A ZETA FUNCTION OF A SEMIREGULAR WEIGHTED BIPARTITE GRAPH

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Abstract

We give determinant expressions of the zeta function and an L-function of a semiregular weighted bipartite graph. As an application, we present a decomposition formula for the weighted complexity of a semiregular weighted bipartite graph.

1 Introduction

Graphs and digraphs treated here are finite.

Let G be a connected graph with a set $V(G)$ of vertices and a set $E(G)$ of edges. Set $D(G) = \{(u,v),(v,u) \mid uv \in E(G)\}$. For $e = (u,v) \in D(G)$, set $u = o(e)$ and $v = t(e)$. Furthermore, let $e^{-1} = (v,u)$ be the *inverse* of $e = (u,v)$.

A *path P of length n* in G is a sequence $P = (e_1, \cdots, e_n)$ of n arcs such that $e_i \in D(G)$, $t(e_i) = o(e_{i+1})(1 \leq i \leq n-1)$. If $e_i = (v_{i-1}, v_i)$ for $i = 1, \cdots, n$, then we can write $P = (v_0, v_1, \cdots, v_{n-1}, v_n)$. Set $\mid P \mid= n$, $o(P) = o(e_1)$ and $t(P) = t(e_n)$. Also, P is called an *$(o(P), t(P))$-path.* We say that a path $P = (e_1, \cdots, e_n)$ has a *backtracking* if $e^{-1}_{i+1} = e_i$ for some $i(1 \leq i \leq n-1)$. A (v,w)-path is called a *v-cycle* (or *v-closed path*) if $v = w$. The *inverse cycle* of a cycle $C = (e_1, \cdots, e_n)$ is the cycle $C^{-1} = (e^{-1}_n, \cdots, e^{-1}_1)$.

We introduce an equivalence relation between cycles. Two cycles $C_1 = (e_1, \cdots, e_m)$ and $C_2 = (f_1, \cdots, f_m)$ are called *equivalent* if $f_j = e_{j+k}$ for all j. The inverse cycle of C is not equivalent to C. Let $[C]$ be the equivalence class which contains a cycle C. Let B^r be the cycle obtained by going r

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times around a cycle B. Such a cycle is called a *multiple* of B. A cycle C is *reduced* if C has no backtracking. Furthermore, a cycle C is *prime* if it is not a multiple of a strictly smaller cycle. Note that each equivalence class of prime, reduced cycles of a graph G corresponds to a unique conjugacy class of the fundamental group $\pi_1(G, v)$ of G at a vertex v of G.

The *(Ihara) zeta function* of a graph G is defined to be a function of a complex variable u with $\mid u \mid$ sufficiently small, by

$$\mathbf{Z}(G, u) = \mathbf{Z}_G(u) = \prod_{[C]} (1 - u^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G, and $\mid C \mid$ is the length of C.

Let G be a connected graph with n vertices $v_1, \cdots, v_n$. The *adjacency matrix* $\mathbf{A}(G) = (a_{ij})$ is the square matrix such that $a_{ij} = \mid \{e \in E(G) \mid e = v_i v_j\} \mid$ if v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. The *degree* $\deg_G v = \deg v$ of a vertex v in G is the number of edges which are adjacent to v. Let $\mathbf{D} = (d_{ij})$ be the diagonal matrix with $d_{ii} = deg_G\ v_i$, and $\mathbf{Q} = \mathbf{D} - \mathbf{I}$.

Ihara [7] defined the Ihara zeta function of a regular graph, and showed that its reciprocal is a polynomial. A zeta function of a regular graph G associated with a unitary representation of the fundamental group of G was developed by Sunada [16,17]. Hashimoto [5] treated multivariable zeta functions of bipartite graphs. Bass [1] generalized Ihara's result on Ihara zeta functions of regular graphs to irregular graphs.

Theorem 1 (Bass) *The reciprocal of the Ihara zeta function of G is given by*

$$\mathbf{Z}_G(u)^{-1} = (1 - u^2)^{l-n} \det(\mathbf{I} - u\mathbf{A}(G) + u^2\mathbf{Q}),$$

where $n = \mid V(G) \mid$ and $l = \mid E(G) \mid$.

Stark and Terras [14] gave an elementary proof of Theorem 1, and discussed three different zeta functions of any graph. Recently, various proofs of Theorem 1 were given by Foata and Zeilberger [3], Kotani and Sunada [8]. Stark and Terras [15], and, independently, Mizuno and Sato [9] obtained a decomposition formula for the Ihara zeta function of a regular covering of a graph.

