, 4) - \alpha'(G_{24}, 4) = -9 \cdot 2^{p-4} + 9 \cdot 2^{q-4} < 0.$$

$\alpha'(G_{24}, 4) - \alpha'(G_{25}, 4)$

$$= \left[\left\{ \sum_{i=1}^{4} \binom{4}{i} (2^{p-i-1} + 2^{q-2} - 2) \right\} + 7 \cdot 2^{p-2} + 2^q + 3 \cdot 2^{q-4} + 91 \right] -$$

$$\left[\left\{ \sum_{i=1}^{4} \binom{4}{i} (2^{q-i-1} + 2^{p-2} - 2) \right\} + 7 \cdot 2^{q-2} + 2^p + 3 \cdot 2^{p-4} + 91 \right]$$

$$= \left\{ \sum_{i=1}^{4} \binom{4}{i} (2^{p-i-1} - 2^{q-i-1})(1 - 2^{i-1}) \right\} + 7(2^{p-2} - 2^{q-2}) -$$

$$(2^p - 2^q) - 3(2^{p-4} - 2^{q-4})$$

$$< -3(2^{p-4} - 2^{q-4}) < 0.$$

$\alpha'(G_{25}, 4) - \alpha'(G_{23}, 4)$

$$= \left[\left\{ \sum_{i=1}^{4} \binom{4}{i} (2^{q-i-1} + 2^{p-2} - 2) \right\} + 7 \cdot 2^{q-2} + 2^p + 3 \cdot 2^{p-4} + 91 \right] -$$

$$\left[\left\{ \sum_{i=1}^{4} \binom{4}{i} (2^{q-i-1} + 2^{p-2} - 2) \right\} + 2^q + 3 \cdot 2^{q-4} + 7 \cdot 2^{p-2} + 91 \right]$$

$$= (2^p + 3 \cdot 2^{p-4} - 7 \cdot 2^{p-2}) + (7 \cdot 2^{q-2} - 2^q - 3 \cdot 2^{q-4})$$

$$= -9 \cdot 2^{p-4} + 9 \cdot 2^{q-4} < 0.$$

Therefore, we have

$$\alpha'(G_{22}, 4) < \alpha'(G_{24}, 4) < \alpha'(G_{25}, 4) < \alpha'(G_{23}, 4).$$

Similarly, we can show that for any two graphs $G_{i_1}, G_{i_2} \in \mathcal{T}_i$ $(6 \leq i \leq 13)$, if $G_{i_1} \not\cong G_{i_2}$, then either $\alpha'(G_{i_1}, 4) \neq \alpha'(G_{i_2}, 4)$ or $\alpha(G_{i_1}, 5) \neq \alpha(G_{i_2}, 5)$ (see [15]). Hence the proof of the theorem is complete. □

Acknowledgments. The authors would like to thank the referee for his helpful comments.

References

1. M. Borowiecki, E. Drgas-Burchardt, Classes of chromatically unique graphs, *Discrete Math.* **111** (1993),71–74.

2. X. Chen, Some families of chromatically unique bipartite graphs, *Discrete Math.* **184** (1998), 245–253.

3. F.M. Dong, K.M. Koh, K.L. Teo, C.H.C. Little, M.D. Hendy, An attempt to classify bipartite graphs by chromatic polynomials. *Discrete Math.* **222** (2000), 73–88.

4. F.M. Dong, K.M. Koh, K.L. Teo, C.H.C. Little, M.D. Hendy, Chromatically unique graphs with low 3-independent partition numbers, *Discrete Math.* **224** (2000), 107–124.

5. F.M. Dong, K.M. Koh, K.L. Teo, C.H.C. Little, M.D. Hendy, Sharp bounds for the numbers of 3-partitions and the chromaticity of bipartite graphs, *J. Graph Theory* **37** (2001) 48–77.

