Distance in the non-commuting graph of groups

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Abstract

Let G be a non-abelian group and let $Z(G)$ be the center of G. Associate with G a graph Γ_G as follows: Take $G \backslash Z(G)$ as vertices of Γ_G and join two distinct vertices x and y whenever $xy \neq yx$. Graph Γ_G is called the non-commuting graph of G and many of graph theoretical properties of Γ_G have been studied. In this paper we study some metric graph properties of Γ_G.

1 Introduction

Let Γ be an undirected connected graph without loops or multiple edges. The sets of vertices and edges of Γ are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. For vertices x and y in $V(\Gamma)$, We denote by $d(x,y)$ the topological distance i.e., the number of edges on the shortest path, joining the two vertices of Γ. Since Γ is connected, $d(x,y)$ exists for all $x, y \in V(\Gamma)$. The name Wiener number or Wiener index is nowadays in standard use in chemistry and is sometimes encountered also in the mathematical literature(see [5, 2, 3, 4]). The Wiener index of the graph Γ is the half sum of distances over all its vertex pairs (u,v): $W(\Gamma) = \frac{1}{2} \sum_{u,v \in V(\Gamma)} d(u,v)$.

For an edge $e(= uv)$ of a graph Γ, let $n_u(e)$ denote the set of vertices of Γ lying closer to u than to v and $n_v(e)$ is the set of vertices of Γ lying closer

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to v than to u.
The sets $n_u(e)$ and $n_v(e)$ play an important role in metric graph theory. For more information on the research in this direction see [6, 10, 11]). Ivan Gutman [7] defined the Szeged index, $Sz(\Gamma)$, of a graph Γ as:

$$Sz(\Gamma) = \sum_{uv=e\in E(\Gamma)} |n_u(e)| \cdot |n_v(e)|.$$

If Γ is a tree, then $Sz(\Gamma) = W(\Gamma)$. We recall that this is not true for any graph.
Let G be a non-abelian group and let $Z(G)$ be the center of G. Associate with G a graph Γ_G as follows: take $G\backslash Z(G)$ as vertices of Γ_G and join two distinct vertices x and y whenever $xy \neq yx$. Graph Γ_G is called the non-commuting graph of G and many of graph theoretical properties of Γ_G have been studied in [1, 12]. In this paper We study some metric properties of Γ_G and compute the Wiener and Szeged indices of some linear groups. Since Wiener and Szeged indices are invariant under graph isomorphism, one of the advantages of this work is that for comparing the non-commuting graph of groups, We can compare their indices at first.
If G and H are two graphs, it may happen that $W(G) = W(H)$ but $Sz(G) \neq Sz(G)$. However, if G and H are two non-abelian groups, We guess that if $W(\Gamma_G) = W(\Gamma_H)$, then $Sz(\Gamma_G) = Sz(\Gamma_H)$!

1 metric properties of the non-commuting graph

We begin with some basic metric properties of the non-commuting graph of group G. The first lemma introduces an important fact about Γ_G.

Lemma 1 *Let G be a non-abelian group. Then Γ_G is a connected graph of diameter* 2 *and girth* 3.

PROOF: See [1].

By the above Lemma it is clear that:

Remark 2 *Let G be a non-abelian group and $x \in G\backslash Z(G)$. Then*

$$d(x,y) = \begin{cases} 1, & \text{if} \quad y \in G\backslash C_G(x)\ ; \\ 2, & \text{if} \quad y \in C_G(x)\backslash Z(G). \end{cases}$$

We prove a key lemma about $n_u(e)$ for an arbitrary finite non-abelian group. It will be used later in the paper.

Lemma 3 *Let G be a non-abelian group and $e = uv \in E(\Gamma_G)$. Then*

$$n_u(e) = ((C_G(v)\backslash C_G(u))\backslash\{v\}) \cup \{u\}.$$

PROOF: Let $B = ((C_G(v)\backslash C_G(u))\backslash\{v\}) \cup \{u\}$. An easy computing shows that: $B \subseteq n_u(e)$. Now suppose that $y \in n_u(e)$ and $d(y,u) = t_1, d(y,v) = t_2$. So $t_1 < t_2$. If $t_2 = 0$, then $t_1 < 0$ which is impossible. If $t_2 = 1$, then $t_1 = 0$ and $y = u$. Finally, if $t_2 = 2$, then $t_1 = 0$ *or* 1. According to the Remark 2, We have $y = u$ or $y \in C_G(v)\backslash C_G(u)$. This completes the proof.