The *complexity* $\kappa(G)$(= the number of spanning trees in G) of a connected graph G is closely related to the Ihara zeta function of G. Northshield [11] showed that the complexity of G is given by the derivative of a determinant contained in the reciprocal of its Ihara zeta function. For a connected graph G, let $f_G(u) = \det(\mathbf{I} - u\mathbf{A}(G) + u^2\mathbf{Q})$.

Theorem 2 (Northshield) *The complexity of G is given as follows:*

$$f'_G(1) = 2(l-n)\kappa(G),$$

where $n = |V(G)|$ *and* $l = |E(G)|$.

Furthermore, Hashimoto [6] and Northshield [11] gave the value of $(1-u)^{n-m}\mathbf{Z}_G(u)^{-1}$ at $u=1$ in terms of the complexity of G.

Sato [12] defined a new zeta function of a graph by using a determinant (c.f., [1]).

Let G be a connected graph and $V(G) = \{v_1, \cdots, v_n\}$. Then we consider an $n \times n$ matrix $\mathbf{W} = (w_{ij})_{1 \le i,j \le n}$ with ij entry the complex number w_{ij} if $(v_i, v_j) \in D(G)$, and $w_{ij} = 0$ otherwise. The matrix $\mathbf{W} = \mathbf{W}(G)$ is called the *weighted matrix* of G. For each path $P = (v_{i_1}, \cdots, v_{i_r})$ of G, the *norm* $w(P)$ of P is defined as follows: $w(P) = w_{i_1 i_2} w_{i_2 i_3} \cdots w_{i_{r-1} i_r}$. Furthermore, let $w(v_i, v_j) = w_{ij}$, $v_i, v_j \in V(G)$ and $w(e) = w_{ij}, e = (v_i, v_j) \in D(G)$. A weight $w : D(G) \longrightarrow \mathbf{C}$ is a *real symmetric* if $w(e)$ is a real number and $w(e^{-1}) = w(e)$ for each $e \in D(G)$. Then, note that $\mathbf{W}(G)$ is a real symmetric matrix.

Let G be a connected graph with n vertices and m unoriented edges, and $\mathbf{W} = \mathbf{W}(G)$ a weighted matrix of G. Two $2m \times 2m$ matrices $\mathbf{B} = \mathbf{B}(G) = (\mathbf{B}_{e,f})_{e,f \in D(G)}$ and $\mathbf{J}_0 = \mathbf{J}_0(G) = (\mathbf{J}_{e,f})_{e,f \in D(G)}$ are defined as follows:

$$\mathbf{B}_{e,f} = \begin{cases} w(f) & \text{if } t(e) = o(f), \\ 0 & \text{otherwise} \end{cases}, \mathbf{J}_{e,f} = \begin{cases} 1 & \text{if } f = e^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then the *zeta function* of G is defined by

$$\mathbf{Z}_1(G, w, u) = \det(\mathbf{I}_n - u(\mathbf{B} - \mathbf{J}_0))^{-1}.$$

If $w(e) = 1$ for any $e \in D(G)$, then the zeta function of G is the Ihara zeta function of G(see [1,5]).

Sato [12] gave a determinant expression for the zeta function $\mathbf{Z}_1(G, w, u)$ of a graph G. Let $\deg_{G,w}(v) = \deg_w(v) = \sum_{o(e)=v} w(e)$ for each vertex $v \in V(G)$.

Theorem 3 (Sato) *Let G be a connected graph, and let $\mathbf{W} = \mathbf{W}(G)$ be a weighted matrix of G. Then the reciprocal of the zeta function of G is given by*

$$\mathbf{Z}_1(G, w, u)^{-1} = (1-u^2)^{m-n} \det(\mathbf{I}_n - u\mathbf{W}(G) + u^2(\mathbf{D} - \mathbf{I}_n)),$$

where $n = |V(G)|$, $m = |E(G)|$ *and* $\mathbf{D} = (d_{ij})$ *is the diagonal matrix with* $d_{ii} = \deg_w(v_i)$, $V(G) = \{v_1, \cdots, v_n\}$.