6. R.E. Giudici, E. Lima de Sa, Chromatic uniqueness of certain bipartite graphs, *Congr. Numer.* **76** (1990), 69–75.

7. Y.H. Peng, Chromatic uniqueness of certain bipartite graphs, *Discrete Math.* **94** (1991),129–140.

8. R.C.Read, W.T. Tutte, Chromatic polynomials, in: L.W. Beineke, R.J. Wilson (Eds.), *Selected Topics in Graph Theory III*, Academic Press, New York, 1988, pp. 15–42.

9. H. Roslan and Y.H. Peng, Chromatic uniqueness of complete bipartite graphs with certain edges deleted, II. (submitted for publication).

10. H. Roslan and Y.H. Peng, Chromatic uniqueness of complete bipartite graphs with certain 3-independent partition numbers. *Matematika* (accepted for publication).

11. H. Roslan and Y.H. Peng, Chromatic uniqueness of complete bipartite graphs with certain 3-independent partition numbers II. *Bull. Malaysian Math. Soc.* (2) **29**(2) (2006), 147–168.

12. H. Roslan and Y.H. Peng, Chromatic uniqueness of complete bipartite graphs with certain 3-independent partition numbers III. *Malaysian J. Math. Sci.* (accepted for publication).

13. H. Roslan and Y.H. Peng, Chromatic uniqueness of complete bipartite graphs with Five or Six Edges Deleted. (submitted for publication).

14. C.P. Teo, K.M. Koh, The chromaticity of complete bipartite graphs with at most one edge deleted, *J. Graph Theory* **14** (1990), 89–99.

15. `http://www.fsas.upm.edu.my/~yhpeng/publish/proofkmn-7e.pdf`

16. `http://www.fsas.upm.edu.my/~yhpeng/publish/table_s7.pdf`

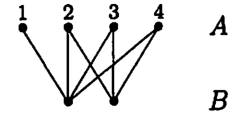
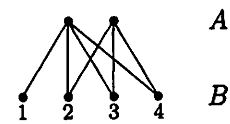
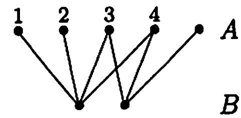
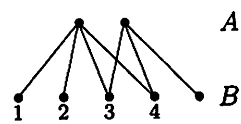
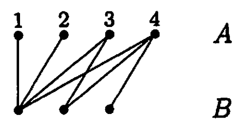
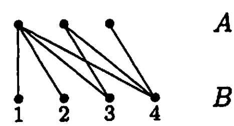
Name of Graph, G_i	Graphs G_i' ($G_i' = K_{p,q} - G_i$) $\lvert A\rvert = p\lvert B\rvert = q$	$\alpha'(G_i, 3)$	$\alpha'(G_i, 4)$
G_2		28	$\sum_{i=1}^{4} \binom{4}{i}(2^{p-i-1} + 2^{q-2} - 2)+$ $2^{p+1} + 5 \cdot 2^{p-4} + 5 \cdot 2^{q-1} + 5$
G_3		28	$\sum_{i=1}^{4} \binom{4}{i}(2^{q-i-1} + 2^{p-2} - 2)+$ $2^{q+1} + 5 \cdot 2^{q-4} + 5 \cdot 2^{p-1} + 5$
G_4		25	$\sum_{i=1}^{4} \binom{4}{i}(2^{p-i-1} + 2^{q-2} - 2)+$ $7 \cdot 2^{p-2} + 2^{p-4} + 2^{q+1} + 2^{q-3}$ $+29$
G_5		25	$\sum_{i=1}^{4} \binom{4}{i}(2^{q-i-1} + 2^{p-2} - 2)+$ $7 \cdot 2^{q-2} + 2^{q-4} + 2^{p+1} + 2^{p-3}$ $+29$
G_6		25	$\sum_{i=1}^{4} \binom{4}{i}(2^{p-i-1} + 2^{q-2} - 2)+$ $9 \cdot 2^{p-2} + 3 \cdot 2^{q-1} + 3 \cdot 2^{q-4}$ $+14$
G_7		25	$\sum_{i=1}^{4} \binom{4}{i}(2^{q-i-1} + 2^{p-2} - 2)+$ $9 \cdot 2^{q-2} + 3 \cdot 2^{p-1} + 3 \cdot 2^{p-4}$ $+14$

