On Candelabra Quadruple Systems

Shaopu Zhang*
Department of Mathematics and Physics,
Shijiazhuang Tiedao University,
Shijiazhuang 050043, China
shaopuzhang@hotmail.com

Abstract

Candelabra quadruple systems, which are usually denoted by $CQS(g^n : s)$, can be used in recursive constructions to build Steiner quadruple systems. In this paper, we introduce some necessary conditions for the existence of a $CQS(g^n : s)$ and settle the existence when $n = 4, 5$ and g is even. Finally, we get that for any $n \in \{n \geq 3 : n \not\equiv 2, 6 \pmod{12}$ *and* $n \neq 8\}$, there exists a $CQS(g^n : s)$ for all $g \equiv 0 \pmod 6$, $s \equiv 0 \pmod 2$ and $0 \leq s \leq g$.

Keywords: Candelabra system, Transverse quadruple system, s-fan design.

1 Introduction

Let v be a non-negative integer, let t be a positive integer and K be a set of positive integers. A *candelabra t-system* (or *t-CS*) of order v, and block sizes from K denoted by $CS(t, K, v)$ is a quadruple $(X, S, \mathcal{G}, \mathcal{A})$ that satisfies the following properties:

1. X is a set of v elements (called *points*);
2. S is a subset (called the *stem* of the candelabra) of X of size s;
3. $\mathcal{G} = (G_1, G_2, \ldots)$ is a set of non-empty subsets (called groups or branches) of $X \backslash S$, which partition $X \backslash S$;
4. $\mathcal{A}$ is family of subsets (called *blocks*) of X, each of cardinality from K;

*Research supported by the NSFC Grant 10771051 and 11001182.

5. every t-subset T of X with $|T \cap (S \cup G_i)| < t$ for all i is contained in a unique block and no t-subset of $S \cup G_i$ for all i is contained in any block.

By the *group type* (or *type*) of a t-$CS(X, S, \mathcal{G}, \mathcal{A})$ we mean the list $(|G||G \in \mathcal{G} : |S|)$ of group sizes and stem size. The stem size is separated from the group sizes by a colon. If a t-CS has n_i groups of size g_i, $1 \leq i \leq r$ and stem size s, then we use the notation $(g_1^{n_1} g_2^{n_2} \cdots g_r^{n_r} : s)$ to denote group type. A candelabra system is called *uniform* if all groups have the same size. A $CS(3, K, v)$ of type T is denoted by K-$CS(T)$. When $K = \{k\}$, we simply write k for K. A candelabra system with $t = 3$ and $K = \{4\}$ is called a candelabra quadruple system and denoted by $CQS(g_1^{n_1} g_2^{n_2} \cdots g_r^{n_r} : s)$.

Example 1.1 *It is known that for any prime power q, there is an $S(3, q+1, q^2+1)$ which contains q blocks intersecting in a fixed point and pairwise disjoint elsewhere [4]. This is indeed a $\{q+1\}$-$CS(q^q : 1)$ if we remove the blocks intersecting in the fixed point. In particular for $q = 3$, there exists a $CQS(3^3 : 1)$.*

Candelabra systems were first introduced by Hanani (see Definition 2 of [7]) who used quite different terminology, in Hanani's notation, a $CQS(g^n : s)$ would be denoted by $P_g[4, 1, ng + s]$.

A candelabra system $CS(t, K, v)$ of type $(1^v : 0)$ $(X, S, \mathcal{G}, \mathcal{A})$ is usually called a *t-wise balanced design* and briefly denoted by $S(t, K, v)$. As well, the stem and the group set are often omitted and we write a pair $(X, \mathcal{A})$ instead of a quadruple $(X, S, \mathcal{G}, \mathcal{A})$. An $S(3, 4, v)$ is called a *Steiner quadruple system* and denoted by $SQS(v)$. It is well known that an $SQS(v)$ exists if and only if $v \equiv 2, 4 \pmod 6$ [6].

In this paper, we concentrate on uniform candelabra quadruple systems. The known results of $CQS(g^n : s)$ are concluded as below:

In [15], Lenz stated an infinite class of CQS with three groups, with which he gave a new proof of the existence of a Steiner quadruple system.

Lemma 1.2 *(Lenz [15]). A $CQS(g^3 : s)$ exists for all even s and all $g \equiv 0, s \pmod 6$ with $s \leq g$.*

Lemma 1.3 *(Phelps [18]). A $CQS(g^3 : 1)$ exists for all $g \equiv 1, 3 \pmod 6$.*

Granville and Hartman [3] got an infinite class of CQS with four groups.

Lemma 1.4 *(Granville and Hartman [3]). A $CQS(g^4 : s)$ exists for all even g and s with $s \leq g$.*

Mills [16] also investigated the special case of candelabra systems with $s = 0$, calling them *G-designs*.

Lemma 1.5 *(Mills [16]). There exists a $CQS(6^n : 0)$ for all $n \geq 0$.*

Recently, Zhuralev et al. [20] investigated the other cases of uniform G-designs (called group divisible Steiner quadruple systems as in [20]) and gave the following result:

Lemma 1.6 *(Zhuralev, Keranen and Kreher [20]). There exists a* $CQS(g^n : 0)$ *if and only if* $g = 1$ *and* $n \equiv 2, 4 \pmod 6$*, or* g *is even and* $g(n-1)(n-2) \equiv 0 \pmod 3$.

