# On $(k,t)$-choosability of graphs

Wongsakorn Charoenpanitseri[1], Narong Punnim [2] and Chariya Uiyyasathian [3]
[1,3]Department of Mathematics, Faculty of Science,
Chulalongkorn University, Bangkok, 10330, Thailand
[2]Department of Mathematics, Srinakharinwirot University,
Sukhumvit 23, Bangkok 10110, Thailand
[3]Center of Excellent in Mathematics, CHE, Sri Ayutthaya Rd.
Bangkok, 10400, Thailand.

Email: [1]idolhere@hotmail.com, [2]narongp@swu.ac.th,[3]chariya.u@chula.ac.th

**Abstract**

A $(k,t)$-list assignment $L$ of a graph $G$ is a mapping which assigns a set of size $k$ to each vertex $v$ of $G$ and $|\bigcup_{v\in V(G)} L(v)| = t$. A graph $G$ is $(k,t)$-choosable if $G$ has a proper coloring $f$ such that $f(v) \in L(v)$ for each $(k,t)$-list assignment $L$.

We determine $t$ in terms of $k$ and $n$ that guarantee $(k,t)$-choosability of any $n$-vertex graph and a better bound if such graph does not contain $(k+1)$-clique.

**Keywords** : list assignment, list coloring, system of distinct representatives

# 1 Introduction

A *k-list assignment* $L$ of a graph $G$ is a mapping which assigns a set of size $k$ to each vertex $v$ of $G$. A *(k,t)-list assignment* of $G$ is a $k$-list assignment with $|\bigcup_{v\in V(G)} L(v)| = t$. Given a list assignment $L$, a proper coloring $f$ of $G$ is an *L-coloring* of $G$ if $f(v)$ is chosen from $L(v)$ for every vertex $v$ of $G$. A graph $G$ is *L-colorable* if $G$ has an $L$-coloring. Particularly, if $L$ is a $(k,k)$-list assignment of $G$, then any $L$-coloring of $G$ is a *k-coloring* of $G$. A graph $G$ is *$(k,t)$-choosable* if $G$ is $L$-colorable for every $(k,t)$-list assignment $L$. If a graph $G$ is $(k,t)$-choosable for each positive number $t$ then $G$ is called *k-choosable* and the smallest number $k$ satisfying this property is called the *list chromatic number* of $G$ denoted by $\chi_l(G)$.

The problem of list assignments is first studied by Vizing[10] and by Erdős, Rubin and Taylor[2]. In [2], the authors give a characterization of 2-choosable graphs. However, there is no literature giving a characterization

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of $k$-choosable graphs for $k \geq 3$. Given a positive integer $k$, $k$-choosable graphs are investigated only for specific classes of graphs, for example, every planar graph is 5-choosable while some planar graphs are 3-choosable. (See [6],[8],[9],[12],[13],[14], [15].) Some authors explore list assignment problems by investigating *uniquely k-list colorable graphs*; for instant, Ganjari et al. [1] use $(k,t)$-choosability of graphs in order to generalize a characterization of uniquely 2-list colorable graphs.

Throughout the paper, $G$ denotes a simple, undirected, finite, connected graph; $V(G)$ and $E(G)$ are the vertex set and the edge set of $G$, respectively. A *clique* is a set of pairwise adjacent vertices in a graph; a *k-clique* is a clique of size $k$. An *independent set* in a graph is a set of pairwise nonadjacent vertices; an independent set of size $n$ is denoted by $S_n$. For $X \subseteq V(G)$, $G-X$ is the graph obtained from deleting all vertices of $X$ from $G$. In case $X = \{v\}$, we write $G - v$ instead of $G - \{v\}$. The *subgraph of G induced by X*, denoted by $G[X]$ is the graph obtained from deleting all vertices of $V(G)$ outside $X$. A graph $G$ is *H-free* if $G$ has no induced subgraph which is isomorphic to a graph $H$. A graph is called *triangle-free* if it is $K_3$-free.

When $t < k$ or $t > kn$, there is no $(k,t)$-list assignment, so $G$ is automatically $(k,t)$-choosable. Unless we say otherwise, our parameters $k$, $n$ and $t$ in this paper are always positive integers such that $t \geq k$. If $k \geq n$ then all of the $n$-vertex graphs are $(k,t)$-choosable. When $k \geq \chi_l(G)$, a graph $G$ is always $(k,t)$-choosable; therefore, we focus on a positive integer $k$ such that $k < \chi_l(G)$.

Let $S \subseteq V(G)$. If $L$ is a list assignment of $G$, we let $L|_S$ denote $L$ restricted to $S$ and $L(S)$ denote $\bigcup_{v\in S} L(v)$. For a color set $A$, let $L - A$ be the new list assignment obtained from $L$ by deleting all colors in $A$ from $L(v)$ for each $v \in V(G)$. When $A$ has only one color $a$, we write $L - a$ instead of $L - \{a\}$.