TABLE 1: Graphs in $\mathcal{T}_i$ for $2 \le i \le 6$ (1 of 4)

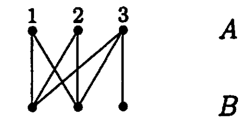
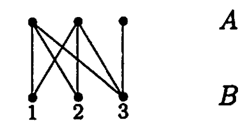
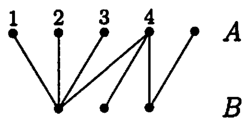
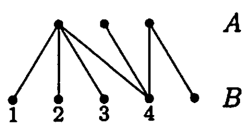
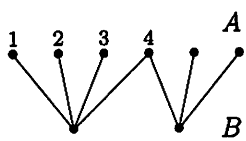
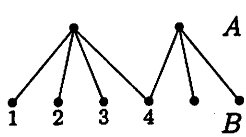
Name of Graph, G_i	Graphs G_i' ($G_i' = K_{p,q} - G_i$) $\|A\| = p\|B\| = q$	$\alpha'(G_i, 3)$	$\alpha'(G_i, 4)$
G_8		24	$\sum_{i=1}^{4} \binom{4}{i}(2^{p-i-1} + 2^{q-2} - 2)+$ $53 \cdot 2^{p-4} + 12 \cdot 2^{q-2} + 2^{q-4} - 7$
G_9		24	$\sum_{i=1}^{4} \binom{4}{i}(2^{q-i-1} + 2^{p-2} - 2)+$ $53 \cdot 2^{q-4} + 12 \cdot 2^{p-2} + 2^{p-4} - 7$
G_{10}		23	$\sum_{i=1}^{4} \binom{4}{i}(2^{p-i-1} + 2^{q-2} - 2)+$ $15 \cdot 2^{p-3} + 2^{q} + 7 \cdot 2^{q-4} + 29$
G_{11}		23	$\sum_{i=1}^{4} \binom{4}{i}(2^{q-i-1} + 2^{p-2} - 2)+$ $15 \cdot 2^{q-3} + 2^{p} + 7 \cdot 2^{p-4} + 29$
G_{12}		23	$\sum_{i=1}^{4} \binom{4}{i}(2^{p-i-1} + 2^{q-2} - 2)+$ $23 \cdot 2^{p-4} + 7 \cdot 2^{q-2} + 2^{q-3} + 57$
G_{13}		23	$\sum_{i=1}^{4} \binom{4}{i}(2^{q-i-1} + 2^{p-2} - 2)+$ $23 \cdot 2^{q-4} + 7 \cdot 2^{p-2} + 2^{p-3} + 57$

TABLE 1: Graphs in $\mathcal{T}_i$ for $2 \le i \le 6$ (2 of 4)