We begin in Section 2 with background material and some necessary conditions for the existence of a $CQS(g^n : s)$. In Sections 3 and 4 we investigate CQS with four and five groups respectively. Finally in Section 5, we get that for any $n \in \{n \geq 3 : n \not\equiv 2, 6 \pmod{12}$ *and* $n \neq 8\}$, there exists a $CQS(g^n : s)$ for all $g \equiv 0 \pmod 6$, $s \equiv 0 \pmod 2$ and $0 \leq s \leq g$.

2 Preliminaries and Background Material

Let v be a non-negative integer, let t be a positive integer and K be a set of positive integers. A *group divisible t-design* (or *t-GDD*) of order v and block sizes from K denoted by $GDD(t, K, v)$ is a triple $(X, \mathcal{G}, \mathcal{B})$ such that

1. X is a set of v elements (called *points*);
2. $\mathcal{G} = \{G_1, G_2, \ldots\}$ is a set of non-empty subsets (called *groups*) of X which partition X;
3. $\mathcal{B}$ is a family of subsets of X (called *blocks*) each of cardinality from K such that each block intersects any given group in at most one point;
4. each t-set of points from t distinct groups is contained in exactly one block.

The *type* of the GDD is defined to be the list $(|G||G \in \mathcal{G})$.

A $GDD(3, 4, v)$ of type $(g_1^{n_1} g_2^{n_2} \cdots g_r^{n_r})$ is usually called a *transverse quadruple system* and denoted by $TRQS(g_1^{n_1} g_2^{n_2} \cdots g_r^{n_r})$. Mills [17] determined the existence of a $TRQS(g^n)$ (It is also called an *H design* as in [17]) except for $n = 5$. Recently, Ji [10] improved these results for $n = 5$. The existence of a $TRQS(g^n)$ can be stated as:

Lemma 2.1 *(Mills [17], Ji [10]). For* $n > 3$ *and* $n \neq 5$*, an* $H(n, g, 4, 3)$ *exists if and only if* ng *is even and* $g(n-1)(n-2)$ *is divisible by* 3*. For* $n = 5$*, an* $H(5, g, 4, 3)$ *exists if* g *is even,* $g \neq 2$ *and* $g \not\equiv 10, 26 \pmod{48}$.

More results on $TRQS$ can be found in [13] and [14].

Lemma 2.2 *(Lauinger, Kreher, Rees and Stinson [14]). There exists a* $TRQS(g^n ((n-2)g)^1)$ *if and only if* $n(n-1)g^2 \equiv 0 \pmod 6$, $(n-1)g \equiv 0 \pmod 2$, *and* $(g, n) \neq (1, 7)$.

Lemma 2.3 *(Hanani [4]). There exists a $GDD(3, q+1, q^2+q)$ of type q^{q+1} for prime power q.*

Lemma 2.4 *(Hanani [4]). If $q = 2^r$, r a non-negative integer. Then there exists a $GDD(3, q+2, q^2+2q)$ of type q^{q+2}.*

Lemma 2.5 *(Ji and Yin [12]). There exists a $GDD(3, 5, 5g)$ of type g^5 for any integer $g \geq 4$, $g \not\equiv 2 \pmod 4$, and a $GDD(3, 6, 6g)$ of type g^6 for any positive integer g satisfying $gcd(g, 4) \neq 2$ and $gcd(g, 18) \neq 3$.*

A candelabra 3-system is equivalent to the *s-fan design* defined by Hartman [9]. A $(s+3)$-tuple $(X, \mathcal{G}, \mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_s, \mathcal{B}_T)$ is an *s-fan design* if $\mathcal{G} = \{G_1, G_2, \ldots\}$ is a set of non-empty subsets of X which partition X, $(X, \mathcal{G} \cup \mathcal{B}_i)$ is a 2-wise balanced design (which is usually called a PBD), for all $i = 1, 2, \ldots, s$ and $(X, \mathcal{G} \cup \bigcup_{i=1}^{s} \mathcal{B}_i \cup \mathcal{B}_T)$ is a 3-wise balanced design.

Now let $(X, S, \mathcal{G}, \mathcal{A})$ be a $CS(3, K, v)$ of type $(g_1^{n_1} g_2^{n_2} \cdots g_r^{n_r} : s)$ with $s > 0$ and let $S = \{\infty_1, \ldots, \infty_s\}$. For $1 \leq i \leq s$, let $\mathcal{A}_i = \{A \backslash \{\infty_i\} : A \in \mathcal{A}, \infty_i \in A\}$ and $\mathcal{A}_T = \{A \in \mathcal{A} : A \cap S = \emptyset\}$. Then $(X, \mathcal{G}, \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_s, \mathcal{A}_T)$ is an s-fan design. If block sizes of $\mathcal{A}_i$ and $\mathcal{A}_T$ are from $K_i (1 \leq i \leq s)$ and K_T, respectively, then the s-fan design is denoted by s-$FG(3, (K_1, K_2, \ldots, K_s, K_T), \sum_{i=1}^{r} n_i g_i)$ of type $g_1^{n_1} g_2^{n_2} \cdots g_r^{n_r}$. On the contrary, if we add ∞_i to every block of $\mathcal{A}_i$ for all $1 \leq i \leq s$, then we get a $CS(3, K, v)$ of type $(g_1^{n_1} g_2^{n_2} \cdots g_r^{n_r} : s)$.