We recall that a graph Γ is called distance-balanced if $|n_u(e)| = |n_v(e)|$ for any edge uv of $E(\Gamma)$. We now have the following proposition that characterizes all finite non-abelian groups so that Γ_G is distance-balanced.

Proposition 4 *Let G be a finite non-abelian group such that Γ_G is distance-balanced. Then G is nilpotent of class at most 3 and $G = P \times A$, where A is an abelian group and P is a p-group (p is a prime) and furthermore Γ_P is a regular graph.*

PROOF: We show that for each $u, v \in G\backslash Z(G)$, $|C_G(u)| = |C_G(v)|$. If $uv \in E(\Gamma_G)$, then, by Lemma 3, We have $|C_G(u)\backslash C_G(v)| = |C_G(v)\backslash C_G(u)|$, since $|n_u(e)| = |n_v(e)|$. Therefore $|C_G(u)| = |C_G(v)|$. Now suppose that $uv \notin E(\Gamma_G)$. Since $diam(\Gamma_G) = 2$, so there exists $y \in V(\Gamma_G)$ such that $uy, vy \in E(\Gamma_G)$. Thus $|C_G(u)| = |C_G(y)| = |C_G(v)|$. Hence Γ_G is regular and by [1, Proposition 2.6], G is nilpotent of class at most 3 and $G = P \times A$, where A is an abelian group and P is a p-group (p is a prime) and furthermore Γ_P is a regular graph.

Corollary 5 *Let G be a finite non-abelian group such that for each $xy \in E(\Gamma_G)$ there exists φ in $Aut(\Gamma_G)$ such that $\varphi(x) = y$ and $\varphi(y) = x$. Then G is nilpotent of class at most 3 and $G = P \times A$, where A is an abelian group and P is a p-group.*

PROOF: Let $e = xy \in \Gamma_G$ and let φ be an automorphism of Γ_G for which $\varphi(x) = y$ and $\varphi(y) = x$. If $a \in n_e(x)$, then $d(a,x) < d(a,y)$. Therefore $d(\varphi(a),\varphi(x)) < d(\varphi(a),\varphi(y))$ and so $d(\varphi(a),y) < d(\varphi(a),x)$. It follows that $\varphi(a) \in n_e(y)$. Likewise, if $a \in n_e(y)$, then $\varphi(a) \in n_e(x)$. Hence $|n_e(x)| = |n_e(y)|$ and the Proposition 4 completes the proof.

Lemma 6 *Let G be a finite group. Then*

$$W(\Gamma_G) = \frac{(|G| - |Z(G)|)(|G| - 2|Z(G)| - 2) + |G|(k(G) - |Z(G)|)}{2},$$

where $k(G)$ is the number of conjugacy classes of G.

PROOF: Suppose that $v \in G \backslash Z(G)$ and $d(v) = \sum_{x \in G \backslash Z(G)} d(v,x)$. Then

$$\begin{aligned} d(v) &= \sum_{x \in G \backslash C_G(v)} d(v,x) + \sum_{x \in C_G(v) \backslash Z(G)} d(v,x) \\ &= |G| - |C_G(v)| + (|C_G(v)| - |Z(G)| - 1)2 \\ &= |G| + |C_G(v)| - 2|Z(G)| - 2. \end{aligned} \tag{1}$$

Now, by definition of the Wiener index, $2W(G) = \sum_{v \in G \backslash Z(G)} d(v)$ and also by (1) We have

$$\begin{aligned} 2W(G) &= \sum_{v \in G \backslash Z(G)} \Big(|G| + |C_G(v)| - 2|Z(G)| - 2 \Big) \\ &= (|G| - 2|Z(G)| - 2)(|G| - |Z(G)|) + \\ & \quad \sum_{v \in G \backslash Z(G)} |C_G(v)|. \end{aligned}$$

But by Burnsid Lemma $\sum_{v \in G} |C_G(v)| = k(G)|G|$, which implies

$$\sum_{v \in G \backslash Z(G)} |C_G(v)| = |G|k(G) - |G||Z(G)|,$$

which completes the proof.