**Example 1.1.**
(i) *Choosability of cycles. The cycle $C_n$ is $(2,t)$-choosable unless $n$ is odd and $t = 2$.*

Note that a graph $G$ is $(2,2)$-choosable if and only if $G$ is 2-colorable. Hence, $C_n$ is $(2,2)$-choosable if and only if $n$ is even. It remains to show that all of the cycles are $(2,t)$-choosable for $t \geq 3$.

Let $t \geq 3$ and $L$ be a $(2,t)$-list assignment of $C_n$. Thus there are two adjacent vertices $v_1, v_n \in V(G)$ such that $L(v_1) \neq L(v_n)$. Let $v_2, v_3 \ldots, v_{n-1}$ be remaining vertices along the cycle $C_n$ where $v_i$ is adjacent to $v_{i+1}$ for $i = 1,2,\ldots,n-1$. First we assign $v_1$ a color $c$ in $L(v_1)$ which is not in $L(v_n)$ and then we assign vertex $v_2$ a color in $L(v_2)$ different from $c$ and so on. This algorithm guarantees that each pair of adjacent vertices receives distinct colors.
(ii) *Choosability of $K_{2,3}$. The complete bipartite graph $K_{2,3}$ is $(2,t)$-choosable*

*for every positive integer t.*

Let $\{u_1, u_2\}$ and $\{v_1, v_2, v_3\}$ be the bipartite sets of $K_{2,3}$ and $L$ a $(2,t)$-list assignment of $K_{2,3}$. If $L(u_1) \cap L(u_2) \neq \emptyset$ then $u_1$ and $u_2$ can be colored by using the same color; hence, the remaining vertices in the other bipartite set can be easily colored. Otherwise, $L(u_1) \cap L(u_2) = \emptyset$. There are four possible ways to pick a color from each of $L(u_1)$ and $L(u_2)$. Thus, we can choose $c_1 \in L(u_1)$ and $c_2 \in L(u_2)$ such that $\{c_1, c_2\}$ are distinct from $L(v_i)$ for $i = 1, 2, 3$. Then, we can assign $v_i$ a color which is neither $c_1$ nor $c_2$ in $L(v_i)$ for every $i = 1, 2, 3$.

# 2 $(k,t)$-choosability of graphs

This section aims to investigate parameters $k, n$ and $t$ which guarantee that every $n$-vertex graph is $(k,t)$-choosable. It may not be true that $(k,t)$-choosability implies $(k,t+1)$-choosability. Example 2.1 illustrates this fact.

**Example 2.1.** Let $X, Y$ be the bipartite sets of $K_{10,10}$. To show that $K_{10,10}$ is $(3,4)$-choosable, let $L$ be a $(3,4)$-list assignment of $K_{10,10}$. For any $u \in X$, at least one of the numbers $1, 2$ is in $L(u)$. Hence, each vertex in $X$ can be colored by only color 1 or 2. For all $v \in Y$, at least one of the numbers $3, 4$ is in $L(v)$. Hence, we can color each vertex in $Y$ by only color 3 or 4.

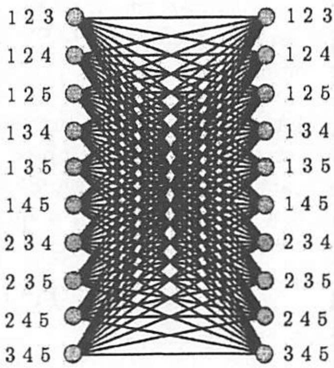



Figure 2.1: A $(3,5)$-list assignment of $K_{10,10}$

Next claim that $K_{10,10}$ is not $(3,5)$-choosable, let $L$ be the $(3,5)$-list assignment shown in Figure 2.1. At least 3 colors must be used to color all vertices in each bipartite set of $K_{10,10}$. However, only 5 colors are available; hence, there are $u \in X$ and $v \in Y$ receiving the same color. It is a contradiction. □

Given a collection of subsets of $X$, $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$, a *System of Distinct Representatives* (SDR) of $\mathcal{A}$ is a set of distinct elements $a_1, a_2, \ldots a_n$ such that $a_i \in A_i$ for all $i$. The following theorem shows the well-known necessary and sufficient condition for the existence of an SDR. Indeed, Hall's Theorem [3] is originally proved in the language of an SDR and is equivalent to Menger's Theorem [7].

