The spectrum of 4-cycles in 2-factorizations of $K_{n,n}$

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Abstract

A 2-factor of a graph G is a 2-regular spanning subgraph of G and a 2-factorization of a graph G is a 2-factor decomposition of G. A complete solution to the problem of determining the spectrum of 4-cycles in 2-factorizations of the complete bipartite graph is presented.

1 Introduction

A 2-factor of a graph G is a 2-regular spanning subgraph of G. If the graph G is simple then necessarily any 2-factor of G consists of a collection of cycles which partition the vertex set of G. A 2-factorization of G is a collection of edge-disjoint 2-factors of G whose union is G. We use the notation $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ to denote a 2-factorization $\mathcal{F}$ with 2-factors $F_1, F_2, \ldots, F_k$. A graph G is said to be 2-factorable, if there exists a 2-factorization of G. Clearly, for G to possess a 2-factorization it must be regular of even degree.

Recently, some papers investigated the possible number of k-cycles in 2-factorizations of K_n. In 1997, Dejter et al. [3] looked at the problem of con-

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structing 2-factorizations of K_n containing a specified number of 3-cycles. Modulo a few exceptions they gave a complete solution for $n \equiv 1$ or $3 \pmod 6$. In 1998, Dejter et al. [4] gave a complete solution to the problem of constructing 2-factorizations of K_n containing a specified number of 4-cycles, where n is odd and $n \geq 11$:

$$\begin{cases} \{0,1,\ldots,(n-1)(n-5)/8\}, & \text{if } n \equiv 1(mod\ 4); \\ \{0,1,\ldots,(n-1)(n-3)/8\}, & \text{if } n \equiv 3(mod\ 4). \end{cases}$$

In 2000, Peter Adams et al. [1] obtained the solution for the number of 4-cycles in 2-factorizations of $K_n \setminus F$, where F is a 1-factor of K_n and n is even, $n \geq 10$:

$$\begin{cases} \{0,1,\ldots,n(n-2)/8\}, & \text{if } n \equiv 0(mod\ 4); \\ \{0,1,\ldots,(n-2)(n-6)/8\}, & \text{if } n \equiv 2(mod\ 4). \end{cases}$$

Peter Adams et al. [2] obtained the solution for the number of 6-cycles in 2-factorizations of K_n, where n is odd:

$$\begin{cases} \{0,1,\ldots,6k(2k-1)\}, & \text{if } n = 12k+1; \\ \{0,1,\ldots,(6k+1)2k\}, & \text{if } n = 12k+3; \\ \{0,1,\ldots,(6k+2)2k\}, & \text{if } n = 12k+5; \\ \{0,1,\ldots,(6k+3)2k\}, & \text{if } n = 12k+7; \\ \{0,1,\ldots,(6k+4)(2k+1)\}, & \text{if } n = 12k+9; \\ \{0,1,\ldots,(6k+5)(2k+1)\}, & \text{if } n = 12k+11. \end{cases}$$

Selda Küçükçifçi [6] obtained the solution for the number of 8-cycles in 2-factorizations of K_n, where n is odd:

$$\begin{cases} \{0,1,\ldots,8k(2k-1)\}, & \text{if } n = 16k+1; \\ \{0,1,\ldots,(8k+1)2k\}, & \text{if } n = 16k+3; \\ \{0,1,\ldots,(8k+2)2k\}, & \text{if } n = 16k+5; \\ \{0,1,\ldots,(8k+3)2k\}, & \text{if } n = 16k+7; \\ \{0,1,\ldots,8k(2k+1)\}, & \text{if } n = 16k+9; \\ \{0,1,\ldots,(8k+5)(2k+1)\}, & \text{if } n = 16k+11; \\ \{0,1,\ldots,(8k+6)(2k+1)\}, & \text{if } n = 16k+13; \\ \{0,1,\ldots,(8k+7)(2k+1)\}, & \text{if } n = 16k+15. \end{cases}$$

Selda Küçükçifçi [7] obtained the solution for the number of 8-cycles in 2-factorizations of K_{2n}:

$$\begin{cases} \{0,1,\ldots,2k(8k-1)\}, & \text{if } n = 16k; \\ \{0,1,\ldots,8k(2k-1)\}, & \text{if } n = 16k+2; \\ \{0,1,\ldots,2k(8k+1)\}, & \text{if } n = 16k+4; \\ \{0,1,\ldots,2k(8k+2)\}, & \text{if } n = 16k+6; \\ \{0,1,\ldots,(2k+1)(8k+3)\}, & \text{if } n = 16k+8; \\ \{0,1,\ldots,8k(2k+1)\}, & \text{if } n = 16k+10; \\ \{0,1,\ldots,(2k+1)(8k+5)\}, & \text{if } n = 16k+12; \\ \{0,1,\ldots,(2k+1)(8k+6)\}, & \text{if } n = 16k+14. \end{cases}$$

The purpose of this article is to approach the same problem for 4-cycles on the complete bipartite graph $K_{n,n}$. Of course, a 2-factorization of $K_{n,n}$ exists if and only if n is even. In this case, the number of 2-factors is $n/2$ and the maximum number of 4-cycles in a 2-factorization of $K_{n,n}$ is $n^2/4$. When n is odd, the graph $K_{n,n}$ cannot be 2-factorable. However, if we remove a 1-factor from the edge set of $K_{n,n}$, there is a different situation. Therefore, a 2-factorization of $K_{n,n}$ when n is odd is a 2-factorization of $K_{n,n} \setminus F$, where F is a 1-factor of $K_{n,n}$. Since the number of 2-factors is $(n-1)/2$ and each 2-factor must contains at least one cycle of length at least 6, the maximum number of 4-cycles in a 2-factorization of $K_{n,n}$ is $(n-1)(n-3)/4$.

Let $S(n)$ be the set of all k such that there exists a 2-factorization of $K_{n,n}$ containing exactly k 4-cycles. We define

$$FC(n) = \begin{cases} \{0, 1, \ldots, n^2/4 - 2, n^2/4\}, & \text{if } n \text{ is even;} \\ \{0, 1, \ldots, (n-1)(n-3)/4\}, & \text{if } n \text{ is odd.} \end{cases}$$

It is obvious that $S(n) \subseteq FC(n)$. Hence to obtain the results $S(n) = FC(n)$, we need to show that $FC(n) \subseteq S(n)$.

2 $n \equiv 1(mod\ 2)$

In this section, we introduce a construction to count the number of 4-cycles in the 2-factorization of $K_{n,n} \setminus F$ for odd n, where F is a 1-factor of $K_{n,n}$.

A latin square $A = (a_{ij})$ of order n is called *idempotent* if $a_{ii} = i$ for each i. Two idempotent latin squares, $L = (l_{ij})$ and $M = (m_{ij})$, are said to have k entries in common off the main diagonal, if there are exactly k cells (i, j), $i \neq j$, such that $l_{ij} = m_{ij}$. Let $J(n)$ be the set of all integers k such that there exists a pair of idempotent latin squares of the order n which have k entries in common off the main diagonal.

Lemma 2.1 [5], $J(n) = \{0, 1, 2, \ldots, n^2 - n - 6, n^2 - n - 4, n^2 - n\}$, $n \geq 6$, *and* $J(3) = \{6\}, J(4) = \{0, 12\}, J(5) = \{0, 2, 4, 6, 8, 10, 12, 20\}$.

Lemma 2.2 [1]. *If there exists a pair of idempotent latin squares of the order n having x entries in common off the main diagonal, then there exists a 2-factorization of $K_{2n+1,2n+1}$ containing exactly x 4-cycles.*

Now, we give the small case of the 2-factorization of $K_{n,n} \setminus F$ for n being odd. This time, let A and B be the partite sets of $K_{n,n} \setminus F$, where $A = \{1, 2, 3, \ldots, n\}$, $B = \{n+1, n+2, n+3, \ldots, 2n\}$ and $F = \{\{1, n+1\}, \{2, n+2\}, \ldots, \{n, 2n\}\}$.

Lemma 2.3 $S(3) = \{0\}$, $S(5) = \{0, 1\}$, $S(i) = FC(i)$ *for* $i = 7, 9$.

Proof:

$n = 3$. From $K_{3,3} \setminus F = (1,5,3,4,2,6)$, we have $0 \in S(3)$.

$n = 5$. $0 \in S(5)$: $(2,6,3,7,4,8,5,9,1,10)$; $(4,6,5,7,1,8,2,9,3,10)$.

$1 \in S(5)$: $(3,6,4,7),(2,8,5,9,1,10)$; $(2,6,5,7,1,8,4,10,3,9)$.

$2 \notin S(5)$: Suppose that $2 \in S(5)$. Let $\mathcal{F}$ be the 2-factorization of $K_{5,5}$ with 2 C_4, where $\mathcal{F} = \{F_1, F_2\}$. Then F_i must be a 2-factor containing a C_4 and a C_6. Let $\{x_1, x_2, z_1, z_2, z_3\}$ and $\{y_1, y_2, w_1, w_2, w_3\}$ be the partite set of $K_{5,5}$ and $F_1 = (x_1, y_1, x_2, y_2) \cup (z_1, w_1, z_2, w_2, z_3, w_3)$. Case 1, one of the vertices of 4-cycle H in F_2 must be one of $\{x_1, x_2, y_1, y_2\}$. Therefore, there is a subgraph (in fact, it is a path of length 2) of H in $K_{3,3} \setminus (z_1, w_1, z_2, w_2, z_3, w_3)$, where the partite set of $K_{3,3}$ is the set $\{z_1, z_2, z_3\}$ and $\{w_1, w_2, w_3\}$. But the edge set of $K_{3,3} \setminus (z_1, w_1, z_2, w_2, z_3, w_3)$ is the set $\{\{z_1, w_2\}, \{z_2, w_3\}, \{z_3, w_1\}\}$, we have a contradiction. Case 2, there are two vertices of H in the set $\{x_1, x_2, y_1, y_2\}$. They must be $\{x_1, x_2\}$ or $\{y_1, y_2\}$, say $\{x_1, x_2\}$. The other 2 vertices of H must be two of $\{w_1, w_2, w_3\}$, say $\{w_1, w_2\}$. On the graph $K_{5,5} \setminus F_1$, we cannot find any 6-cycle containing the vertices $\{z_1, z_2, z_3, y_1, y_2, w_3\}$.