2 Wiener and Szeged indices of $PSL(2,\ q)$

In this section We first obtain Wiener index of $\Gamma_{PSL(2,q)}$ and then Szeged index of $\Gamma_{PSL(2,q)}$, in which $q \equiv o \pmod 4$.

Theorem 1

$$W(\Gamma_{PSL(2,\ q)}) = \begin{cases} 11 & q=2, \\ 62 & q=3, \\ 1772 & q=5, \\ \frac{q^6-q^4-5q^3+5q+4}{2} & q \equiv 0 \pmod 4, \\ \frac{q^6-q^4-7q^3+7q+16}{8} & q > 5 \ \text{and} \ q \not\equiv 0 \pmod 4. \end{cases}$$

PROOF: By [8, Satz 6.14, p. 183], $PSL(2,2) \cong S_3$ and $PSL(2,3) \cong A_4$. So in the cases $q = 2$ and $q = 3$, the computation of $W(\Gamma_{PSL(2,q)})$ is

straightforward.
However, by [9, Theorems 5.5, 5.6 and 5.7], We have

$$k(G) = \begin{cases} q+1 & q \equiv 0 \pmod{4}; \\ \frac{q+5}{2} & q > 5 \text{ and } q \not\equiv 0 \pmod{4}. \end{cases}$$

We now that $PSL(2,4) \cong PSL(2,5)$ (see [8, Satz 6.14, p. 183]).
If $q \equiv 0 \pmod{4}$, then $|G| = q(q+1)(q-1)$ and $|Z(G)| = 1$. So, by Lemma 6,

$$W(\Gamma_{PSL(2,q)}) = \frac{q^6 - q^4 - 5q^3 + 5q + 4}{2}.$$

If $q \not\equiv 0 \pmod{4}$, then $|G| = \frac{q(q+1)(q-1)}{2}$ and $|Z(G)| = 1$. Hence, by Lemma 6,

$$W(\Gamma_{PSL(2,q)}) = \frac{q^6 - q^4 - 7q^3 + 7q + 16}{8},$$

which completes the proof.

Proposition 7 *Let $G = PSL(2,q)$, where q is a power of a prime p and let $k = gcd(q-1, 2)$. Then*

(1) *a Sylow p-subgroup P of G is an elementary abelian group of order q and the number of Sylow p-subgroups of G is $q+1$.*

(2) *G contains a cyclic subgroup A of order $t = \frac{q-1}{k}$ such that $N_G(\langle u \rangle)$ is a dihedral group of order $2t$ for every non-trivial element $u \in A$.*

(3) *G contains a cyclic subgroup B of order $s = \frac{q+1}{k}$ such that $N_G(\langle u \rangle)$ is a dihedral group of order $2s$ for every non-trivial element $u \in B$.*

(4) *The set $\{P^x, A^x, B^x | x \in G\}$ is a partition for G.*

Suppose that a is a non-trivial element of G.

(5) *If $q \equiv 0 \pmod{4}$, then*

$$C_G(a) = \begin{cases} A^x & \text{if } a \in A^x \text{ for some } x \in G \\ B^x & \text{if } a \in B^x \text{ for some } x \in G \\ P^x & \text{if } a \in P^x \text{ for some } x \in G \end{cases}$$

PROOF: See Proposition 3.21 of [1].

Now, it is clear, from the Proposition 7, that the number of conjugates of P, A and B are $q+1$, $\frac{q(q+1)}{2}$ and $\frac{q(q-1)}{2}$, respectively.

Definition 1 *Let Γ be a graph and $A, B \subseteq V(\Gamma)$. We define*

$$\begin{aligned} E_{A,B} &= \{ab \in E(G) \mid a \in A, \ b \in B\}, \\ S_{A,B} &= \sum_{ab \in E_{A,B}} |n_a(ab)| \cdot |n_a(ab)|. \end{aligned} \tag{2}$$

If $E_{A,B} = \emptyset$, then We define $S_{A,B} = 0$. Also, for $a \in V(G)$ We put $E_{\{a\},B} = E_{a,B}$.