**Theorem 2.2.** [11] *Given a collection of sets of $X$, $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$, an SDR of $\mathcal{A}$ exists if and only if $|\bigcup_{i\in J} A_i| \geq |J|$ for all $J \subseteq \{1, 2, \ldots, n\}$.*

**Corollary 2.3.** *Let $L$ be a list assignment of a graph $G$. If $|L(S)| \geq |S|$ for all $S \subseteq V(G)$, then $G$ is $L$-colorable. Moreover, there exists an $L$-coloring such that each vertex of $G$ assigned by distinct colors.*

*Proof.* Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Assume that $|L(S)| \geq |S|$ for all $S \subseteq V(G)$. From Theorem 2.2, there exist $c_1 \in L(v_1), c_2 \in L(v_2) \ldots, c_n \in L(v_n)$ such that $c_1, c_2, \ldots, c_n$ are distinct. Thus we define $f : V(G) \to \{1, 2, \ldots, n\}$ by $f(v_i) = c_i$; hence, $f$ is an $L$-coloring. □

Theorem 2.4 studies a more profound condition than one in Corollary 2.3 to conclude an $L$-colorable graph. Kierstead [5] and He et al. [4] use it to investigate the list chromatic number on some complete multipartite graphs.

**Theorem 2.4.** [4] *Let $L$ be a list assignment of a graph $G$ and let $S \subseteq V(G)$ be a maximal non-empty subset such that $|L(S)| < |S|$. If $G[S]$ is $L|_S$-colorable then $G$ is $L$-colorable.*

To utilize Theorem 2.4 as well as simplify our proof, throughout the rest of our paper, we will use a stronger assumption by considering all nonempty subsets $S \subseteq V(G)$ such that $|L(S)| < |S|$. In addition, we combine it with the next lemma, which has a simple proof but is quite useful, to obtain the desired results.

**Lemma 2.5.** *Let $A_1, A_2, \ldots, A_n$ be $k$-sets and $J \subseteq \{1, 2, \ldots, n\}$. If $|\bigcup_{i=1}^{n} A_i| \geq p$, then $|\bigcup_{i\in J} A_i| \geq p - (n - |J|)k$.*

*Proof.* Let $A_1, A_2, \ldots, A_n$ be $k$-sets such that $|\bigcup_{i=1}^{n} A_i| \geq p$ and $J \subseteq \{1, 2, \ldots, n\}$. Suppose that $|\bigcup_{i\in J} A_i| < p - (n - |J|)k$. Thus $|\bigcup_{i=1}^{n} A_i| \leq |\bigcup_{i\in J} A_i| + |\bigcup_{i\notin J} A_i| < p - nk + |J|k + k(n - |J|) = p$. It is a contradiction. □

The next theorem is our first main result.

**Theorem A.** *For an $n$-vertex graph $G$, if $t \geq kn - k^2 + 1$ then $G$ is $(k, t)$-choosable.*

*Proof.* Let $G$ be an $n$-vertex graph. Suppose $t \geq kn - k^2 + 1$. Let $L$ be a $(k,t)$-list assignment of $G$; that is, we obtain $|L(V(G))| = t \geq kn - k^2 + 1$. Let $S \subseteq V(G)$. If $|S| \leq k$, then, together with $|L(S)| \geq k$ always, $|L(S)| \geq |S|$. Otherwise, $|S| \geq k+1$. By Lemma 2.5, $|L(S)| \geq kn - k^2 + 1 - (n - |S|)k = k|S| - k^2 + 1 = |S| + (k-1)|S| - k^2 + 1 \geq |S| + (k-1)(k+1) - k^2 + 1 = |S|$. Hence $|L(S)| \geq |S|$ for all $S \subseteq V(G)$; therefore, by Corollary 2.3, $G$ is $L$-colorable. □

In particular, Theorem A can be rephrased in terms of a sufficient condition of the existence of an SDR on $k$-sets, concluded in Corollary 2.6.

**Corollary 2.6.** *Let $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ be a collection of $k$-subsets of a universal set $X$. If $|\bigcup_{i=1}^{n} A_i| \geq kn - k^2 + 1$, then $\mathcal{A}$ has an SDR.*

# 3 $(k,t)$-choosability of $K_{k+1}$-free graphs

Theorem A shows that if $t \geq kn - k^2 + 1$ then every $n$-vertex graph is $(k,t)$-choosable. In this section, we focus on $k \leq t \leq kn - k^2$. Theorems and lemmas are provided in order to prove Theorem B as our second main result.

**Theorem B.** *Let $n, k, t$ be positive integers such that $nk - k^2 - 2k + 1 \leq t \leq nk - k^2$ and $3 \leq k \leq n-3$. An $n$-vertex graph is $(k,t)$-choosable if and only if it is $K_{k+1}$-free. Moreover, for $k = 2$ and $2n - 6 \leq t \leq 2n - 4$, an $n$-vertex graph is $(2,t)$-choosable if and only if it is triangle-free.*

Recall that if $t < k$, $G$ is always $(k,t)$-choosable. First, we present an $n$-vertex graph which is not $(k,t)$-choosable for $k \leq t \leq kn - k^2$.