In this paper, we consider the zeta function of some bipartite graph with arc weights which are real and symmetric. In Section 2, we give a determinant expression of the zeta function of a semiregular weighted bipartite graph. In Section 3, we present a determinant expression for the L-function of a semiregular weighted bipartite graph. In Section 4, we give a decomposition formula for the weighted complexity of a semiregular weighted bipartite graph. In Section 5, we give a decomposition formula of the zeta function of a regular covering of a semiregular weighted bipartite graph.

A general theory of the representation of groups, the reader is referred to [13].

2 Zeta functions of semiregular weighted bipartite graphs

A graph G is called *bipartite*, denoted by $G = (V_1, V_2)$ if there exists a partition $V(G) = V_1 \cup V_2$ of $V(G)$ such that $uv \in E(G)$ if and only if $u \in V_1$ and $v \in V_2$. A bipartite graph $G = (V_1, V_2)$ is called (q_1+1, q_2+1)-*semiregular* if $\deg_G v = q_i + 1$ for each $v \in V_i (i = 1, 2)$. For a (q_1+1, q_2+1)-semiregular bipartite graph $G = (V_1, V_2)$, let $G^{[i]}$ be the graph with vertex set V_i and edge set $\{P :$ reduced path $\mid\ \mid P \mid = 2; o(P), t(P) \in V_i\}$ for $i = 1, 2$. Then $G^{[1]}$ is $(q_1+1)q_2$-regular, and $G^{[2]}$ is $(q_2+1)q_1$-regular.

Hashimoto [5] treated multivariable zeta functions of bipartite graphs. For a graph G, let $Spec(G)$ be the set of all eigenvalues of the adjacency matrix of G.

Theorem 4 (Hashimoto) *Let $G = (V_1, V_2)$ be a connected (q_1+1, q_2+1)-semiregular bipartite graph with ν vertices and ϵ edges, $\mid V_1 \mid = n$ and $\mid V_2 \mid = m (n \leq m)$. Then*

$$\mathbf{Z}(G,u)^{-1} = (1-u^2)^{\epsilon-\nu}(1+q_2u^2)^{m-n}\prod_{j=1}^{n}(1-(\lambda_j^2-q_1-q_2)u^2+q_1q_2u^4)$$

$$= (1-u^2)^{\epsilon-\nu}(1+q_2u^2)^{m-n}\det(\mathbf{I}_n-(\mathbf{A}^{[1]}-(q_2-1)\mathbf{I}_n)u^2+q_1q_2u^4\mathbf{I}_n)$$

$$= (1-u^2)^{\epsilon-\nu}(1+q_1u^2)^{n-m}\det(\mathbf{I}_m-(\mathbf{A}^{[2]}-(q_1-1)\mathbf{I}_m)u^2+q_1q_2u^4\mathbf{I}_m),$$

where $Spec(G) = \{\pm\lambda_1, \cdots, \pm\lambda_n, 0, \cdots, 0\}$ and $\mathbf{A}^{[i]} = \mathbf{A}(G^{[i]})(i = 1, 2)$.

Let $G = (V_1, V_2)$ be a connected bipartite graph. Then G is called a (q_1+1, q_2+1)-*semiregular weighted bipartite graph* if $\deg_w(v_i) = q_i + 1$ for each $i = 1, 2$ and each $v_i \in V_i$. Then we obtain an analogue of Hashimoto's Theorem. For a square matrix $\mathbf{F}$, let $Spec(\mathbf{F})$ be the set of all eigenvalues of $\mathbf{F}$.

Theorem 5 *Let $G=(V_1,V_2)$ be a connected (q_1+1,q_2+1)-semiregular weighted bipartite graph with ν vertices and ϵ edges, $|V_1|=n$, $|V_2|=m(n\le m)$ and $\mathbf{W}=\mathbf{W}(G)$ a real symmetric weighted matrix of G. Then*

$$\mathbf{Z}_1(G,u)^{-1}=(1-u^2)^{\epsilon-\nu}(1+q_2u^2)^{m-n}\prod_{j=1}^{n}(1-(\lambda_j^2-q_1-q_2)u^2+q_1q_2u^4),$$

where $Spec(\mathbf{W}(G))=\{\pm\lambda_1,\cdots,\pm\lambda_n,0,\cdots,0\}$.

Proof. Similar to the proof of Theorem 4. Q.E.D.

Let G be a connected graph and $\mathbf{W}=\mathbf{W}(G)$ a weighted matrix of G. Then G is called a *r-regular weighted graph* if $\deg_w(v)=r$ for each $v\in V(G)$. By Theorem 3, we obtain the following result.