Lemma 8 *Let $G = PSL(2, q)$, where $q \equiv 0 \pmod 4$. Suppose that $\{P_i : \ i = 1, 2, \ldots, q+1\}$, $\{A_i : \ i = 1, 2, \ldots, \frac{q(q+1)}{2}\}$ and $\{B_i : \ i = 1, 2, \ldots, \frac{q(q-1)}{2}\}$ are the set of all conjugates of P, A and B, respectively. Then*

(1) $S_{a,b} = S_{b,a}$.

(2) $S_{a,b} = |P_j - 1| \cdot |P_i - 1| = (q-1)^2$, *where $a \in P_i$, $b \in P_j$ and $i \neq j$.*

(3) $S_{a,b} = |A_j - 1| \cdot |P_i - 1| = (q-2) \cdot (q-1)$, *where $a \in P_i$ and $b \in A_j$.*

(4) $S_{a,b} = |B_j - 1| \cdot |P_i - 1| = q \cdot (q-1)$, *where $a \in P_i$ and $b \in B_j$.*

(5) $S_{a,b} = |A_j - 1| \cdot |A_i - 1| = |q-2|^2$, *where $a \in A_i$, $b \in A_j$ and $i \neq j$.*

(6) $S_{a,b} = |B_j - 1| \cdot |A_i - 1| = q \cdot (q-2)$, *where $a \in A_i$ and $b \in B_j$.*

(7) $S_{a,b} = |B_j - 1| \cdot |B_i - 1| = q^2$, *where $a \in B_i$, $b \in B_j$ and $i \neq j$.*

PROOF: Part (1) is trivial and the proofs of the other parts are similar. So We prove only the part (2). Let $a \in P_i$ and $b \in A_j$. By Lemma 3, $n_a(ab) = ((C_G(b) \backslash C_G(a)) \backslash \{b\}) \cup \{a\}$. Now by part 5 of Proposition 7, $C_G(a) = P_i$ and $C_G(b) = A_j$. So $n_a(ab) = ((A_j \backslash P_i) \backslash \{b\}) \cup \{a\}$. By part 4 of Proposition 7, the set $\{P^x, \ A^x, \ B^x \mid x \in G\}$ is a partition of G, hence $n_a(ab) = (A_j - \{1, b\}) \cup \{a\}$. Thus $|n_a(ab)| = |A_j| - 1 = (q-1) - 1$. Likewise, $|n_b(ab)| = |P_i| - 1 = q - 1$. Therefore $S_{a,b} = (q-2) \cdot (q-1)$.

Lemma 9 *Under the assumptions of Lemma 8 and if $F \neq E \in \{P^x, A^x, B^x | x \in G\}$, then*

$$S_{F,E} = (|E| - 1)^2 \cdot (|F| - 1)^2.$$

PROOF: We give the proof only for the case $F = P_i$ and $E = A_j$, the other cases are similar.
Let $a \in P_i$ and $b \in A_j$. By Lemma 3, $n_a(ab) = ((C_G(b) \backslash C_G(a)) \backslash \{b\}) \cup \{a\}$. Now by part 5 of Proposition 7, $C_G(a) = P_i$ and $C_G(b) = A_j$. So $n_a(ab) = ((A_j \backslash P_i) \backslash \{b\}) \cup \{a\}$. By part 4 of Proposition 7, the set

$\{P^x, \ A^x, \ B^x \ | \ x \in G\}$ is a partition of G, hence $n_a(ab) = (A_j - \{1, b\}) \cup \{a\}$. Thus $|n_a(ab)| = |A_j| - 1$. Likewise, $|n_a(ab)| = |P_i| - 1$. Thus $S_{a,b} = (|A_j| - 1) \cdot (|P_i| - 1)$. Therefore

$$\begin{aligned} S_{P_i,A_j} &= \sum_{a \in P_i \setminus \{1\}} \sum_{b \in A_j \setminus \{1\}} S_{a,b} \\ &= \sum_{a \in P_i \setminus \{1\}} \sum_{b \in A_j \setminus \{1\}} (|A_j| - 1) \cdot (|P_i| - 1) \\ &= (|A_j| - 1)^2 \cdot (|P_i| - 1)^2. \end{aligned}$$

Corollary 10 *Under the assumptions of Lemma 8, We have*

(1) $S_{P_i,P_j} = (|P_j| - 1)^2 \cdot (|P_i| - 1)^2$, *where* $1 \leq i \neq j \leq q + 1$.