**Theorem 3.1.** *An $n$-vertex graph containing a $(k+1)$-clique is not $(k,t)$-choosable where $k \leq t \leq kn - k^2$.*

*Proof.* Let $G$ be an $n$-vertex graph containing $(k+1)$-clique $K$ and $k \leq t \leq kn - k^2$. Consider a $(k,t)$-list assignment $L$ of $G$ such that $L(v) = \{1, 2, \ldots, k\}$ for each vertex $v$ in $K$. Because $t - k \leq k(n-k-1)$, it is possible to construct a $(k,t)$-list assignment $L$ in which the union of lists for the $n-k-1$ vertices outside $K$ is $\{k+1, k+2, \ldots, t\}$. However, since every vertex in $K$ receives the same list of size $k$, we cannot color all vertices in this $(k+1)$-clique. Therefore, $G$ is not $L$-colorable. □

Theorem 3.1 shows the necessity of the first part in Theorem B. The sufficiency will be held by Theorem 3.8. Furthermore, Theorems 3.9 and 3.10 are provided to claim the statement for the case $k = 2$ of the main theorem. To simplify the proofs of our desired theorems, we prove a number of lemmas along the way.

**Lemma 3.2.** *Let $G$ be an $n$-vertex graph. If $k \geq n-2$ and $G$ is $K_{k+1}$-free, then $G$ is $(k,t)$-choosable for any positive integer $k$.*

*Proof.* Let $G$ be a $K_{k+1}$-free graph with $n$ vertices where $k \geq n-2$. Let $L$ be a $(k,t)$-list assignment of $G$ where $t \geq k$. By Theorem 2.4, it suffices to show that $\forall S \subseteq V(G)$, if $|L(S)| < |S|$ then $G[S]$ is $L|_S$-colorable.

Let $S \subseteq V(G)$ such that $|L(S)| < |S|$. Recall that $|L(S)| \geq k$ and $|S| \leq n \leq k+2$; hence, $|S| = k+1$ or $|S| = k+2$.

*Case* 1. $|S| = k+1$. We obtain $|L(S)| = k$. Since $G$ is $K_{k+1}$-free, $G[S]$ is $k$-colorable. Therefore, $G[S]$ is $L|_S$-colorable.

*Case* 2. $|S| = k+2$. Then $S = V(G)$, so $|L(S)| = k$ or $k+1$. Let $u, v$ be nonadjacent vertices of $G$. If $L(u) \cap L(v) = \emptyset$ then $2k = |L(u) \cup L(v)| \leq t \leq k+1$. Hence $k \leq 1$, which is a trivial case. Suppose that $c \in L(u) \cap L(v)$.

*Case* 2.1 $G - \{u,v\}$ is not a complete graph. It is easy to check that a $k$-vertex graph which is not a complete graph is always $L'$-colorable for every $(k-1)$-list assignment $L'$. Therefore, $G - \{u,v\}$ is $(L-c)$-colorable. Together with coloring $u$ and $v$ by $c$, we have that $G$ is $L$-colorable.

*Case* 2.2. $G - \{u,v\}$ is a complete graph. Since $G - \{u,v\}$ has $k$ vertices, $G - \{u,v\}$ is $L|_{V(G-\{u,v\})}$-colorable. Since $G$ does not contain $K_{k+1}$, each of vertices $u, v$ is adjacent to at most $k-1$ vertices in $G - \{u,v\}$. Therefore, $u, v$ can be colored. □

Corollary 3.3 follows from Lemma 3.2 which gives a characterization of an upper bound on some graphs. It then suggests a simple proof to conclude that $\chi_l(K_n - e_1 - e_2) = \begin{cases} n-1 & \text{if } e_1, e_2 \in E(K_n) \text{ are incident;} \\ n-2 & \text{otherwise.} \end{cases}$

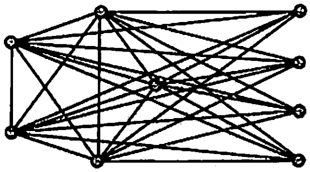
**Corollary 3.3.** *Let $G$ be an $n$-vertex graph. $\chi_l(G) \leq n-2$ if and only if $G$ contains two pairs of nonadjacent vertices or an independent set of size 3.*

*Proof.* Let $G$ be an $n$-vertex graph and $k = |V(G)| - 2$. Assume that $G$ contains two pairs of nonadjacent vertices or an independent set of size 3. Since $G$ has $k+2$ vertices, it is $K_{k+1}$-free. By Lemma 3.2, $G$ is $(k,t)$-choosable for every positive integer $t \geq k$, i.e. $\chi_l(G) \leq k = n-2$.

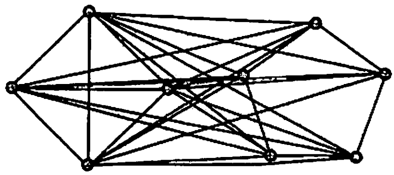
Conversely, assume that $\chi_l(G) \leq k$. Then $G$ is $k$-colorable. Since $k = n-2$, there exist three vertices assigned the same color or two pairs of vertices such that each pair assigned the same color. □

The *join* of graphs $G$ and $H$, written $G \vee H$, is the graph obtained from $G$ and $H$ by adding the edges between all vertices of $G$ and all vertices of $H$.