Corollary 1 *Let G be a connected $(r+1)$-regular weighted graph with ν vertices and ϵ edges, and $\mathbf{W}=\mathbf{W}(G)$ a real symmetric weighted matrix of G. Then*

$$\mathbf{Z}_1(G,u)^{-1}=(1-u^2)^{\epsilon-\nu}\prod_{j=1}^{\nu}(1-\lambda_ju+ru^2),$$

where $Spec(\mathbf{W}(G))=\{\lambda_1,\cdots,\lambda_\nu\}$.

3 L-functions of graphs

Let G be a connected graph with n vertices and l unoriented edges, $\mathbf{W}=\mathbf{W}(G)$ a weighted matrix of G and Γ a finite group. Then a mapping $\alpha: D(G)\longrightarrow\Gamma$ is called an *ordinary voltage assignment* if $\alpha(v,u)=\alpha(u,v)^{-1}$ for each $(u,v)\in D(G)$. For each path $P=(e_1,\cdots,e_r)$ of G, set $\alpha(P)=\alpha(e_1)\cdots\alpha(e_r)$. This is called the *net voltage* of P. Furthermore, let ρ be a unitary representation of Γ and d its degree. The *Kronecker product* $\mathbf{A}\bigotimes\mathbf{B}$ of matrices $\mathbf{A}$ and $\mathbf{B}$ is considered as the matrix $\mathbf{A}$ having the element a_{ij} replaced by the matrix $a_{ij}\mathbf{B}$.

The *L-function* of G associated with ρ and α is defined by

$$\mathbf{Z}_1(G,w,u,\rho,\alpha)=\det(\mathbf{I}_{2ld}-u\sum_{h\in\Gamma}\rho(h)\bigotimes(\mathbf{B}_h-\mathbf{J}_h))^{-1},$$

where, for $g\in\Gamma$, two matrices $\mathbf{B}_g=(b_{ef}^{(g)})$ and $\mathbf{J}_g=(c_{ef}^{(g)})$ are given by

$$b_{ef}^{(g)}:=\begin{cases} w(f) & \text{if } \alpha(e)=g \text{ and } t(e)=o(f),\\ 0 & \text{otherwise.}\end{cases}$$

and

$$c_{ef}^{(g)}:=\begin{cases} 1 & \text{if } \alpha(f)=g \text{ and } e^{-1}=f,\\ 0 & \text{otherwise.}\end{cases}$$

If $\rho = \mathbf{1}$ is the identity representation of Γ, then the L-function of G is the zeta function of G:

$$\mathbf{Z}_1(G, w, u, \rho, \alpha) = \mathbf{Z}_1(G, w, u).$$

Sato [12] gave a determinant expression for the L-function of G associated with ρ and α. Let $1 \leq i, j \leq n$. For $g \in \Gamma$, let the matrix $\mathbf{W}_g = (w_{uv}^{(g)})$ be defined by

$$w_{uv}^{(g)} := \begin{cases} w(u,v) & \text{if } \alpha(u,v) = g \text{ and } (u,v) \in D(G), \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 6 (Sato) *Let G be a connected graph with n vertices and l unoriented edges, $\mathbf{W}(G)$ a weighted matrix of G, Γ a finite group and $\alpha : D(G) \longrightarrow \Gamma$ an ordinary voltage assignment. Furthermore, let ρ a representation of Γ and d the degree of ρ. Then the reciprocal of the L-function of G associated with ρ and α is*

$$\mathbf{Z}_1(G, w, u, \rho, \alpha)^{-1} = (1-u^2)^{(l-n)d} \det(\mathbf{I}_{nd} - u \sum_{h \in \Gamma} \rho(h) \bigotimes \mathbf{W}_h + u^2(\mathbf{I}_d \bigotimes \mathbf{Q})),$$

where $\mathbf{Q} = \mathbf{D} - \mathbf{I}_n$.

By Theorem 6, we obtain the following result.