$n = 7, 9$. By Lemma 2.1 and 2.2, we have $6 \in S(7)$ and $\{0, 12\} \subseteq S(9)$. The remaining data is given in Appendix A. ∎

Lemma 2.4 $\{0, 1, 2, \ldots, t-6, t-4, t\} \subseteq S(n)$ *for* $n \equiv 1 (mod\ 2)$ *and* $n \geq 13$, *where* $t = (n-1)(n-3)/4$.

Proof: Using Lemma 2.1 and Lemma 2.2 completes the proof. ∎

To solve the problem of the missing data of $S(n)$, we need to describe the construction methods referred to as prolongation. Prolongation enables us to produce from a latin square of order n with k cell-disjoint transversals a latin square of order $n+k$ with a subsquare of order k. Let A be a latin square of order n based on the symbols $1, 2, \ldots, n$ with k cell-disjoint transversals $T_1, T_2, \ldots, T_k$. Adding k new rows and k new columns produce a square B of order $n+k$ as follows: if $(i,j) \in T_r$, we put $B_{i,j} = n + r$, $B_{i,n+r} = B_{n+r,j} = A_{i,j}$ and if $(i,j) \notin T_r$, we put $B_{i,j} = A_{i,j}$. Finally, in the remaining subarray of order k, we insert a latin square of order k based on the set $\{n+1, n+2, \ldots, n+k\}$. We say that the transversal T_r has been projected onto $n + r^{th}$ row and $n + r^{th}$ column.

Lemma 2.5 $\{t-1, t-2, t-3, t-5\} \subseteq S(n)$ *for* $n \equiv 1 (mod\ 2)$ *and* $n \geq 17$, *where* $t = (n-1)(n-3)/4$.

Proof: Case 1, $n = 4k+1$ and $k \geq 4$. Let $A = (a_{ij})$ be a matrix of order $2k-3$, where $a_{ij} \equiv 2i - j \pmod{2k-3}$, then A is an idempotent latin square of order $2k-3$ with 3 cell-disjoint transversals T_1, T_2, T_3, where $T_r = \{(i, i+r) \mid i = 1, 2, \ldots, 2k-3\}$ and the entry sum is modulo $2k-3$. Applying prolongation and projecting T_i onto $2k-3+i^{th}$ row and $2k-3+i^{th}$ column, we have a latin square B of order $2k$ with a hole H size 3 based on $\{2k-2, 2k-1, 2k\}$. Let G be a complete bipartite graph with partite sets X and Y, where $X = \{x_i \mid i = 1, 2, 3, \ldots, 4k\} \cup \{x_\infty\}$ and $Y = \{y_i \mid i = 1, 2, 3, \ldots, 4k\} \cup \{y_\infty\}$. Let $O_1(B; i, j)$ be the 4-cycle $(x_i, y_j, x_{i+2k}, y_{j+2k})$ and $O_2(B; i, i)$ be the 6-cycle $(x_i, y_{2k+i}, x_\infty, y_i, x_{i+2k}, y_\infty)$. Set $F_r = \{O_1(B; i, j) \mid i \neq j, B_{ij} = r\} \cup \{O_2(B; r, r)\}$, for $r = 1, 2, \ldots, 2k-3$. Then F_r is a 2-factor of G with $2k-1$ 4-cycles and one 6-cycle. For $r = 2k-2, 2k-1, 2k$, set $F_r = \{O_1(B; i, j) \mid B_{ij} = r\}$. Up to now, we have $4k^2 - 2k - 6$ 4-cycles and $2k-3$ 6-cycles in the almost 2-factorization of G. In fact, the unused edges are the edges of $K_{7,7}$ whose partite set are $\{x_{2k-2}, x_{2k-1}, x_{2k}, x_{4k-2}, x_{4k-1}, x_{4k}, x_\infty\}$ and $\{y_{2k-2}, y_{2k-1}, y_{2k}, y_{4k-2}, y_{4k-1}, y_{4k}, y_\infty\}$. Taking a 2-factorization of $K_{7,7}$ from Lemma 2.3, we obtain the 2-factorization of G. Therefore, we have a 2-factorization of G with $4k^2 - 2k - 6 + s$ 4-cycles, where $s \in S(7)$. Hence $t-1, t-2, t-3, t-5 \in S(n)$, where $t = 4k^2 - 2k$.

Case 2, $n = 4k+3$ and $k \geq 4$. Let $A = (a_{ij})$ be a matrix of order $2k-3$, where $a_{ij} \equiv 2i - j \pmod{2k-3}$, then A is an idempotent latin square of order $2k-3$ with 4 cell-disjoint transversals T_1, T_2, T_3, T_4, where $T_r = \{(i, i+r) \mid i = , 1, 2, \ldots, 2k-3\}$ and the entry sum is modulo $2k-3$. Applying prolongation and projecting T_i onto $2k-3+i^{th}$ row and $2k-3+i^{th}$ column, we have a latin square B of order $2k+1$ with a hole H size 4 based on $\{2k-2, 2k-1, 2k, 2k+1\}$. Let G be a complete bipartite graph with partite sets X and Y, where $X = \{x_i \mid i = 1, 2, 3, \ldots, 4k+2\} \cup \{x_\infty\}$ and $Y = \{y_i \mid i = 1, 2, 3, \ldots, 4k+2\} \cup \{y_\infty\}$. Let $O_1(B; i, j)$ be the 4-cycle $(x_i, y_j, x_{i+2k+1}, y_{j+2k+1})$ and $O_2(B; i, i)$ be the 6-cycle $(x_i, y_{2k+1+i}, x_\infty, y_i, x_{i+2k+1}, y_\infty)$. Set $F_r = \{O_1(B; i, j) \mid i \neq j, B_{ij} = r\} \cup \{O_2(B; r, r)\}$, for $r = 1, 2, \ldots, 2k-3$. Then F_r is a 2-factor of G with $2k$ 4-cycles and one 6-cycle. For $r = 2k-2, 2k-1, 2k, 2k+1$, set $F_r = \{O_1(B; i, j) \mid B_{ij} = r\}$. Up to now, we have $4k^2 + 2k - 12$ 4-cycles in the almost 2-factorization of G. Taking a 2-factorization of $K_{9,9}$, we obtain the 2-factorization of G. Therefore, we have a 2-factorization of G with $4k^2 + 2k - 12 + s$ 4-cycles, where $s \in S(9)$. Hence $t-1, t-2, t-3, t-5 \in S(n)$, where $t = 4k^2 + 2k$. ∎

Lemma 2.6 $S(i) = FC(i)$ *for* $i = 11, 13, 15$.

Proof:

$n = 11$. Consider the idempotent latin square A of order 5 with a subsquare of size 2, where

$$A = \begin{pmatrix} 1 & 4 & 5 & 3 & 2 \\ 5 & 2 & 4 & 1 & 3 \\ 4 & 5 & 3 & 2 & 1 \\ 2 & 3 & 1 & & \\ 3 & 1 & 2 & & \end{pmatrix}$$

Using the same construction of Lemma 2.5, we can obtained an almost 2-factorization of $K_{11,11} \setminus K_{5,5}$ with 3 C_6 and 18 C_4. From the results $S(5) = \{0, 1\}$, we have $18, 19 \in S(11)$ Using Lemma 2.1 and 2.2, we have $0, 2, 4, 6, 8, 10, 12, 20 \in S(11)$. The remaining data is given in Appendix A.