**Lemma 3.4.** *Let $G$ be a $K_{k+1}$-free graph with $k+3$ vertices. $G$ is isomorphic to either $K_{k-1} \vee S_4$ or $K_{k-2} \vee C_5$ if and only if $G - \{u,v\}$ contains a $k$-clique for every pair of nonadjacent vertices $u, v$.*



$K_5 \vee S_4$



$K_4 \vee C_5$

Figure 3.1: Examples of $K_{k-1} \vee S_4$ or $K_{k-2} \vee C_5$

*Proof.* Let $G$ be a $K_{k+1}$-free graph with $k+3$ vertices. It is easy to check that the necessity is true. For sufficiency, assume that $G - \{u, v\}$ contains a $k$-clique for every pair of nonadjacent vertices $u, v$.

Since $G$ has $k+3$ vertices and does not contain any $(k+1)$-clique, $G$ contains four distinct vertices $u_1, u_2, v_1, v_2$ such that $u_i$ is not adjacent to $v_i$ for $i = 1, 2$. Let $X = \{u_1, u_2, v_1, v_2\}$ and $H = G - X$. By the assumption, $G - \{u_1, v_1\}$ contains a $k$-clique. Since $G - \{u_1, v_1\}$ has $k+1$ vertices, exactly one vertex among nonadjacent vertices $u_2, v_2$ must be in such $k$-clique, say $v_2$. That is, $V(H) \cup \{v_2\}$ is a $k$-clique. Similarly, we may assume that $V(H) \cup \{v_1\}$ is a $k$-clique by considering $G - \{u_2, v_2\}$. As a consequence, $v_1$ is not adjacent to $v_2$; otherwise, $G$ contains a $(k+1)$-clique. (See Figure 3.2.)

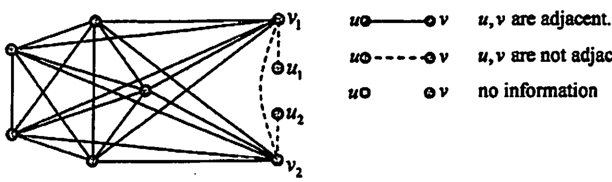



Figure 3.2: $V(H) \cup \{v_1\}$ and $V(H) \cup \{v_2\}$ are $k$-cliques while $v_1 \not\leftrightarrow u_1$, $v_2 \not\leftrightarrow u_2$ and $v_1 \not\leftrightarrow v_2$.

Suppose both $u_1$ and $u_2$ are adjacent to every vertex in $H$. If $X$ is not an independent set, then $G$ contains a $(k+1)$-clique which is a contradiction. If $X$ is an independent set, then $G$ is isomorphic to $K_{k-1} \vee S_4$. Now, we can suppose that there is $w \in V(H)$ such that $w$ is not adjacent to $u_1$.

We know that $G - \{u_1, w\}$ has $k+1$ vertices and contains a $k$-clique. Since $v_2$ is not adjacent to $v_1$ and $u_2$, the vertex $v_2$ cannot be in the $k$-clique. Therefore, $V(H - w) \cup \{v_1, u_2\}$ forms a $k$-clique. Besides, $u_2$ is not adjacent to $w$; otherwise, $V(H) \cup \{v_1, u_2\}$ forms a $(k+1)$-clique. (See Figure 3.3.)

Similarly, considering $G - \{w, u_2\}$, we obtain that $V(H - w) \cup \{v_2, u_1\}$ forms a $k$-clique.

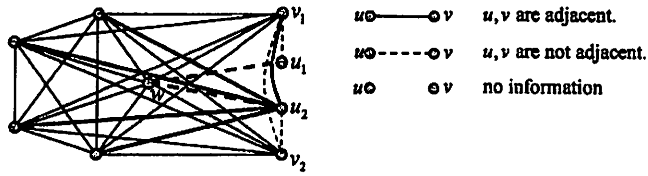



Figure 3.3: $G - \{w, u_1, v_2\}$ is a complete graph with $k$ vertices.

Finally, we consider $G - \{v_1, v_2\}$. Then $w$ cannot be in any $k$-clique of $G - \{v_1, v_2\}$ because $w$ is not adjacent to both $u_1$ and $u_2$. Then $V(H - w) \cup \{u_1, u_2\}$ forms a $k$-clique. That is, $u_1$ is adjacent to $u_2$. (See Figure 3.4.) Therefore, $\{w, v_1, u_2, u_1, v_2\}$ forms a cycle of length 5 and $H - w$ is a complete graph with $k-2$ vertices; moreover, all vertices of $C_5$ are adjacent to all vertices of $H - w$. □

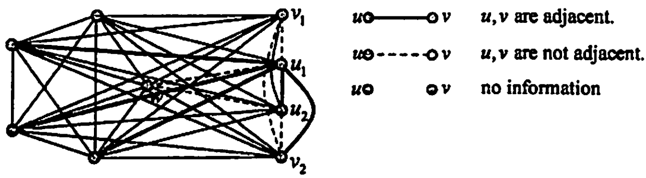



Figure 3.4: $\{w, v_1, u_2, u_1, v_2\}$ forms a cycle of length 5.