Theorem 7 *Let $G = (V_1, V_2)$ be a connected $(q_1 + 1, q_2 + 1)$-semiregular weighted bipartite graph with ν vertices and ϵ edges, $\mathbf{W} = \mathbf{W}(G)$ a real symmetric weighted matrix of G, Γ be a finite group and $\alpha : D(G) \longrightarrow \Gamma$ an ordinary voltage assignment. Furthermore, let ρ be a unitary representation of Γ and $d = \deg \rho$. Set $\mid V_1 \mid = n$, $\mid V_2 \mid = m (n \leq m)$ and $\mathbf{W}_\rho = \sum_{g \in \Gamma} \mathbf{W}_g \bigotimes \rho(g)$. Then*

$$\begin{aligned} & \mathbf{Z}_1(G, w, u, \rho, \alpha)^{-1} \\ = \ & (1 - u^2)^{(\epsilon - \nu)d} (1 + q_2 u^2)^{(m-n)d} \prod_{j=1}^{nd} (1 - (\lambda_j^2 - q_1 - q_2) u^2 + q_1 q_2 u^4), \end{aligned}$$

where $Spec(\mathbf{W}_\rho) = \{\pm\lambda_1, \cdots, \pm\lambda_{nd}, 0, \cdots, 0\}$.

Proof. The argument is an analogue of Hashimoto's method [5].
By Theorem 6, we have

$$\begin{aligned} & \mathbf{Z}_1(G, w, u, \rho, \alpha)^{-1} \\ = \ & (1 - u^2)^{(\epsilon - \nu)d} \det(\mathbf{I}_{\nu d} - u \textstyle\sum_{g \in A} \rho(g) \bigotimes \mathbf{W}_g + u^2(\mathbf{I}_d \bigotimes \mathbf{Q})) \\ = \ & (1 - u^2)^{(\epsilon - \nu)d} \det(\mathbf{I}_{\nu d} - u \textstyle\sum_{g \in A} \mathbf{W}_g \bigotimes \rho(g) + u^2 \mathbf{Q} \bigotimes \mathbf{I}_d). \end{aligned}$$

Let $V_1 = \{u_1, \cdots, u_n\}$ and $V_2 = \{v_1, \cdots, v_m\}$. Arrange vertices of G as follows: $u_1, \cdots, u_n; v_1, \cdots, v_m$. We consider the matrix $\mathbf{W}_\rho$ under this order. Then, with the definition, we can see that

$$\mathbf{W}_\rho = \begin{bmatrix} \mathbf{0} & \mathbf{B}_\rho \\ {}^t\bar{\mathbf{B}}_\rho & \mathbf{0} \end{bmatrix}.$$

For $e \in D(G)$, we have $\rho(\alpha(e^{-1}))\rho(\alpha(e)) = \rho(1) = \mathbf{I}_d$. Since ρ is unitary,

$$\rho(\alpha(e^{-1})) = \rho(\alpha(e))^{-1} = {}^t\overline{\rho(\alpha(e))}.$$

Thus, $\mathbf{W}_\rho$ is Hermitian. Therefore, there exists a unitary matrix $\mathbf{U} \in U(md)$ such that

$$\mathbf{B}_\rho \mathbf{U} = \begin{bmatrix} \mathbf{C}_\rho & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mu_1 & & 0 & 0 & \cdots & 0 \\ & \ddots & & \vdots & & \vdots \\ \star & & \mu_{nd} & 0 & \cdots & 0 \end{bmatrix}.$$

Now, let

$$\mathbf{P} = \begin{bmatrix} \mathbf{I}_{nd} & \mathbf{0} \\ \mathbf{0} & \mathbf{U} \end{bmatrix}.$$

Then we have

$${}^t\bar{\mathbf{P}}\mathbf{W}_\rho \mathbf{P} = \begin{bmatrix} \mathbf{0} & \mathbf{C}_\rho & \mathbf{0} \\ {}^t\bar{\mathbf{C}}_\rho & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Furthermore, we have

$${}^t\bar{\mathbf{P}}(\mathbf{Q} \bigotimes \mathbf{I}_d)\mathbf{P} = \mathbf{Q} \bigotimes \mathbf{I}_d.$$

Thus,

$$\mathbf{Z}_1(G, w, u, \rho, \alpha)^{-1} = (1-u^2)^{(\epsilon-\nu)d}(1+q_2u^2)^{(m-n)d}$$

$$\times \quad \det \begin{bmatrix} (1+q_1u^2)\mathbf{I}_{nd} & -u\mathbf{C}_\rho \\ -u\,{}^t\bar{\mathbf{C}}_\rho & (1+q_2u^2)\mathbf{I}_{nd} \end{bmatrix}$$