$n = 13$. By the above same method, we consider the idempotent latin square B of order 6 with a hole size 2, where

$$B = \begin{pmatrix} 1 & 5 & 6 & 3 & 4 & 2 \\ 5 & 2 & 4 & 6 & 3 & 1 \\ 6 & 1 & 3 & 5 & 2 & 4 \\ 2 & 6 & 5 & 4 & 1 & 3 \\ 3 & 4 & 1 & 2 & & \\ 4 & 3 & 2 & 1 & & \end{pmatrix}$$

We obtain an almost 2-factorization of $K_{13,13} \setminus K_{5,5}$ with 4 C_6 and 28 C_4. From $S(5) = \{0, 1\}$, we have $28, 29 \in S(13)$. For the idempotent latin square C of order 6, where

$$C = \begin{pmatrix} 1 & 6 & 5 & 3 & 4 & 2 \\ 4 & 2 & 1 & 6 & 3 & 5 \\ 2 & 5 & 3 & 1 & 6 & 4 \\ 5 & 1 & 6 & 4 & 2 & 3 \\ 6 & 3 & 4 & 2 & 5 & 1 \\ 3 & 4 & 2 & 5 & 1 & 6 \end{pmatrix},$$

we have a 2-factorization $\mathcal{F}$ of $K_{13,13} \setminus F$, where $\mathcal{F} = \{F_1, F_2, \ldots, F_6\}$ and $F_r = \{O_1(C; i, j) \mid i \neq j, C_{ij} = r\} \cup \{O_2(C; r, r)\}$. $F_1 = \{(x_2, y_3, x_8, y_9)$, (x_3, y_4, x_9, y_{10}), (x_4, y_2, x_{10}, y_8), $(x_5, y_6, x_{11}, y_{12})$, $(x_6, y_5, x_{12}, y_{11})$, $(x_1, y_7, x_\infty, y_1, x_7, y_\infty\}$. Interchanging the edges (x_2, y_3), (x_3, y_4), (x_4, y_2) from the edge set of F_1 and (x_2, y_2), (x_3, y_3), (x_4, y_4) from F, we obtain a new F_1', where $F_1' = \{(x_2, y_2, x_{10}, y_8, x_4, y_4, x_9, y_{10}, x_3, y_3, x_8, y_9)$, $(x_5, y_6, x_{11}, y_{12})$, $(x_6, y_5, x_{12}, y_{11})$, $(x_1, y_7, x_\infty, y_1, x_7, y_\infty\}$. From the 2-factorization $\{F_1', F_2, \ldots, F_6\}$, we have $27 \in S(13)$. Interchanging the edges (x_5, y_6), (x_6, y_5) from the edge set of F_1' and (x_5, y_5), (x_6, y_6) from F, we obtain a new

F_1'', where $F_1'' = \{(x_2, y_2, x_{10}, y_8, x_4, y_4, x_9, y_{10}, x_3, y_3, x_8, y_9), (x_5, y_5, x_{12}, y_{11}, x_6, y_6, x_{11}, y_{12}), (x_1, y_7, x_\infty, y_1, x_7, y_\infty\}$. From the 2-factorization $\{F_1'', F_2, \ldots, F_6\}$, we have $25 \in S(13)$. Combining these results and Lemma 2.4, we have $S(13) = FC(13)$.

$n = 15$. For the idempotent latin square D of order 7 with a hole size 3, where

$$D = \begin{pmatrix} 1 & 5 & 6 & 7 & 4 & 2 & 3 \\ 5 & 2 & 7 & 6 & 3 & 1 & 4 \\ 6 & 7 & 3 & 5 & 2 & 4 & 1 \\ 7 & 6 & 5 & 4 & 1 & 3 & 2 \\ 3 & 4 & 1 & 2 & & & \\ 4 & 3 & 2 & 1 & & & \\ 2 & 1 & 4 & 3 & & & \end{pmatrix}$$

We obtain an almost 2-factorization of $K_{15,15} \setminus K_{7,7}$ with 4 C_6 and 36 C_4. From $S(7) = FC(7)$, we have $37, 39, 40, 41 \in S(15)$. Combining these results and Lemma 2.4, we have $S(15) = FC(15)$.∎

From Lemma 2.3, 2.4, 2.5 and 2.6, we obtain the following theorem.

Theorem 2.7 *$S(3) = \{0\}$, $S(5) = \{0, 1\}$ and $S(n) = FC(n)$ for odd n, $n \geq 7$.*

3 $n \equiv 0 (mod\ 2)$

We can now give the recursive method to count the number of 4-cycles of 2-factorization for the complete bipartite graph $K_{n,n}$, for even n.

Let A and B be two sets of integers. We define $A+B = \{a+b \mid a \in A, b \in B\}$.

Lemma 3.1 *If $S(2k) = FC(2k)$, then $S(4k) = FC(4k)$ for all $k \geq 5$.*

Proof. Let $X = A \cup B$ and $Y = C \cup D$ be the partite sets of $K_{4k,4k}$, where $|A| = |B| = |C| = |D| = 2k$. Consider two complete bipartite graphs $K_{2k,2k}$, one with partite sets $A \cup C$ and another with $B \cup D$. Combining two 2-factors on two graphs $K_{2k,2k}$, we have a 2-factor of $K_{4k,4k}$. Therefore, we can obtain k 2-factors of $K_{4k,4k}$. Similarly, consider two complete bipartite graphs $K_{2k,2k}$, one with partite sets $A \cup D$ and another with $B \cup C$. Thus, we obtain another k 2-factors of $K_{4k,4k}$.

Those $2k$ 2-factors of $K_{4k,4k}$ form a 2-factorization of $K_{4k,4k}$. Let k_1, k_2, k_3 and k_4 be the number of 4-cycles in the 2-factorizations of $K_{2k,2k}$ with partite sets $A \cup C$, $B \cup D$, $A \cup D$ and $B \cup C$, respectively. Then $k_1 + k_2 + k_3 + k_4$ is the number of 4-cycles in this 2-factorizations of $K_{4k,4k}$. Therefore,

$$S(4k) \supseteq \underbrace{S(2k)}_{A,C} + \underbrace{S(2k)}_{B,D} + \underbrace{S(2k)}_{A,D} + \underbrace{S(2k)}_{B,C}$$

Since $S(2k) = FC(2k)$, we have

$$S(4k) \supseteq FC(2k) + FC(2k) + FC(2k) + FC(2k) = FC(4k).$$

This implies that $S(4k) = FC(4k)$. ∎

Lemma 3.2 *If $S(2k) = FC(2k)$ and $S(2k+1) = FC(2k+1)$, then $S(4k+2) = FC(4k+2)$ for all $k \geq 6$.*

Proof: Let $A \cup B$ and $C \cup D$ be the partite sets of $K_{4k+2,4k+2}$, where $A = \{x_1, x_2, \ldots, x_{2k+1}\}$, $B = \{y_1, y_2, \ldots, y_{2k+1}\}$, $C = \{z_1, z_2, \ldots, z_{2k+1}\}$ and $D = \{w_1, w_2, \ldots, w_{2k+1}\}$. Consider two complete bipartite graphs $K_{2k+1,2k+1}$ embedded in the graph $K_{4k+2,4k+2}$, where A and C are the partite sets of the first graph; and B and D are the partite sets of second graph. When we remove a 1-factor, the remaining edges of $K_{2k+1,2k+1}$ can be partitioned into 2-factors. Combining two 2-factors on two graphs $K_{2k+1,2k+1}$ produces a 2-factor of $K_{4k+2,4k+2}$. Thus we obtain k 2-factors and one 1-factors F_1 of $K_{4k+2,4k+2}$, where $F_1 = \{\{x_i, z_i\}, \{y_i, w_i\} \mid i = 1, 2, \ldots, 2k+1\}$. Similarly, consider two complete bipartite graphs $K_{2k+1,2k+1}$ embedded in the graph $K_{4k+2,4k+2}$, where A and D are the partite sets of the first graph; and B and C are the partite sets of second graph. We have k 2-factors and one 1-factors F_2 of $K_{4k+2,4k+2}$, where $F_2 = \{\{x_i, w_i\}, \{y_i, z_i\} \mid i = 1, 2, \ldots, 2k+1\}$ or $F_2 = \{\{x_i, w_i\}, \{y_i, z_{i+1}\} \mid i = 1, 2, \ldots, 2k+1\}$. Combining the 1-factors F_1 and F_2, we have a 2-factor of $K_{4k+2,4k+2}$ with $2k+1$ 4-cycles $\{(x_i, w_i, y_i, z_i) \mid i = 1, 2, \ldots, 2k+1\}$ or a 8k+4-cycle $(x_1, w_1, y_1, z_2, x_2, w_2, y_2, z_3, \ldots, x_i, w_i, y_i, z_{i+1}, \ldots, x_{2k+1}, w_{2k+1}, y_{2k+1}, z_1)$.

The $2k+1$ 2-factors of $K_{4k+2,4k+2}$ form a 2-factorization of $K_{4k+2,4k+2}$. Let k_1, k_2, k_3 and k_4 be the number of 4-cycles in the 2-factorizations of $K_{2k+1,2k+1}$ with partite sets $A \cup C$, $B \cup D$, $A \cup D$ and $B \cup C$, respectively. Then $k_1 + k_2 + k_3 + k_4 + s$ is the number of 4-cycles in the 2-factorizations of $K_{4k+2,4k+2}$, where $s = 0$ or $2k+1$. Since $S(2k+1) = FC(2k+1)$, we have $S(4k+2) \supseteq \{k_1 + k_2 + k_3 + k_4 + s \mid k_1, k_2, k_3, k_4 \in FC(2k+1), s \in \{0, 2k+1\}\} = \{0, 1, 2, \ldots, 4k^2 - 2k + 1\}$.

We now use another construction for the remaining data. Let $A \cup B$ and $C \cup D$ be the partite sets of $K_{4k+2,4k+2}$, where $A = \{x_1, x_2, \ldots, x_{2k}\}$, $B = \{y_1, y_2, \ldots, y_{2k+2}\}$, $C = \{z_1, z_2, \ldots, z_{2k}\}$ and $D = \{w_1, w_2, \ldots, w_{2k+2}\}$. Consider two complete bipartite graphs $K_{2k,2k}$ and $K_{2k+2,2k+2}$ with partite sets

$A \cup C$ and $B \cup D$, respectively. Let $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$ be a 2-factorization of $K_{2k,2k}$ with partite sets A and C. Given an unipotent quasigroup, $(X, \circ)$, of order $k+1$ with diagonal entries $k+1$. For $1 \le a \le k+1$, we define $F_a = \{C(y_i, w_j) \mid i \circ j = a, 1 \le i, j \le k+1\}$, where $C(y_i, w_j) = (y_{2i-1}, w_{2j}, y_{2i}, w_{2j-1})$ is a 4-cycle. Then, F_a is a 2-factor of $K_{2k+2,2k+2}$. Combining the 2-factors on graphs $K_{2k,2k}$ and $K_{2k+2,2k+2}$, we obtain k 2-factors of $K_{4k+2,4k+2}$ (i.e. $G_i \cup F_i, i = 1, 2, \ldots, k$) and a 2-factor, F_{k+1}, of $K_{2k+2,2k+2}$. In fact, $F_{k+1} = \{(y_{2i-1}, w_{2i}, y_{2i}, w_{2i-1}) \mid i = 1, 2, \ldots, k+1\}$.