**Lemma 3.5.** *If a $(k+3)$-vertex graph is $K_{k+1}$-free, then it is $(k,t)$-choosable for $t \geq k+1$.*

*Proof.* Let $G$ be a graph with $k+3$ vertices and $L$ be a $(k,t)$-list assignment of $G$. Assume that $G$ does not contain $K_{k+1}$ as a subgraph and $t \geq k+1$. Let $S \subseteq V(G)$ be such that $|L(S)| < |S|$. It suffices to show by Theorem 2.4 that $G[S]$ is $L|_S$-colorable. If $k = 1$ then $G$ has no edges. So, it is $(1,t)$-choosable for every positive integer $t$. If $k = 2$, then $G$ is triangle-free and has five vertices which could be only $C_5$ or a subgraph of $K_{2,3}$. By Example 1.1, $G$ is $(2,t)$-choosable for $t \geq 3$. If $|S| = k+1$ or $k+2$, then the statement holds by Lemma 3.2.

Now, assume that $k \geq 3$ and $|S| = k+3$; that is, $S = V(G)$.

*Case* 1. There exists a pair of nonadjacent vertices $u, v \in V(G)$ such that $G - \{u, v\}$ does not contain a $k$-clique. Since $t = |L(V(G))| < |V(G)| \leq k+3$, we obtain $t \leq k+2$. Moreover, $L(u) \cap L(v) \neq \emptyset$ since $k \geq 3$. Let $c \in L(u) \cap L(v)$. By Lemma 3.2, $G - \{u, v\}$ is $(L-c)|_{V(G-\{u,v\})}$-colorable. Extend this to an $L$-coloring of $G$ by coloring vertices $u, v$ with color $c$.

*Case* 2. $G - \{u, v\}$ contains a $k$-clique for every pair of nonadjacent vertices

$u, v$. Apply Lemma 3.4; $G$ can be only two possible graphs. If $G \cong K_{k-1} \vee S_4$, then we first color all vertices in $K_{k-1}$ and next choose a remaining color in $L(v)$ to color $v$ for each $v \in S_4$. Otherwise, $G \cong K_{k-2} \vee C_5$. Begin with coloring all vertices of $K_{k-2}$; each vertex of $C_5$ has at least two remaining colors. The total number of the remaining colors is at least $t-(k-2) \geq 3$. So, by Example 1.1, every vertex of $C_5$ can be colored. Therefore, $G$ is $L$-colorable. □

In the next two following lemmas, we focus on 2-list assignments. Both lemmas are prepared for Theorem 3.9.

**Lemma 3.6.** *Graphs $G_1$ and $G_2$ defined below are $(2,5)$-choosable.*

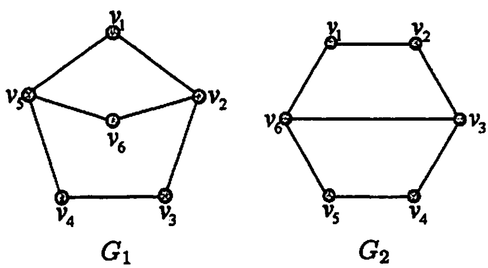



Figure 3.5: $(2,5)$-choosable graphs

*Proof.* Let $L$ be a $(2,5)$-list assignment of $G_1$. Since $|L(V(G_1-v_6))| \geq 3$, $G_1-v_6$ has an $L|_{V(G_1-v_6)}$-coloring, namely $f_1$. Then $f_1$ can be extended to be an $L$-coloring unless, without loss of generality, $L(v_6)=\{1,2\}$ and $f_1(v_2)=1, f_1(v_5)=2$. In such case, let $f_2$ be a new coloring on of $G_1-v_6$ such that $f_2(v_2)=a \in L(v_2)-\{f_1(v_2)\}$ and $f_2(v)=f_1(v)$ for each of the remaining vertices $v$. If $f_2$ is a proper coloring, then it is done. Otherwise, suppose $f_2$ is not proper. That is, $f_2(v_1)=a$ or $f_2(v_3)=a$. We may assume that $f_2(v_3)=a$. Again, we let $f_3$ be a new coloring of $G_1-v_6$ such that $f_3(v_3)=b \in L(v_3)-\{f_2(v_3)\}$ and $f_3(v)=f_2(v)$ for each of the remaining vertices $v$. If $f_3$ is still not proper, we keep defining a new coloring of $G_1-v_6$ and so on. Finally, we either have a proper coloring or know the list assignment $L$ of $G_1$ shown in Figure 3.6. Since $|L(V(G_1))|=5$, it yields $\{a,b,c\}=\{3,4,5\}$. Then we can easily obtain an $L$-coloring of $G_1$.