$$= \quad (1-u^2)^{(\epsilon-\nu)d}(1+q_2u^2)^{(m-n)d}$$

$$\times \det \begin{bmatrix} (1+q_1u^2)\mathbf{I}_{nd} & 0 \\ -u\,{}^t\bar{\mathbf{C}}_\rho & (1+q_2u^2)\mathbf{I}_{nd} - (1+q_1u^2)^{-1}u^2\,{}^t\bar{\mathbf{C}}_\rho \mathbf{C}_\rho \end{bmatrix}$$

$$= (1-u^2)^{(\epsilon-\nu)d}(1+q_2u^2)^{(m-n)d}\det((1+q_1u^2)(1+q_2u^2)\mathbf{I}_{nd} - u^2\,{}^t\bar{\mathbf{C}}_\rho \mathbf{C}_\rho).$$

Since $\mathbf{W}_\rho$ is Hermitian, ${}^t\bar{\mathbf{C}}_\rho\mathbf{C}_\rho$ is Hermitian and positive definite, i.e., the eigenvalues of ${}^t\bar{\mathbf{C}}_\rho\mathbf{C}_\rho$ are of form:

$$\lambda_1^2, \cdots, \lambda_{nd}^2 (\lambda_1, \cdots, \lambda_{nd} > 0).$$

Therefore it follows that

$$\mathbf{Z}_1(G, w, u, \rho, \alpha)^{-1} = (1-u^2)^{(\epsilon-\nu)d}(1+q_2u^2)^{(m-n)d}\prod_{j=1}^{nd}(1-(\lambda_j^2-q_1-q_2)u^2+q_1q_2u^4)$$

But, we have

$$\det(\lambda\mathbf{I} - \mathbf{W}_\rho) = \lambda^{(m-n)d}\det(\lambda^2\mathbf{I} - {}^t\bar{\mathbf{C}}_\rho\mathbf{C}_\rho),$$

and so

$$Spec(\mathbf{W}_\rho) = \{\pm\lambda_1, \cdots, \pm\lambda_{nd}, 0, \cdots, 0\}.$$

Therefore, the result follows. Q.E.D.

In the case that $\rho = \mathbf{1}$(the trivial representation of Γ), we obtain Theorem 5.

4 Weighted complexity of a semiregular weighted bipartite graph

Let G be a connected graph with n vertices and m edges, and $\mathbf{W} = \mathbf{W}(G)$ a weighted matrix of G. Then, let

$$f_G(w, u) = \det(\mathbf{I}_n - u\mathbf{W} + (\mathbf{D} - \mathbf{I}_n)u^2).$$

When $w = 1$, i.e., $w(e) = 1$ for each $e \in D(G)$, Theorem 2 implies that

$$\kappa(G) = \frac{1}{2(m-n)}f'(1,1)$$

if $m \neq n$.

In the case that w is symmetric, ie., $w(e^{-1}) = w(e)$ for each $e \in D(G)$, we consider all spanning arborescences of G rooted at any fixed vertex $v \in V(G)$ (spanning trees of G oriented so that all edges point to v). The sum of the product of weights of all arcs in those spanning arborescences of G are not depended on a vertex v of G. Then this sum is called the *weighted complexity* of G, denoted by $\kappa_w(G)$. Mizuno and Sato [10] showed the following result.

Theorem 8

$$\kappa_w(G) = \frac{1}{2(w(G) - n)} f'(w,1),$$

where $w(G) = \sum_{xy \in E(G)} w(x,y)$.

Furthermore, they presented a formula for the weighted complexity of a regular covering H of G in terms of that of G and a product of determinants over all distinct irreducible representations of the covering transformation group of H.

Next, we present a decomposition formula for the weighted complexity of a semiregular weighted bipartite graph G.

Theorem 9 *Let* $G = (V_1, V_2)$ *be a connected* (q_1+1, q_2+1)*-semiregular weighted bipartite graph with* ν *vertices and* ϵ *edges,* $|V_1| = n$, $|V_2| = m (n \leq m)$ *and* $\mathbf{W}(G)$ *a real symmetric weighted matrix of* G. *Then*

$$\kappa_w(G) = \frac{1}{nq_1 - m}(q_2+1)^{m-n}(q_1q_2 - 1)\prod_{j \neq 1}((q_1+1)(q_2+1) - \lambda_j^2),$$

where $Spec(\mathbf{W}(G)) = \{\pm\lambda_1, \cdots, \pm\lambda_n, 0, \cdots, 0\}$ *and* $\lambda_1 = \sqrt{(q_1+1)(q_2+1)}$.