For the other $k+1$ 2-factors of $K_{4k+2,4k+2}$, we construct it by using the two complete bipartite graphs $K_{2k,2k+2}$ with partite sets $A \cup D$ and $C \cup B$ combining the 4-cycles in F_{k+1} as follows.

Let $(Y, \circ)$ be a quasigroup of order $k+1$ with $(k+1) \circ i = i$. For each $a \in Y$, set $H_a = C(y_a, w_a) \cup \{C(z_i, y_j), C(x_i, w_j) \mid i \circ j = a, i, j \in Y, i \ne k+1\}$. Then H_a is a 2-factor of $K_{4k+2,4k+2}$, for $a = 1, 2, \ldots, k+1$. By this construction, we have $S(4k+2) \supseteq \underbrace{S(2k)}_{A,C} + \underbrace{(k+1)^2}_{B,D} + \underbrace{k(k+1)}_{A,D} + \underbrace{k(k+1)}_{B,C}$
$= \{3k^2 + 4k + 1, 3k^2 + 4k + 2, \ldots, 4k^2 + 4k - 1, 4k^2 + 4k + 1\}$.

From $4k^2 - 2k + 1 \ge 3k^2 + 4k + 1$, we have $k \ge 6$. Hence $S(4k+2) = FC(4k+2)$ for $k \ge 6$. ▮

We now give the small case of the 2-factorization of $K_{n,n}$ for even n. In this time, let A and B be the partite sets of $K_{n,n}$, where $A = \{1, 2, 3, \ldots, n\}$ and $B = \{n+1, n+2, n+3, \ldots, 2n\}$.

Lemma 3.3 *$S(4) = FC(4) \setminus \{1, 2\}$, $S(6) = FC(6) \setminus \{4, 6, 7\}$, $S(8) = FC(4) \setminus \{14\}$ and $S(n) = FC(n)$, $n = 10, 12, 14, 16, 18, 22$.*

Proof:

$n = 4$. $0 \in S(4)$: (1, 5, 2, 6, 3, 7, 4, 8); (3, 5, 4, 6, 1, 7, 2, 8).

$4 \in S(4)$: (1, 5, 2, 6), (3, 7, 4, 8); (3, 5, 4, 6), (1, 7, 2, 8).

$1 \notin S(4)$: If $1 \in S(4)$, then there is one 2-factor (with 8 edges) containing the 4-cycle. The remaining edges of the 2-factor must be a 4-cycle. However, this is a contradiction.

$2 \notin S(4)$: If $2 \in Q(4)$, then the 4-cycles are contained in the same 2-factor or are different 2-factors. The remaining edges must be two 4-cycles. However, this is a contradiction.

From the above, we conclude that $S(4) = FC(4) \setminus \{1, 2\}$.

$n = 6$. $4, 6, 7 \notin S(6)$: Suppose that $4, 6, 7 \in S(6)$. Let $\mathcal{F}$ be the 2-factorization of $K_{6,6}$ with 4, 6 or 7 4-cycles, where $\mathcal{F} = \{F_1, F_2, F_3\}$. Since a 2-factor with two C_4 implies that the remaining edges form a C_4, we can only consider the following cases. Let the number of 4-cycles in each 2-factor be $\{3,1,0\}$, $\{3,3,0\}$ or $\{3,3,1\}$. Let the bipartite set of $K_{6,6}$ be $\{x_1,x_2,x_3,x_4,x_5,x_6\} \cup \{y_1,y_2,y_3,y_4,y_5,y_6\}$ and the 4-cycles of F_1 denoted by (x_1,y_1,x_2,y_2), (x_3,y_3,x_4,y_4) and (x_5,y_5,x_6,y_6). Taking any one 4-cycle in F_2, there is a correspondence 4-cycle in F_3. When we take another 4-cycle in F_2, the results of F_2 and F_3 must be three 4-cycles. Therefore, we have $4,6,7 \notin S(6)$. According to the remaining data in Appendix B, we conclude that $S(6) = FC(6) \setminus \{4,6,7\}$.

$n = 8$. Using the same construction of Lemma 3.1 and the result $S(4) = \{0,4\}$, we obtain $0,4,8,12,16 \in S(8)$. From data in Appendix B, we have $FC(8) \setminus \{14\} \subseteq S(8)$. As for $14 \notin S(8)$, it can be obtained by using exhaustive computing checking. Therefore, we conclude that $S(8) = FC(8) \setminus \{14\}$.

$n = 10$. Using the method in the proof of the first part of Lemma 3.2 with $S(5) = \{0,1\}$, we can obtain $0,1,2,\ldots,9 \in S(10)$. Using the proof in the second part of Lemma 3.2 with $S(4) = \{0,4\}$, we have $21,25 \in S(10)$.

Furthermore, using the modified method in the second construction of Lemma 3.2 by replacing 2-factorization $\mathcal{F}$ of $K_{6,6}$ by 2-factorization $\mathcal{F}$ with 3 or 5 C_4, we can obtain $15,17,19 \in S(10)$. According to the remaining data in Appendix B, we conclude that $S(10) = FC(10)$.

$n = 12$. Using the same construction of Lemma 3.1 and the result $S(6) = \{0,1,2,3,5,9\}$, we can obtain $S(12) \supseteq \{0,1,2,\ldots,30, 32, 36\}$. According to the remaining data in Appendix B, we conclude that $S(12) = FC(12)$.

$n = 14$. Using the method in the proof of Lemma 3.2, we have $0,1,\ldots,31 \in S(14)$ and $40,41,42,43,45,49 \in S(14)$.

Again, using the modified method in the second construction of Lemma 3.2 by replacing 2-factorization $\mathcal{F}$ of $K_{8,8}$ by 2-factorization $\mathcal{F}$ with 8 or 12 C_4, we can obtain $32,33,34,35,36,37,38,39 \in S(14)$. According to the remaining data in Appendix B, we conclude that $S(14) = FC(14)$.

$n = 16$. Using the proof in Lemma 3.1, we have $0,1,2,\ldots,61,64 \in S(16)$. According to the remaining data in Appendix B, we conclude that $S(16) = FC(16)$.

$n = 18$. Using the method in the proof of Lemma 3.2, we have $0,1,2,\ldots,57 \in S(18)$ and $65,66,67,\ldots,78,81 \in S(18)$.

Using the modified method in the second construction of Lemma 3.2 by replacing 2-factorization $\mathcal{F}$ of $K_{10,10}$ by 2-factorization $\mathcal{F}$ with 17 C_4, we can obtain $58,59,\ldots,64 \in S(18)$. Replacing $\mathcal{F}$ by the 2-factorization of $K_{10,10}$ containing 23 C_4 and the 2-factorization of $K_{8,8}$ with 16 C_4, we can obtain $79 \in S(18)$. Therefore, we have $S(18) = FC(18)$.

$n = 22$. Using the method in the proof of Lemma 3.2, we have $0, 1, 2, \dots, 91 \in S(22)$ and $96, 97, 98, \dots, 119, 121 \in S(22)$.

Again, using the modified method in the second construction of Lemma 3.2 by replacing 2-factorization $\mathcal{F}$ of $K_{12,12}$ by 2-factorization $\mathcal{F}$ with 32 C_4, we can obtain $92, 93, 94, \dots, 115, 117 \in S(22)$. Therefore, we have $S(22) = FC(22)$.

Applying the small cases to Lemma 3.1, and 3.2 recursively, we obtained the following results:

Theorem 3.4 *$S(4) = \{0, 4\}$, $S(6) = FC(6)\backslash\{4, 6, 7\}$, $S(8) = FC(8)\backslash\{11, 13, 14\}$ and $S(n) = FC(n)$ for even n, $n \geq 10$.*

4 Summary

From Theorems 2.7 and 3.4, we obtained the following Main Theorem.

Main Theorem $S(n) = FC(n)$ for even n and $n \geq 10$; odd n and $n \geq 7$.

Appendix A

$n = 7$.

$0 \in S(7)$: (1, 9, 3, 11, 5, 13, 7, 8, 2, 10, 4, 12, 6, 14); (1, 10, 5, 14, 2, 11, 6, 8, 3, 12, 7, 9, 4, 13); (1, 11, 7, 10, 6, 9, 5, 8, 4, 14, 3, 13, 2, 12).

$1 \in S(7)$: (1, 13, 2, 14), (3, 8, 4, 9, 5, 10, 6, 11, 7, 12); (5, 8, 6, 9, 7, 10, 1, 11, 2, 12, 4, 13, 3, 14); (2, 8, 7, 13, 5, 11, 3, 9, 1, 12, 6, 14, 4, 10).

$2 \in S(7)$: (1, 13, 2, 14), (6, 8, 7, 9), (4, 10, 5, 11, 3, 12); (3, 8, 4, 9, 5, 13, 7, 10, 2, 11, 1, 12, 6, 14); (2, 8, 5, 14, 4, 13, 3, 9, 1, 10, 6, 11, 7, 12).