Now for $G_2$, let $L$ be a $(2,5)$-list assignment of $G_2$. Since $C_6$ is 2-choosable, we obtain an $L$-coloring of $G_2-e$, namely $f_1$ where $e$ is the edge whose endpoints $v_3$ and $v_6$. The $L$-coloring $f_1$ is also an $L$-coloring of $G_2$ unless $f_1(v_3)=f_1(v_6)$. Without loss of generality, suppose that $f_1(v_3)=f_1(v_6)=1$. In such case, let $f_2$ be a new coloring of $G_2-e$ such that $f_2(v_3)=a \in L(v_3)-\{f_1(v_3)\}$ and $f_2(v)=f_1(v)$ for each of the

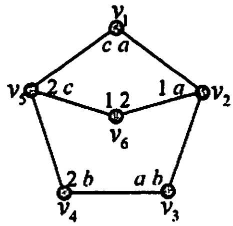



Figure 3.6: A list assignment $L$ of $G_1$

other vertices $v$. If $f_2$ is proper, then it is done. Otherwise, we define a new coloring of $G_2 - e$ similar to the proof of $G_1$. Eventually, we obtain an $L$-coloring of $G_2$ or the list assignment $L$ of $G_2$ shown in Figure 3.7. Since $L$ have five colors, it forces that $\{a, b, c, d\} = \{2, 3, 4, 5\}$. Therefore, we easily obtain an $L$-coloring of $G_2$. □

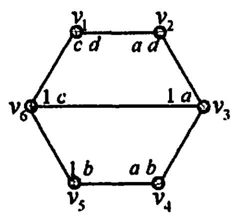



Figure 3.7: A list assignment $L$ of $G_2$

**Lemma 3.7.** *A triangle-free graph with six vertices is* $(2, 5)$*-choosable if and only if it is neither* $K_{3,3}$ *nor* $K_{3,3} - e$.

*Proof.* The $(2, 5)$-list assignments of $K_{3,3}$ and $K_{3,3} - e$ shown in Figure 3.8 do not have a proper coloring.

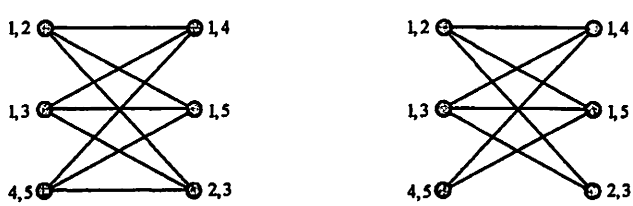



Figure 3.8: $(2, 5)$-list assignments of $K_{3,3}$ and $K_{3,3} - e$

Let $G$ be a triangle-free graph with six vertices and $L$ a $(2, 5)$-list assignment of $G$. Assume that $G$ is neither $K_{3,3}$ nor $K_{3,3} - e$. If $G$ has no cycle, then $G$ can be easily colored. If $G$ contains only one cycle, because there are all together at least three available colors, then we can first color such cycle. Then, the remaining vertices outside the cycle can be easily colored. Now assume $G$ contains at least two cycles. Since $G$ is a triangle-free graph, $G$ is one of the graphs in Lemma 3.6. Therefore, $G$ is $L$-colorable. □

We are now ready to prove our theorems.

**Theorem 3.8.** *Let $k \geq 3$. A $K_{k+1}$-free graph with $n$ vertices is $(k,t)$-choosable for $t \geq kn - k^2 - 2k + 1$.*

*Proof.* Let $k \geq 3$, $t \geq kn - k^2 - 2k + 1$ and $G$ be a $K_{k+1}$-free graph with $n$ vertices. Let $S \subseteq V(G)$ be such that $|L(S)| < |S|$. We prove that $G[S]$ is $L|_S$-colorable in order to utilize Theorem 2.4. By Lemma 2.5, $|S|k - k^2 - 2k + 1 \leq |L(S)| < |S|$. Hence $|S| < k + 3 + \frac{2}{k-1}$; i.e. $|S| \leq k + 3$.

If $|S| \leq k + 2$, then $G[S]$ is $L|_S$-colorable by Lemma 3.2. If $|S| = k + 3$ and $|L(S)| = k$ then by Lemma 2.5 we obtain $t = |L(V(G))| \leq kn - k^2 - 2k$, a contradiction. Otherwise, $|S| = k + 3$ and $|L(S)| \geq k + 1$; hence $G[S]$ is also $L|_S$-colorable by Lemma 3.5. □

It is worth mentioning that Theorem 3.8 is not true when $k = 2$. However, the statement is correct if the bound is slightly improved. This is illustrated in Theorem 3.9. Furthermore, Theorem 3.10 reveals all graphs forbidding the case for which Theorem 3.8 fails when $k = 2$.