(2) $S_{P_i,A_j} = (|A_j| - 1)^2 \cdot (|P_i| - 1)^2$, *where* $1 \leq i \leq q + 1$ *and* $1 \leq j \leq \alpha$.

(3) $S_{P_i,B_j} = (|B_j| - 1)^2 \cdot (|P_i| - 1)^2$, *where* $1 \leq i \leq q + 1$ *and* $1 \leq j \leq \beta$.

(4) $S_{A_i,A_j} = (|A_j| - 1)^2 \cdot (|A_i| - 1)^2$, *where* $1 \leq i \neq j \leq \alpha$.

(5) $S_{A_i,B_j} = (|B_j| - 1)^2 \cdot (|A_i| - 1)^2$, *where* $1 \leq i \leq \alpha$ *and* $1 \leq j \leq \beta$.

(6) $S_{B_i,B_j} = (|B_j| - 1)^2 \cdot (|B_i| - 1)^2$, *where* $1 \leq i \neq j \leq \beta$.

PROOF: By using the Lemma 9 and an easy computation the proof is obtained.

Theorem 2 *Let* $G = PSL(2, q)$, *where* $q \equiv 0 \pmod 4$. *Then*

$$Sz(\Gamma_G) = \frac{q(q-1)(q+1)(q^5 - 2q^4 - q^3 + 5q^2 - 5q + 3)}{2}.$$

PROOF: By relation (2) of Definition 1,

$$\begin{aligned} 2Sz(G) &= S_{G,\ G} \\ &= S_{\cup_{i=1}^{q+1} P_i,\ G} + S_{\cup_{i=1}^{\alpha} A_i,\ G} + S_{\cup_{i=1}^{\beta} B_i,\ G} \\ &= \sum_{i=1}^{q+1} S_{P_i,\ G \setminus P_i} + \sum_{i=1}^{\alpha} S_{A_i,\ G \setminus A_i} + \sum_{i=1}^{\beta} S_{B_i,\ G \setminus B_i}. \end{aligned} \tag{3}$$

But, by Lemma 9,

$$\begin{aligned} S_{P_i,\ G \setminus P_i} &= S_{P_i,\ \cup_{j=1}^{q+1} P_j \setminus P_i} + S_{P_i,\ \cup_{j=1}^{\alpha} A_j} + S_{P_i,\ \cup_{j=1}^{\beta} B_j} \\ &= \sum_{j=1,\ j \neq i}^{q+1} S_{P_i,\ P_j} + \sum_{j=1}^{\alpha} S_{P_i,\ A_j} + \sum_{j=1}^{\beta} S_{P_i,\ B_j} \\ &= q \cdot (|P_i| - 1)^4 + \alpha \cdot (|A_i| - 1)^2 (|P_i| - 1)^2 + \\ &\quad \beta \cdot (|B_i| - 1)^2 (|P_i| - 1)^2. \end{aligned}$$

Repeated applications of Lemma 9 enables us to write

$$\begin{aligned} S_{A_i,\, G\setminus A_i} &= (q+1)\cdot(|P_i|-1)^2(|A_i|-1)^2 + \\ &\quad (\alpha-1)\cdot(|A_i|-1)^4 + \beta\cdot(|B_i|-1)^2(|A_i|-1)^2, \\ S_{B_i,\, G\setminus B_i} &= (q+1)\cdot(|P_i|-1)^2(|B_i|-1)^2 + \\ &\quad \alpha\cdot(|A_i|-1)^2(|B_i|-1)^2 + (\beta-1)\cdot(|B_i|-1)^4. \end{aligned}$$

The proof is completed by replacing the above equations in (3).

3 Questions

Let us end the paper by a question and a problem.

(1) Let G and H be two non-abelian groups. Is it true that if $W(\Gamma_G) = W(\Gamma_H)$, then $Sz(\Gamma_G) = Sz(\Gamma_H)$?

(2) Find $Sz(\Gamma_{PSL(2,q)})$ when $q \not\equiv 0 \pmod 4$.

Acknowledgement: The first author wishes to thank the Arak university for financial support. The second author was partially supported by the Iran National Science Foundation (Grant No. 88000140). The authors would like to thank the anonymous referee for his/her helpful comments.

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