$3 \in S(7)$: (1, 13, 2, 14), (3, 8, 4, 9, 5, 10, 6, 11, 7, 12); (3, 13, 4, 14), (5, 8, 6, 9, 7, 10, 1, 12, 2, 11); (1, 9, 3, 11), (2, 8, 7, 13, 5, 14, 6, 12, 4).

$4 \in S(7)$: (1, 13, 2, 14), (6, 8, 7, 9), (4, 10, 5, 11, 3, 12); (3, 8, 4, 9), (1, 10, 7, 13, 5, 14, 6, 12, 2, 11); (3, 13, 4, 14), (2, 8, 5, 9, 1, 12, 7, 11, 6, 10).

$5 \in S(7)$: (1, 9, 3, 13), (6, 11, 7, 12), (2, 8, 5, 14, 4, 10); (4, 9, 5, 13), (6, 8, 7, 10), (1, 12, 3, 11, 2, 14); (1, 11, 5, 10), (2, 12, 4, 8, 3, 14, 6, 9, 7, 13).

$n = 9$.

$1 \in S(9)$: (8, 10, 9, 11)(12, 1, 13, 2, 14, 3, 15, 4, 16, 5, 17, 6, 18, 7); (1, 14, 9, 15, 2, 16)(4, 10, 5, 11, 6, 12, 8, 13, 7, 17, 3, 18); (2, 10, 3, 11, 4, 12, 5, 13, 6, 14, 7, 15, 8, 16, 9, 17, 1, 18); (6, 10, 7, 11, 1, 15, 5, 18, 8, 14, 4, 17, 2, 12, 9, 13, 3, 16).

$2 \in S(9)$: (8, 15, 9, 16), (1, 17, 2, 18), (3, 10, 4, 11, 5, 12, 6, 13, 7, 14); (5, 10, 6, 11, 7, 12, 8, 13, 9, 14, 1, 15, 2, 16, 3, 17, 4, 18); (7, 10, 8, 11, 9, 12, 1, 13, 2, 14, 6, 17, 5, 16, 4, 15, 3, 18); (2, 10, 9, 17, 7, 15, 5, 13, 3, 11, 1, 16, 6, 18, 8, 14, 4, 12).

$3 \in S(9)$: (8, 10, 9, 11)(12, 1, 13, 2, 14, 3, 15, 4, 16, 5, 17, 6, 18, 7); (1, 17, 2, 18)(3, 10, 4, 11, 5, 12, 6, 13, 7, 14, 8, 15, 9, 16); (3, 13, 9, 17)(6, 10, 7, 11, 1, 14, 4, 12, 8, 18, 5, 15, 2, 16); (2, 10, 5, 13, 8, 16, 1, 15, 7, 17, 4, 18, 3, 11, 6, 14, 9, 12).

Let $A = \{$(8, 15, 9, 16), (1, 17, 2, 18), (3, 10, 4, 11, 5, 12, 6, 13, 7, 14); (2, 15, 3, 16), (4, 17, 5, 18), (6, 10, 7, 11, 8, 12, 9, 13, 1, 14)$\}$ and $B = \{$(6, 13, 7, 14), (8, 15, 9, 16), (4, 17, 5, 18), (2, 10, 3, 11, 1, 12); (6, 10, 7, 11), (4, 12, 5, 15), (8, 13, 9, 14), (2, 16, 3, 17, 1, 18)$\}$.

$4 \in S(9) : A \cup \{$(5, 10, 9, 11, 1, 12, 2, 13, 3, 17, 7, 15, 4, 14, 8, 18, 6, 16); (2, 10, 8, 13, 5, 15, 1, 16, 4, 12, 7, 18, 3, 11, 6, 17, 9, 14)$\}$.

$5 \in S(9) : A \cup \{$(4, 12, 7, 15), (2, 10, 5, 13, 8, 18, 6, 16, 1, 11, 3, 17, 9, 14); (8, 10, 9, 11, 6, 17, 7, 18, 3, 13, 2, 12, 1, 15, 5, 16, 4, 14)$\}$.

$6 \in S(9) : B \cup \{$(1, 13, 2, 14, 3, 15), (4, 10, 5, 11, 8, 12, 9, 17, 7, 18, 6, 16); (6, 12, 7, 15, 2, 17), (8, 10, 9, 11, 4, 14, 1, 16, 5, 13, 3, 18)$\}$.

$7 \in S(9) : B \cup \{$(2, 13, 3, 15), (4, 10, 5, 11, 8, 12, 9, 17, 7, 18, 6, 16, 1, 14); (8, 10, 9, 11, 4, 16, 5, 13, 1, 15, 7, 12, 6, 17, 2, 14, 3, 18)$\}$.

$8 \in S(9) : B \cup \{$(4, 10, 5, 16), (8, 11, 9, 12), (1, 13, 3, 14, 2, 17, 6, 18, 7, 15); (8, 10, 9, 17, 7, 12, 6, 16, 1, 14, 4, 11, 5, 13, 2, 15, 3, 18)$\}$.

$9 \in S(9)$: (3, 10, 4, 11), (5, 12, 6, 13), (1, 14, 2, 15), (8, 16, 9, 17, 7, 18); (5, 10, 6, 11), (7, 14, 8, 15), (1, 16, 2, 17), (4, 12, 9, 13, 3, 18); (7, 10, 8, 11), (1, 12, 2, 13), (6, 14, 9, 15, 3, 16, 4, 17, 5, 18); (7, 12, 8, 13), (2, 10, 9, 11, 1, 18), (3, 14, 4, 15, 5, 16, 6, 17).

$10 \in S(9)$: (1, 13, 8, 14), (4, 15, 9, 16), (2, 17, 3, 18), (6, 10, 7, 11, 5, 12); (7, 13, 9, 14), (1, 15, 8, 16), (5, 17, 6, 18), (2, 10, 3, 11, 4, 12); (8, 11, 9, 12), (2, 15, 3, 16), (1, 17, 7, 18), (4, 10, 5, 13, 6, 14); (2, 13, 3, 14), (8, 10, 9, 17, 4, 18)(1, 11, 6, 16, 5, 15, 7, 12).

$11 \in S(9)$: (4, 10, 6, 11), (1, 12, 5, 13), (2, 14, 3, 15), (8, 16, 9, 17, 7, 18); (7, 10, 8, 11), (2, 13, 3, 16), (1, 17, 5, 18), (4, 12, 6, 14, 9, 15); (7, 13, 8, 14), (1, 15, 5, 16), (4, 17, 6, 18), (2, 10, 3, 11, 9, 12); (2, 17, 3, 18), (7, 12, 8, 15), (5, 10, 9, 13, 6, 16, 4, 14, 1, 12).

$n = 11$.

Let $A = \{$(1, 21, 2, 22), (3, 12, 4, 13, 5, 14, 6, 15, 7, 16, 8, 17, 9, 18, 10, 19, 11, 20)}, $B = \{(6, 15, 7, 16), (4, 17, 5, 18), (10, 19, 11, 20), (8, 21, 9, 22), (2, 12, 3, 13, 1, 14); (3, 21, 4, 22), (5, 12, 6, 13, 7, 14, 8, 15, 9, 16, 10, 17, 11, 18, 1, 19, 2, 20)\}$, $C = \{(6, 15, 7, 16), (4, 17, 5, 18), (10, 19, 11, 20), (8, 21, 9, 22), (2, 12, 3, 13, 1, 14); (10, 12, 11, 13), (8, 17, 9, 18), (6, 19, 7, 20), (1, 21, 2, 22), (4, 14, 5, 15, 3, 16)\}$.

$1 \in S(11) : A \cup \{(5, 12, 6, 13, 7, 14, 8, 15, 9, 16, 10, 17, 11, 18, 1, 19, 2, 20, 4, 21, 3, 22); (7, 12, 8, 13, 9, 14, 10, 15, 11, 16, 1, 17, 2, 18, 3, 19, 4, 22, 6, 21, 5, 20); (9, 12, 10, 13, 11, 14, 1, 15, 2, 16, 3, 17, 4, 18, 5, 19, 6, 20, 8, 21, 7, 22); (2, 12, 11, 21, 9, 19, 7, 17, 5, 15, 3, 13, 1, 20, 10, 22, 8, 18, 6, 16, 4, 14)\}$.

$3 \in S(11) : A \cup \{(3, 21, 4, 22), (5, 12, 6, 13, 7, 14, 8, 15, 9, 16, 10, 17, 11, 18, 1, 19, 2, 20); (5, 21, 6, 22), (7, 12, 8, 13, 9, 14, 10, 15, 11, 16, 1, 17, 2, 18, 3, 19, 4, 20); (9, 12, 10, 13, 11, 14, 1, 15, 2, 16, 3, 17, 4, 18, 5, 19, 6, 20, 8, 21, 7, 22); (2, 12, 11, 21, 9, 19, 7, 17, 5, 15, 3, 13, 1, 20, 10, 22, 8, 18, 6, 16, 4, 14)\}$.

$5 \in S(11) : B \cup \{(4, 12, 9, 18, 8, 16), (5, 13, 10, 22, 6, 14, 11, 21, 2, 15, 1, 20, 3, 17, 7, 19); (7, 12, 8, 13, 9, 14, 10, 15, 11, 16, 1, 17, 2, 18, 3, 19, 4, 20, 6, 21, 5, 22); (10, 12, 11, 13, 4, 14, 5, 15, 3, 16, 2, 22, 1, 21, 7, 20, 8, 17, 9, 19, 6, 18)\}$.