**Theorem 3.9.** *A triangle-free graph with $n$ vertices is $(2,t)$-choosable where $t \geq 2n - 6$.*

*Proof.* Assume that $G$ is a triangle-free graph with $n$ vertices. Let $S \subseteq V(G)$ such that $|L(S)| < |S|$. Again, it suffices by Theorem 2.4 to show that $G[S]$ is $L|_S$-colorable. By Lemma 2.5, $2|S| - 6 \leq |L(S)| < |S|$. Hence $|S| < 6$. If $|S| \leq 4$ then $G[S]$ is $L|_S$-colorable by Lemma 3.2. Now assume that $|S| = 5$. By Lemma 2.5, $|L(S)| \geq 2n - 6 - 2(n - |S|) = 4$; therefore, $G[S]$ is $L|_S$-colorable by Lemma 3.5. □

**Theorem 3.10.** *A triangle-free graph with $n$ vertices is $(2, 2n-7)$-choosable if and only if it does not contain $K_{3,3} - e$ as a subgraph.*

*Proof.* Let $G$ be a triangle-free graph with $n$ vertices.
*Necessity.* Assume that $G$ contains $K_{3,3} - e$ as a subgraph. We will find a $(2, 2n-7)$-list assignment of $G$ such that $G$ is not $L$-colorable. First, assign lists of colors for vertices in $K_{3,3} - e$ shown in Figure 3.8. Assign disjoint sets of colors to each of the remaining $n - 6$ vertices; this uses $2n - 12$ colors. Thus we obtain $(2, 2n-7)$-list assignment $L$ of $G$. Since $K_{3,3} - e$ is not $L|_{V(K_{3,3}-e)}$-colorable, $G$ is not $L$-colorable.
*Sufficiency.* Assume that $G$ does not contain $K_{3,3} - e$ as a subgraph. Let $L$

be a $(2, 2n-7)$-list assignment of $G$. Let $S \subseteq V(G)$ such that $|L(S)| < |S|$. By Theorem 2.4, it suffices to show that $G[S]$ is $L|_S$-colorable.

By Lemma 2.5, $2|S| - 7 \leq |L(S)| < |S|$; therefore, $|S| \leq 6$. If $|S| = 6$, then $|L(S)| \geq 2\cdot 6 - 7 = 5$; hence, the proof is done by Lemma 3.7. If $|S| = 5$, then $|L(S)| \geq 2 \cdot 5 - 7 = 3$, so the proof is done by Lemma 3.5. Otherwise, $|S| \leq 4$. Since $G[S]$ is triangle-free, it is a subgraph of $K_{2,3}$; hence, it is $L$-colorable by Example 1.1. Therefore, $G[S]$ is $L|_S$-colorable. □

In conclusion, Theorem 3.1 and Theorem 3.8 are the necessity and sufficiency for the case $k \geq 3$ of Theorem B. Furthermore, Theorems 3.1, 3.9 and 3.10 prove the remaining case which complete our main theorem.

We next step further to the case $k \leq t \leq nk - k^2 - 2k$. Some $K_{k+1}$-free graphs with $n$ vertices are $(k,t)$-choosable. Theorem C gives us forbidden graphs.

**Theorem C.** *Let $G$ be an $n$-vertex graph and $k \leq t \leq nk - k^2 - 2k$ where $k \geq 2$. If $G$ contains $C_5 \vee K_{k-2}$ then $G$ is not $(k,t)$-choosable.*

*Proof.* Let $G$ be an $n$-vertex graph and $k \leq t \leq nk - k^2 - 2k$ where $k \geq 2$. Suppose that $G$ contains $C_5 \vee K_{k-2}$. Consider a $(k,t)$-list assignment $L$ of $G$ such that $L(v) = \{1, 2, \ldots, k\}$ for every vertex $v$ in $C_5 \vee K_{k-2}$. It is possible to construct such $(k,t)$-list assignment $L$ because $t - k \leq k(n-k-3)$. Notice that the union of lists for the $n-k-3$ vertices outside $C_5 \vee K_{k-2}$ is $\{k+1, k+2, \ldots, t\}$. However, since every vertex in $C_5 \vee K_{k-2}$ receives the same list of size $k$, we cannot color all vertices in $C_5 \vee K_{k-2}$. Therefore, $G$ is not $L$-colorable. □

As our result, an $n$-vertex graph containing $K_{k-2} \vee C_5$ or $K_{k+1}$ is not $(k,t)$-choosable for $k \leq t \leq nk - k^2 - 2k$, the next natural question is whether these graphs are all graphs which are not $(k,t)$-choosable for $k \leq t \leq nk - k^2 - 2k$. We propose the following conjecture.

**Conjecture** *Let $G$ be an $n$-vertex graph. If $G$ contains neither $K_{k-2} \vee C_5$ nor $K_{k+1}$, then it is $(k, nk - k^2 - 2k)$-choosable.*

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