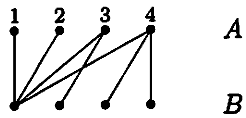
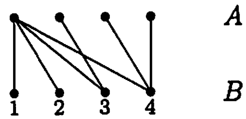
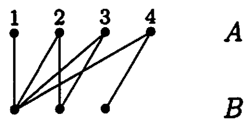
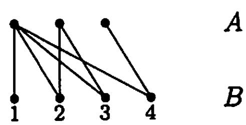
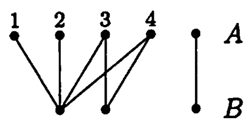
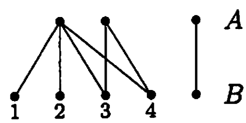
Name of Graph, G_i	Graphs G'_i ($G'_i = K_{p,q} - G_i$) $\lvert A\rvert = p \lvert B\rvert = q$	$\alpha'(G_i, 3)$	$\alpha'(G_i, 4)$
G_{14}		23	$\sum_{i=1}^{4} \binom{4}{i}(2^{p-i-1} + 2^{q-2} - 2) + 2^{p+1} + 5 \cdot 2^{q-2} + 2^{q-4} + 21$
G_{15}		23	$\sum_{i=1}^{4} \binom{4}{i}(2^{q-i-1} + 2^{p-2} - 2) + 2^{q+1} + 5 \cdot 2^{p-2} + 2^{p-4} + 21$
G_{16}		23	$\sum_{i=1}^{4} \binom{4}{i}(2^{p-i-1} + 2^{q-2} - 2) + 7 \cdot 2^{p-2} + 2^{q} + 4 \cdot 2^{q-3} + 17$
G_{17}		23	$\sum_{i=1}^{4} \binom{4}{i}(2^{q-i-1} + 2^{p-2} - 2) + 7 \cdot 2^{q-2} + 2^{p} + 4 \cdot 2^{p-3} + 17$
G_{18}		22	$\sum_{i=1}^{4} \binom{4}{i}(2^{p-i-1} + 2^{q-2} - 2) + 6 \cdot 2^{p-2} + 2^{q} + 3 \cdot 2^{q-3} + 24$
G_{19}		22	$\sum_{i=1}^{4} \binom{4}{i}(2^{q-i-1} + 2^{p-2} - 2) + 6 \cdot 2^{q-2} + 2^{p} + 3 \cdot 2^{p-3} + 24$

TABLE 1: Graphs in $\mathcal{T}_i$ for $2 \le i \le 6$ (3 of 4)

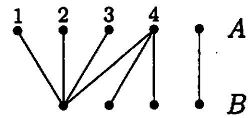
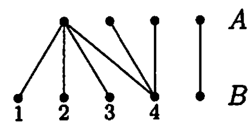
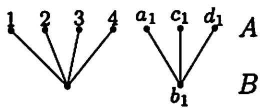
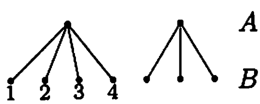
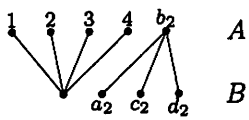
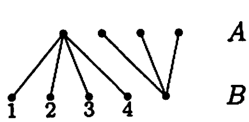
Name of Graph, G_i	Graphs G'_i ($G'_i = K_{p,q} - G_i$) $\lvert A\rvert = p\lvert B\rvert = q$	$\alpha'(G_i, 3)$	$\alpha'(G_i, 4)$
G_{20}		22	$\sum_{i=1}^{4}\binom{4}{i}(2^{p-i-1}+2^{q-2}-2)+$ $7\cdot 2^{p-2}+2^{q}+3\cdot 2^{q-4}+28$
G_{21}		22	$\sum_{i=1}^{4}\binom{4}{i}(2^{q-i-1}+2^{p-2}-2)+$ $7\cdot 2^{q-2}+2^{p}+3\cdot 2^{p-4}+28$
G_{22}		22	$\sum_{i=1}^{4}\binom{4}{i}(2^{p-i-1}+2^{q-2}-2)+$ $2^{p}+3\cdot 2^{p-4}+7\cdot 2^{q-2}+91$
G_{23}		22	$\sum_{i=1}^{4}\binom{4}{i}(2^{q-i-1}+2^{p-2}-2)+$ $2^{q}+3\cdot 2^{q-4}+7\cdot 2^{p-2}+91$
G_{24}		22	$\sum_{i=1}^{4}\binom{4}{i}(2^{p-i-1}+2^{q-2}-2)+$ $7\cdot 2^{p-2}+2^{q}+3\cdot 2^{q-4}+91$
G_{25}		22	$\sum_{i=1}^{4}\binom{4}{i}(2^{q-i-1}+2^{p-2}-2)+$ $7\cdot 2^{q-2}+2^{p}+3\cdot 2^{p-4}+91$

TABLE 1: Graphs in $\mathcal{T}_i$ for $2 \le i \le 6$ (4 of 4)