$7 \in S(11) : B \cup \{(5, 21, 6, 22), (7, 12, 8, 13, 9, 14, 10, 15, 11, 16, 1, 17, 2, 18, 3, 19, 4, 20); (1, 21, 2, 22), (9, 12, 10, 13, 11, 14, 4, 16, 3, 15, 5, 19, 7, 17, 8, 20, 6, 18); (4, 12, 11, 21, 7, 22, 10, 18, 8, 16, 2, 15, 1, 20, 3, 17, 9, 19, 6, 14, 5, 13)\}$.

$9 \in S(11) : C \cup \{(3, 21, 4, 22), (5, 12, 6, 13, 7, 14, 8, 15, 9, 16, 10, 17, 11, 18, 1, 19, 2, 20); (1, 15, 2, 16, 8, 20), (4, 12, 9, 19, 3, 17, 7, 21, 11, 14, 6, 18, 10, 22, 5, 13); (7, 12, 8, 13, 9, 14, 10, 15, 11, 16, 1, 17, 2, 18, 3, 20, 4, 19, 5, 21, 6, 22)\}$.

$11 \in S(11) : C \cup \{(1, 17, 2, 18), (3, 19, 4, 20), (5, 21, 6, 22), (7, 12, 9, 14, 10, 15, 11, 16, 8, 13); (6, 12, 8, 20, 5, 13, 9, 19, 1, 15, 2, 16, 10, 17, 3, 18, 11, 21, 4, 22, 7, 14); (4, 12, 5, 19, 2, 20, 1, 16, 9, 15, 8, 14, 11, 17, 7, 21, 3, 22, 10, 18, 6, 13)\}$.

$13 \in S(11)$: (6, 15, 9, 16), (5, 17, 10, 18), (4, 19, 11, 20), (7, 21, 8, 22), (2, 12, 3, 13, 1, 14); (10, 15, 11, 16), (3, 17, 9, 18), (5, 19, 6, 20), (1, 21, 2, 22), (7, 12, 8, 14, 4, 13); (10, 12, 11, 13), (5, 14, 7, 15), (4, 16, 8, 17), (1, 18, 6, 22, 9, 21, 3, 20, 2, 19); (6, 12, 9, 13), (1, 15, 2, 16), (10, 14, 11, 17, 7, 20, 8, 18, 4, 21, 5, 22, 3, 19); (4, 12, 5, 13, 8, 15, 3, 16, 7, 19, 9, 14, 6, 21, 11, 18, 2, 17, 1, 20, 10, 22).

$14 \in S(11)$: (6, 15, 7, 16), (4, 17, 5, 18), (10, 19, 11, 20), (8, 21, 9, 22), (2, 12, 3, 13, 1, 14); (10, 12, 11, 13), (8, 17, 9, 18), (6, 19, 7, 20), (1, 21, 2, 22), (4, 14, 5, 15, 3, 16); (7, 12, 8, 13), (1, 17, 2, 18), (3, 19, 4, 20), (5, 21, 6, 22), (9, 14, 10, 15, 11, 16); (1, 15, 8, 16), (2, 19, 5, 20), (6, 12, 9, 13, 4, 21, 11, 14, 7, 22, 10, 17, 3, 18); (4, 12, 5, 13, 6, 14, 8, 20, 1, 19, 9, 15, 2, 16, 10, 18, 11, 17, 7, 21, 3, 22).

$15 \in S(11)$: (6, 14, 8, 15), (2, 16, 9, 17), (4, 20, 5, 21), (3, 12, 11, 18, 10, 19, 7, 22, 1, 13); (6, 12, 8, 13), (7, 16, 10, 17), (4, 18, 5, 19), (1, 14, 11, 20, 2, 21, 9, 22, 3, 15); (7, 14, 10, 15), (1, 18, 3, 19), (6, 20, 8, 21), (2, 12, 9, 13, 11, 16, 4, 17, 5, 22); (7, 12, 10, 13), (2, 18, 9, 19), (1, 20, 3, 21), (4, 14, 5, 15, 11, 17, 8, 16, 6, 22); (4, 12, 5, 13), (2, 14, 9, 15), (1, 16, 3, 17), (6, 18, 8, 22, 10, 20, 7, 21, 11, 19).

16 ∈ S(11) : (2, 15, 10, 16), (1, 17, 11, 18), (5, 19, 6, 20), (4, 21, 7, 22), (3, 12, 9, 14, 8, 13); (1, 15, 6, 16), (8, 17, 9, 18), (10, 19, 11, 20), (2, 21, 3, 22), (4, 12, 7, 14, 5, 13); (8, 15, 9, 16), (3, 17, 5, 18), (4, 19, 7, 20), (1, 21, 6, 22), (2, 12, 10, 13, 11, 14); (1, 13, 6, 14), (4, 17, 10, 18), (2, 19, 3, 20), (8, 21, 9, 22), (5, 12, 11, 16, 7, 15); (4, 14, 10, 22, 5, 21, 11, 15, 3, 16)(6, 12, 8, 20, 1, 19, 9, 13, 7, 17, 2, 18).

17 ∈ S(11) : (6, 15, 11, 16), (5, 17, 9, 18), (7, 19, 10, 20), (4, 21, 8, 22), (2, 12, 3, 13, 1, 14); (9, 15, 10, 16), (3, 17, 11, 18), (5, 19, 6, 20), (1, 21, 2, 22), (7, 12, 8, 14, 4, 13); (9, 12, 10, 13), (5, 14, 7, 15), (4, 16, 8, 17), (1, 18, 2, 19), (3, 20, 11, 21, 6, 22); (1, 15, 2, 16), (7, 17, 10, 22), (4, 18, 8, 20), (3, 19, 9, 21), (5, 12, 11, 14, 6, 13); (1, 17, 2, 20), (4, 12, 6, 18, 10, 14, 9, 22, 5, 21, 7, 16, 3, 15, 8, 13, 11, 19).

Appendix B

$n = 6$.

0 ∈ S(6) : (1, 7, 2, 8, 3, 9, 4, 10, 5, 11, 6, 12); (3, 7, 4, 8, 5, 9, 6, 10, 1, 11, 2, 12); (5, 7, 6, 8, 1, 9, 2, 10, 3, 11, 4, 12).

1 ∈ S(6) : (1, 7, 3, 9, 5, 11), (2, 8, 4, 10, 6, 12); (4, 7, 5, 12, 1, 8, 6, 9, 2, 10, 3, 11); (2, 7, 6, 11), (3, 8, 5, 10, 1, 9, 4, 12).

2 ∈ S(6) : (1, 7, 3, 9, 4, 11, 5, 12), (2, 8, 6, 10); (1, 8, 4, 10, 3, 12, 2, 11), (5, 7, 6, 9); (2, 7, 4, 12, 6, 11, 3, 8, 5, 10, 1, 9).

3 ∈ S(6) : (1, 9, 4, 12), (2, 8, 5, 11), (3, 7, 6, 10); (1, 7, 2, 9, 3, 8), (4, 10, 5, 12, 6, 11); (1, 10, 2, 12, 3, 11), (4, 7, 5, 9, 6, 8).

5 ∈ S(6) : (1, 8, 6, 10), (2, 7, 5, 9), (3, 11, 4, 12); (1, 7, 4, 10, 2, 8, 3, 9), (5, 11, 6, 12); (3, 7, 6, 9, 4, 8, 5, 10), (1, 11, 2, 12).

9 ∈ S(6) : (5, 7, 6, 8), (1, 9, 2, 10), (3, 11, 4, 12); (1, 7, 2, 8), (3, 9, 4, 10), (5, 11, 6, 12), (3, 7, 4, 8), (5, 9, 6, 10), (1, 11, 2, 12).

$n = 8$.

1 ∈ S(8) : (1, 9, 2, 10), (3, 11, 4, 12, 5, 13, 6, 14, 7, 15, 8, 16); (3, 9, 4, 10, 5, 11, 6, 12, 7, 13, 8, 14, 1, 16, 2, 15); (6, 9, 7, 10, 8, 11, 1, 12, 2, 13, 3, 14, 4, 15, 5, 16); (5, 9, 8, 12, 3, 10, 6, 15, 1, 13, 4, 16, 7, 11, 2, 14).

2 ∈ S(8) : (1, 9, 2, 10), (3, 11, 4, 12), (5, 13, 6, 14, 7, 15, 8, 16); (3, 9, 4, 10, 5, 11, 6, 12, 7, 13, 8, 14, 1, 15, 2, 16); (5, 9, 6, 10, 7, 11, 8, 12, 1, 16, 4, 13, 2, 14, 3, 15); (7, 9, 8, 10, 3, 13, 1, 11, 2, 12, 5, 14, 4, 15, 6, 16).

$3 \in S(8)$: (1, 9, 2, 10), (3, 11, 4, 12), (5, 13, 6, 14, 7, 15, 8, 16); (11, 1, 12, 2, 13, 3, 16, 4, 14, 5, 15, 6)(7, 9, 8, 10); (3, 9, 6, 16, 2, 15, 1, 14, 8, 11, 5, 12, 7, 13, 4, 10); (4, 9, 5, 10, 6, 12, 8, 13, 1, 16, 7, 11, 2, 14, 3, 15).

Let A = {(1, 9, 2, 10), (3, 11, 4, 12), (5, 13, 6, 14), (7, 15, 8, 16)} and B = {(1, 9, 2, 10), (3, 11, 4, 12), (5, 13, 6, 14), (7, 15, 8, 16); (3, 9, 4, 10), (1, 11, 2, 12), (7, 13, 8, 14), (5, 15, 6, 16)}.