Proof. Theorem 8 implies that

$$\kappa_w(G) = \frac{f'_G(w,1)}{2(w(G) - \nu)} = \frac{f'_G(w,1)}{2(nq_1 - m)}.$$

By Theorem 5, we have

$$f_G(w,u) = (1 + q_2u^2)^{m-n}\prod_{j=1}^{n}(1 - (\lambda_j^2 - q_1 - q_2)u^2 + q_1q_2u^4),$$

where $Spec(\mathbf{W}(G)) = \{\pm\lambda_1, \cdots, \pm\lambda_n, 0, \cdots, 0\}$. Since

$$f_G(w,1) = \det(\mathbf{I}_\nu - \mathbf{W}(G) + (\mathbf{D} - \mathbf{I}_\nu)) = 0,$$

we have

$$(1 + q_2)^{m-n}\prod_{j=1}^{n}((q_1+1)(q_2+1) - \lambda_j^2) = 0,$$

i.e.,

$$\pm\sqrt{(q_1+1)(q_2+1)} \in Spec(\mathbf{W}(G)).$$

Set $\lambda_1 = \sqrt{(q_1+1)(q_2+1)}$.

But,

$$f'_G(w,u) = (m-n)(1+q_2u^2)^{m-n-1} \cdot 2q_2u \prod_{j=1}^{n}(1-(\lambda_j^2-q_1-q_2)u^2+q_1q_2u^4)$$

$$+(1+q_2u^2)^{m-n}\sum_{i=1}^{n}(-2(\lambda_i^2-q_1-q_2)u+4q_1q_2u^3)\prod_{j\neq i}(1-(\lambda_j^2-q_1-q_2)u^2+q_1q_2u^4).$$

Since $\lambda_1 = \sqrt{(q_1+1)(q_2+1)}$, we have

$$f'_G(w,1) = 2(1+q_2)^{m-n}(q_1q_2-1)\prod_{j\neq 1}((q_1+1)(q_2+1)-\lambda_j^2).$$

Therefore, it follows that

$$\kappa_w(G) = \frac{(1+q_2)^{m-n}}{nq_1-m}(q_1q_2-1)\prod_{j\neq 1}((q_1+1)(q_2+1)-\lambda_j^2).$$

Q.E.D.

From Theorem 9, we obtain a formula for the complexity of a complete bipartite graph $K_{m,m}$(see [2]).

Corollary 2 *Let $K_{m,n}$ be a complete bipartite graph and $2 \leq n \leq m$. Then*

$$\kappa(K_{m,n}) = m^{n-1}n^{m-1}.$$

Proof. At first, let $w(e) = 1$ for each $e \in D(K_{m.n})$. Then $K_{m,n}$ is a (m,n)-semiregular weighted bipartite graph. Let V_1, V_2 be the partite set of $K_{m,n}$, $\mid V_1 \mid= n$ and $\mid V_2 \mid= m$. Then we have $q_1 = m-1$ and $q_2 = n-1$. Furthermore,

$$Spec(K_{m,n}) = \{\pm\sqrt{mn}, 0, \cdots, 0\}.$$

By Theorem 9, it follows that

$$\begin{aligned}\kappa(K_{m,n}) &= \tfrac{1}{n(m-1)-m}n^{m-n}((m-1)(n-1)-1)(mn)^{n-1} \\ &= \tfrac{1}{mn-m-n}m^{n-1}n^{m-1}(mn-m-n) = m^{n-1}n^{m-1}.\end{aligned}$$

Q.E.D.

5 Zeta functions of regular coverings of graphs

Let G be a connected graph, and let $N(v) = \{w \in V(G) \mid (v,w) \in D(G)\}$ denote the neighbourhood of a vertex v in G. A graph H is called a *covering* of G with projection $\pi : H \longrightarrow G$ if there is a surjection $\pi : V(H) \longrightarrow V(G)$ such that $\pi|_{N(v')} : N(v') \longrightarrow N(v)$ is a bijection for all vertices $v \in V(G)$ and $v' \in \pi^{-1}(v)$. When a finite group Π acts on a graph G, the *quotient graph* G/Π is a graph whose vertices are the Π-orbits on $V(G)$, with two vertices adjacent in G/Π if and only if some two of their representatives are adjacent in G. A covering $\pi : H \longrightarrow G$ is said to be *regular* if there is a subgroup B of the automorphism group $AutH$ of H acting freely on H such that the quotient graph H/B is isomorphic to G.