$5 \in S(8)$: $A \cup$ {(1, 11, 6, 16, 5, 15, 4, 14, 3, 13, 2, 12), (7, 9, 8, 10); (13, 1, 14, 2, 15, 3, 16, 4, 9, 5, 10, 6, 12, 7, 11, 8); (3, 9, 6, 15, 1, 16, 2, 11, 5, 12, 8, 14, 7, 13, 4, 10)}.

$6 \in S(8)$: $A \cup$ {(1, 11, 6, 16, 5, 15, 4, 14, 3, 13, 2, 12), (7, 9, 8, 10); (5, 9, 6, 12, 8, 11), (3, 10, 4, 16, 2, 15), (1, 13, 7, 14); (3, 9, 4, 13, 8, 14, 2, 11, 7, 12, 5, 10, 6, 15, 1, 16)}.

$7 \in S(8)$: $A \cup$ {(11, 1, 12, 2, 13, 3, 14, 4, 15, 5, 16, 6), (7, 9, 8, 10); (7, 11, 8, 12), (5, 9, 6, 15, 3, 10), (13, 1, 14, 2, 16, 4); (7, 13, 8, 14), (3, 9, 4, 10, 6, 12, 5, 11, 2, 15, 1, 16)}.

$9 \in S(8)$: $A \cup$ {(3, 9, 4, 10, 5, 11, 6, 12, 7, 13, 8, 14), (1, 15, 2, 16); (5, 9, 6, 10, 7, 11, 8, 12), (1, 13, 2, 14), (3, 15, 4, 16); (7, 9, 8, 10, 3, 13, 4, 14), (1, 11, 2, 12), (5, 15, 6, 16)}.

$10 \in S(8)$: $B \cup$ {(5, 9, 8, 12, 7, 10), (1, 11, 2, 13, 4, 14), (3, 15, 6, 16); (6, 9, 7, 11, 8, 10), (1, 12, 2, 14, 3, 13), (4, 15, 5, 16)}.

$11 \in S(8)$: $A \cup$ {(1, 11, 2, 15), (3, 10, 6, 16), (4, 13, 8, 14), (5, 9, 7, 12); (1, 12, 2, 16), (3, 13, 7, 14), (4, 9, 6, 11, 8, 10, 5, 15); (1, 13, 2, 14), (3, 9, 8, 12, 6, 15), (4, 10, 7, 11, 5, 16)}.

$13 \in S(8)$: $A \cup$ {(1, 11, 2, 12), (3, 9, 7, 14), (4, 10, 8, 13), (5, 15, 6, 16); (1, 13, 2, 14), (3, 15, 4, 16), (5, 10, 7, 12), (6, 9, 8, 11); (1, 15, 2, 16), (3, 10, 6, 12, 8, 14, 4, 9, 5, 11, 7, 13)}.

$n = 10$.

Let A = {(3, 11, 4, 12), (5, 13, 6, 14), (7, 15, 8, 16), (9, 17, 10, 18), (1, 19, 2, 20); (1, 11, 2, 12), (3, 13, 4, 14), (5, 15, 6, 16), (7, 17, 8, 18), (9, 19, 10, 20)}.

$10 \in S(10)$: $A \cup$ {(5, 11, 6, 12, 7, 13, 8, 14, 9, 15, 10, 16, 1, 17, 2, 18, 3, 19, 4, 20); (7, 11, 8, 12, 9, 13, 10, 14, 1, 15, 2, 16, 3, 17, 4, 18, 5, 19, 6, 20); (9, 11, 10, 12, 5, 17, 6, 18, 1, 13, 2, 14, 7, 19, 8, 20, 3, 15, 4, 16)}.

$11 \in S(10)$: $A \cup$ {(5, 11, 6, 12), (7, 13, 8, 14, 9, 15, 10, 16, 1, 17, 2, 18, 3, 19, 4, 20); (7, 11, 8, 12, 9, 13, 10, 14, 1, 15, 2, 16, 3, 17, 4, 18, 5, 20, 6, 19); (9, 11, 10, 12, 7, 14, 2, 13, 1, 18, 6, 17, 5, 19, 8, 20, 3, 15, 4, 16)}.

$12 \in S(10)$: $A \cup$ {(3, 19, 4, 20), (5, 11, 6, 12, 7, 13, 8, 14, 9, 15, 10, 16, 1, 17, 2, 18); (9, 11, 10, 12), (1, 13, 2, 15, 3, 16, 4, 18, 6, 17, 5, 20, 8, 19, 7, 14); (7, 11, 8, 12, 5, 19, 6, 20), (9, 13, 10, 14, 2, 16), (1, 15, 4, 17, 3, 18)}.

$13 \in S(10) : A \cup \{$(7, 11, 8, 12), (5, 17, 6, 18), (3, 19, 4, 20), (1, 13, 2, 14, 9, 15, 10, 16); (5, 11, 6, 12, 9, 13, 10, 14, 8, 19, 7, 20), (1, 15, 2, 16, 3, 17, 4, 18); (9, 11, 10, 12, 5, 19, 6, 20, 8, 13, 7, 14, 1, 17, 2, 18, 3, 15, 4, 16)$\}$.

$14 \in S(10) : A \cup \{$(4, 17, 5, 18), (6, 19, 7, 20), (8, 11, 10, 12, 9, 14), (1, 13, 2, 15, 3, 16); (3, 19, 4, 20), (5, 11, 7, 13, 8, 12), (1, 15, 9, 16, 10, 14, 2, 17, 6, 18); (5, 19, 8, 20), (6, 11, 9, 13, 10, 15, 4, 16, 2, 18, 3, 17, 1, 14, 7, 12)$\}$.

$16 \in S(10) : A \cup \{$(7, 11, 8, 12), (3, 19, 5, 20), (1, 13, 10, 16, 9, 14, 2, 15, 4, 17, 6, 18); (5, 11, 10, 12), (6, 19, 7, 20), (8, 13, 9, 15, 3, 18, 4, 16, 2, 17, 1, 14); (6, 11, 9, 12), (4, 19, 8, 20), (2, 13, 7, 14, 10, 15, 1, 16, 3, 17, 5, 18)$\}$.

$18 \in S(10)$: (5, 11, 8, 12), (1, 13, 2, 14), (3, 15, 10, 16), (6, 17, 9, 20), (4, 18, 7, 19); (3, 11, 7, 12), (6, 13, 8, 14), (4, 15, 9, 16), (5, 17, 10, 18), (1, 19, 2, 20); (4, 11, 10, 13, 5, 14, 9, 12), (1, 16, 2, 17), (7, 15, 8, 20), (3, 18, 6, 19); (6, 11, 9, 13, 7, 14, 10, 12), (1, 15, 2, 18), (3, 17, 4, 20), (5, 16, 8, 19); (1, 11, 2, 12), (3, 13, 4, 14), (5, 15, 6, 16, 7, 17, 8, 18, 9, 19, 10, 20).

$20 \in S(10)$: (4, 11, 7, 12), (6, 13, 10, 14), (3, 15, 8, 16), (1, 18, 2, 19), (5, 17, 9, 20); (9, 11, 10, 12), (5, 13, 7, 14), (1, 15, 2, 16), (3, 17, 4, 18), (6, 19, 8, 20); (3, 11, 6, 12), (8, 13, 9, 14), (4, 15, 10, 16), (1, 17, 2, 20), (5, 18, 7, 19); (5, 11, 8, 12), (1, 13, 2, 14), (3, 19, 4, 20), (7, 15, 9, 16, 6, 18, 10, 17); (1, 11, 2, 12), (3, 13, 4, 14), (5, 15, 6, 17, 8, 18, 9, 19, 10, 20, 7, 16).

$22 \in S(10)$: (6, 11, 10, 12), (7, 13, 9, 14), (1, 15, 2, 18), (3, 17, 4, 20), (5, 16, 8, 19); (5, 11, 8, 12), (1, 13, 2, 14), (3, 15, 10, 16), (4, 18, 7, 19), (6, 17, 9, 20); (3, 11, 7, 12), (6, 13, 8, 14), (4, 15, 9, 16), (5, 17, 10, 18), (1, 19, 2, 20); (4, 11, 9, 12), (5, 13, 10, 14), (7, 15, 8, 20), (1, 16, 2, 17), (3, 18, 6, 19); (1, 11, 2, 12), (3, 13, 4, 14), (5, 15, 6, 16, 7, 17, 8, 18, 9, 19, 10, 20).

$23 \in S(10)$: (4, 11, 7, 12), (6, 13, 10, 14), (3, 15, 8, 16), (1, 18, 2, 19)(5, 17, 9, 20); (9, 11, 10, 12), (5, 13, 7, 14), (1, 15, 2, 16), (3, 17, 4, 18), (6, 19, 8, 20); (3, 11, 6, 12), (8, 13, 9, 14), (4, 15, 10, 16), (5, 18, 7, 19), (1, 17, 2, 20); (5, 11, 8, 12), (1, 13, 2, 14), (7, 15, 9, 16), (6, 17, 10, 18), (3, 19, 4, 20); (1, 11, 2, 12), (3, 13, 4, 14), (5, 15, 6, 16), (7, 17, 8, 18, 9, 19, 10, 20).

$n = 12$.

$31 \in S(12)$: (4, 13, 7, 14), (5, 15, 8, 16), (2, 17, 11, 18), (3, 19, 10, 24), (6, 20, 9, 23), (1, 21, 12, 22); (5, 13, 8, 14), (6, 15, 7, 16), (1, 17, 12, 18), (4, 19, 9, 22), (3, 20, 10, 23), (2, 21, 11, 24); (3, 13, 6, 14), (1, 15, 2, 16), (4, 23, 5, 24), (11, 19, 12, 20), (7, 21, 8, 22), (9, 17, 10, 18); (7, 23, 8, 24), (11, 15, 12, 16), (3, 17, 4, 18), (1, 19, 2, 20), (5, 21, 6, 22), (9, 13, 10, 14); (1, 13, 2, 14), (3, 15, 4, 16), (5, 17, 6, 18), (9, 21, 10, 22, 11, 23, 12, 24), (7, 19, 8, 20); (11, 13, 12, 14), (9, 15, 10, 16), (7, 17, 8, 18), (6, 19, 5, 20, 4, 21, 3, 22, 2, 23, 1, 24).

$33 \in S(12)$: (3, 13, 4, 14), (8, 15, 9, 16), (11, 17, 12, 18), (1, 19, 2, 24), (5, 21, 6, 22), (7, 20, 10, 23); (9, 13, 10, 14), (7, 15, 12, 16), (3, 17, 4, 18), (5, 19, 6, 20), (1, 22, 2, 23), (8, 21, 11, 24); (8, 13, 11, 14), (1, 15, 2, 16), (9, 17, 10, 18), (3, 19, 4, 20), (5, 23, 6, 24), (7, 21, 12, 22); (6, 13, 12, 14), (5, 15, 11, 16), (7, 17, 8, 18), (1, 20, 2, 21), (3, 22, 4, 23), (9, 19, 10, 24); (5, 13, 7, 14), (1, 17, 2, 18), (11, 19, 12, 20), (8, 22, 9, 23), (3, 21, 4, 24), (6, 15, 10, 16); (1, 13, 2, 14), (5, 17, 6, 18), (3, 15, 4, 16), (7, 19, 8, 20, 9, 21, 10, 22, 11, 23, 12, 24).

34 $\in S(12)$: (3, 13, 4, 14), (5, 15, 6, 16), (8, 17, 12, 18), (9, 19, 10, 20), (1, 22, 2, 23), (7, 21, 11, 24); (11, 13, 12, 14), (7, 15, 9, 16), (1, 17, 2, 18), (5, 19, 6, 20), (3, 21, 4, 22), (8, 23, 10, 24); (5, 13, 6, 14), (3, 17, 4, 18), (11, 19, 12, 20), (1, 21, 2, 24), (7, 22, 9, 23), (8, 15, 10, 16); (7, 13, 9, 14), (1, 15, 2, 16), (10, 17, 11, 18), (3, 19, 4, 20), (5, 23, 6, 24), (8, 21, 12, 22); (11, 15, 12, 16), (7, 17, 9, 18), (1, 19, 2, 20), (5, 21, 6, 22), (3, 23, 4, 24), (8, 13, 10, 14); (1, 13, 2, 14), (3, 15, 4, 16), (5, 17, 6, 18), (9, 21, 10, 22, 11, 23, 12, 24), (7, 19, 8, 20).

$n = 14$.

44 $\in S(14)$: (4, 15, 8, 16), (13, 17, 14, 18), (10, 20, 11, 21), (3, 19, 7, 22), (5, 23, 6, 28), (1, 25, 2, 26), (9, 24, 12, 27); (3, 15, 5, 16), (11, 17, 12, 18), (4, 21, 6, 22), (7, 26, 8, 27), (10, 25, 13, 28), (1, 23, 2, 24), (9, 1, 14, 20); (12, 15, 13, 16), (6, 17, 7, 18), (10, 19, 11, 22), (3, 20, 5, 21), (4, 23, 8, 24), (1, 27, 2, 28), (9, 25, 14, 26); (1, 17, 2, 18), (4, 19, 8, 20), (13, 21, 14, 22), (11, 23, 12, 28), (3, 24, 7, 25), (5, 26, 6, 27), (9, 15, 10, 16); (6, 15, 7, 16), (1, 19, 12, 20), (2, 21, 9, 22), (13, 23, 14, 24), (4, 25, 8, 28), (3, 26, 11, 27), (5, 17, 10, 18); (11, 15, 14, 16), (8, 17, 9, 18), (1, 21, 12, 22), (3, 23, 7, 28), (5, 24, 6, 25), (4, 26, 10, 27), (2, 19, 13, 20); (5, 19, 6, 20, 7, 21, 8, 22), (9, 23, 10, 24, 11, 25, 12, 26, 13, 27, 14, 28), (3, 17, 4, 18), (1, 15, 2, 16).

46 $\in S(14)$: (10, 15, 11, 16), (1, 17, 2, 18), (3, 19, 4, 20), (13, 21, 14, 22), (5, 23, 6, 28), (7, 25, 8, 26), (9, 24, 12, 27); (7, 15, 8, 16), (11, 19, 12, 20), (1, 21, 2, 22), (3, 23, 4, 24), (10, 25, 13, 28), (5, 26, 6, 27), (9, 17, 14, 18); (5, 15, 6, 16), (12, 17, 13, 18), (1, 19, 2, 20), (10, 21, 11, 22), (3, 27, 4, 28), (7, 23, 8, 24), (9, 25, 14, 26); (13, 15, 14, 16), (7, 17, 8, 18), (3, 21, 4, 22), (5, 24, 6, 25), (11, 23, 12, 28), (1, 26, 2, 27), (9, 19, 10, 20); (3, 15, 4, 16), (5, 17, 6, 18), (7, 19, 8, 20), (13, 23, 14, 24), (10, 26, 11, 27), (1, 25, 2, 28), (9, 21, 12, 22); (10, 17, 11, 18), (13, 19, 14, 20), (5, 21, 6, 22), (1, 23, 2, 24), (3, 25, 4, 26), (7, 27, 8, 28), (9, 15, 12, 16); (3, 17, 4, 18), (7, 21, 8, 22), (9, 23, 10, 24, 11, 25, 12, 26, 13, 27, 14, 28), (5, 19, 6, 20), (1, 15, 2, 16).

47 $\in S(14)$: (12, 15, 14, 16), (5, 17, 6, 18), (10, 19, 11, 20), (3, 21, 4, 22), (7, 23, 8, 24), (1, 26, 2, 27), (9, 25, 13, 28); (11, 15, 13, 16), (7, 17, 8, 18), (1, 19, 2, 20), (3, 23, 4, 24), (10, 27, 12, 28), (5, 25, 6, 26), (9, 21, 14, 22); (3, 15, 4, 16), (13, 17, 14, 18), (7, 19, 8, 20), (10, 21, 12, 22), (5, 23, 6, 24), (1, 25, 2, 28), (9, 26, 11, 27); (7, 15, 8, 16), (11, 17, 12, 18), (5, 21, 6, 22), (1, 23, 2, 24), (10, 25, 14, 26), (3, 27, 4, 28), (9, 19, 13, 20); (5, 15, 6, 16), (12, 19, 14, 20), (1, 21, 2, 22), (11, 23, 13, 24), (3, 25, 4, 26), (7, 27, 8, 28), (9, 17, 10, 18); (1, 17, 2, 18), (3, 19, 4, 20), (11, 21, 13, 22), (12, 23, 14, 24), (7, 25, 8, 26), (5, 27, 6, 28), (9,

15, 10, 16); (1, 15, 2, 16), (5, 19, 6, 20), (7, 21, 8, 22), (11, 25, 12, 26, 13, 27, 14, 28), (3, 17, 4, 18), (9, 23, 10, 24).

$n = 16$.

$62 \in S(16)$: (7, 17, 8, 18), (9, 19, 10, 20), (3, 21, 4, 22), (13, 23, 14, 24), (12, 25, 16, 26), (5, 27, 6, 28), (11, 29, 15, 32), (1, 30, 2, 31); (3, 17, 4, 18), (1, 19, 2, 20), (11, 21, 16, 22), (13, 25, 15, 26), (9, 29, 10, 30), (5, 23, 6, 24), (7, 27, 8, 28), (12, 31, 14, 32); (14, 17, 15, 18), (7, 19, 8, 20), (9, 21, 10, 22), (12, 23, 16, 24), (5, 25, 6, 26), (3, 27, 4, 28), (1, 29, 2, 32), (11, 30, 13, 31); (9, 17, 10, 18), (5, 19, 6, 20), (13, 21, 14, 22), (11, 23, 15, 24), (3, 25, 4, 26), (1, 27, 2, 28), (7, 31, 8, 32), (12, 29, 16, 30); (11, 17, 12, 18), (13, 19, 16, 20), (7, 21, 8, 22), (9, 23, 10, 24), (1, 25, 2, 26), (5, 31, 6, 32), (3, 29, 4, 30), (14, 27, 15, 28); (5, 17, 6, 18), (1, 21, 2, 22), (11, 25, 14, 26), (13, 27, 16, 28), (9, 31, 10, 32), (3, 23, 4, 24), (7, 29, 8, 30), (12, 19, 15, 20); (13, 17, 16, 18), (11, 19, 14, 20), (1, 23, 2, 24), (7, 25, 8, 26), (9, 27, 10, 28), (3, 31, 4, 32), (5, 29, 6, 30), (12, 21, 15, 22); (13, 29, 14, 30, 15, 31, 16, 32), (1, 17, 2, 18), (3, 19, 4, 20), (11, 27, 12, 28), (5, 21, 6, 22), (7, 23, 8, 24), (9, 25, 10, 26).

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