Let G be a graph, Γ a finite group and $\alpha : D(G) \longrightarrow \Gamma$ an ordinary voltage assignment. The pair (G, α) is called an *ordinary voltage graph.* The *derived graph* G^{α} of the ordinary voltage graph (G, α) is defined as follows: $V(G^{\alpha}) = V(G) \times \Gamma$ and $((u,h),(v,k)) \in D(G^{\alpha})$ if and only if $(u,v) \in D(G)$ and $k = h\alpha(u,v)$. The *natural projection* $\pi : G^{\alpha} \longrightarrow G$ is defined by $\pi(u,h) = u$. The graph G^{α} is called a *derived graph covering* of G with voltages in Γ or a Γ*-covering* of G. The natural projection π commutes with the right multiplication action of the $\alpha(e), e \in D(G)$ and the left action of Γ on the fibers: $g(u,h) = (u, gh), g \in \Gamma$, which is free and transitive. Thus, the Γ-covering G^{α} is a $|\Gamma|$-fold regular covering of G with covering transformation group Γ. Furthermore, every regular covering of a graph G is a Γ-covering of G for some group Γ(see [4]).

Let G be a connected graph, Γ a finite group and $\alpha : D(G) \longrightarrow \Gamma$ an ordinary voltage assignment. In the Γ-covering G^{α}, set $v_g = (v,g)$ and $e_g = (e,g)$, where $v \in V(G), e \in D(G), g \in \Gamma$. For $e = (u,v) \in D(G)$, the arc e_g emanates from u_g and terminates at $v_{g\alpha(e)}$. Note that $e_g^{-1} = (e^{-1})_{g\alpha(e)}$.

Let $\mathbf{W} = \mathbf{W}(G)$ be a weighted matrix of G. Then we define the *weighted matrix* $\tilde{\mathbf{W}} = \mathbf{W}(G^{\alpha}) = (\tilde{w}(u_g, v_h))$ of G^{α} *derived from* $\mathbf{W}$ as follows:

$$\tilde{w}(u_g, v_h) := \begin{cases} w(u,v) & \text{if } (u,v) \in D(G) \text{ and } h = g\alpha(u,v), \\ 0 & \text{otherwise.} \end{cases}$$

Sato [12] presented a formula for the zeta function of a regular covering H of G in terms of a product of L-functions of G over all distinct irreducible representations of the covering transformation group of H.

Theorem 10 (Sato) *Let G be a connected graph, $\mathbf{W}(G)$ a weighted matrix of G, Γ a finite group and $\alpha : D(G) \longrightarrow \Gamma$ an ordinary voltage assignment. Then we have*

$$\mathbf{Z}_1(G^{\alpha}, \tilde{w}, u) = \prod_{\rho} \mathbf{Z}_1(G, w, u, \rho, \alpha)^{\deg \rho},$$

where ρ runs over all inequivalent irreducible representations of Γ.

By Theorems 8,10, we obtain the following result.

Corollary 3 *Let $G = (V_1, V_2)$ be a connected (q_1+1, q_2+1)-semiregular weighted bipartite graph with ν vertices and ϵ edges, $\mathbf{W} = \mathbf{W}(G)$ a real symmetric weighted matrix of G, Γ be a finite group and $\alpha : D(G) \longrightarrow \Gamma$ an ordinary voltage assignment. Furthermore, let $\rho_1 = 1, \rho_2, \ldots, \rho_k$ be inequivalent irreducible representations of Γ, and $f_i = \deg \rho_i$ for each $i = 1, \cdots, k$, where $f_1 = 1$. Set $\mid V_1 \mid = n$, $\mid V_2 \mid = m (n \leq m)$ and $\mid \Gamma \mid = r$. Then*

$$\mathbf{Z}_1(G^{\alpha}, \tilde{w}, u)^{-1} = (1-u^2)^{(\epsilon-\nu)r}(1+q_2u^2)^{(m-n)r}$$

$$\times \quad \prod_{i=1}^{k} \prod_{j=1}^{nf_i} (1 - (\lambda_{i,j}^2 - q_1 - q_2)u^2 + q_1q_2u^4),$$

where $Spec(\mathbf{W}_{\rho_i}) = \{\pm\lambda_{i,1}, \cdots, \pm\lambda_{i,nf_i}, 0, \cdots, 0\}(1 \leq i \leq k)$.

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