A $GDD(3, K, v)$ of type $(g_1^{n_1} g_2^{n_2} \cdots g_r^{n_r})$ is called *s-fan* if its block set $\mathcal{B}$ can be partitioned into disjoint subsets $\mathcal{B}_1, \ldots, \mathcal{B}_s$ and $\mathcal{B}_T$ such that for each i, $1 \leq i \leq s$, $\mathcal{B}_i$ is the block set of a $GDD(2, K_i, v)$ of the same type. If block sizes of $\mathcal{B}_T$ are all from K_T, then it is denoted by s-fan $GDD(3, (K_1, K_2, \ldots, K_s, K_T), v)$ of type $g_1^{n_1} g_2^{n_2} \cdots g_r^{n_r}$.

With the known results of $GDD(3, K, v)$ we can get the following lemma:

Lemma 2.6 *Let $g = 2^i 3^j \prod_k p_k^{a_k}$, where $p_k \geq 5$ is a prime and a_k is a non-negative integer. If $i \geq 2$ and $j \geq 2$, then there exist a $GDD(3, 5, 5g)$ of type g^5 and a $GDD(3, 6, 6g)$ of type g^6. And then there exists a g-fan $GDD(3, (4, \ldots, 4), 4g)$ of type g^4 and a g-fan $GDD(3, (5, \ldots, 5), 5g)$ of type g^5.*

Now we introduce two constructions for CQS which will be used frequently in this paper. The following one is a special case of Hartman's fundamental construction [9].

Theorem 2.7 *Suppose there is an e-$FG(3, (K_1, \ldots, K_e, K_T), gn)$ of type g^n with $e \geq 1$. Suppose there exists a $CQS(m^{k_1} : s_1)$ for any $k_1 \in K_1$, a $TRQS(m^{k_i} s_i^1)$ for any $k_i \in K_i$ $(2 \leq i \leq e)$, and a $TRQS(m^k)$ for any $k \in K_T$. Then there exists a $CQS((mg)^n : \sum_{1 \leq i \leq e} s_i)$.*

Lemma 2.8 *(Stern and Lenz [19]). Let G be a graph with vertex set Z_{2k} and let L be a set of integers in the range $1, 2, \ldots, k$, such that $\{a, b\}$ is an edge of G if and only if $|b-a| \in L$, where $|b-a| = b-a$ if $0 \le b-a \le k$ and $|b-a| = a-b$ if $k < b-a < 2k$. Then G has a one-factorization if and only if $2k/gcd(j, 2k)$ is even for some $j \in L$.*

Theorem 2.9 *Suppose there is an e-fan $GDD(3, (K_1, \ldots, K_e, K_T), gn)$ of type g^n with $e \ge 1$ and $g > 1$. Suppose there exists a $CQS(m^{k_1} : s_1)$ for any $k_1 \in K_1$, a $TRQS(m^{k_i} s_i^1)$ for any $k_i \in K_i$ $(2 \le i \le e)$, and a $TRQS(m^k)$ for any $k \in K_T$. If gm is even, then there exists a $CQS((mg)^n : \sum_{1 \le i \le e} s_i)$.*

Proof: Suppose $(X, \mathcal{G}, \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_e, \mathcal{A}_T)$ is the given e-fan GDD, where $\mathcal{G} = \{G_1, \ldots, G_n\}$. Let $s = \sum_{1 \le i \le e} s_i$ and $S = \{\infty\} \times Z_s$, where $S \cap (X \times Z_m) = \emptyset$. We will construct a $CQS((mg)^n : \sum_{1 \le i \le e} s_i)$ on point set $X' = (X \times Z_m) \cup S$ with group set $\mathcal{G}' = \{G'_1, \ldots, G'_n\}$ and stem S, where $G'_i = G_i \times Z_m$ $(1 \le i \le n)$. Block set $\mathcal{F}$ is stated blow.

For the simplicity of description, we let $G_x = \{x\} \times Z_m$ $(x \in X)$, $S = S_1 \cup S_2 \cup \cdots \cup S_e$, where $S_1 = \{\infty\} \times Z_{s_1}$, $S_j = \{(\infty, \sum_{i=1}^{j-1} s_i), (\infty, \sum_{i=1}^{j-1} s_i + 1), \ldots, (\infty, \sum_{i=1}^{j} s_i - 1)\}$ $(2 \le j \le e)$.

For every block $A \in \mathcal{A}_1$, we construct a $CQS(m^{|A|} : s_1)$ on point set $(A \times Z_m) \cup S_1$ with groups $\{G_x : x \in A\}$ and stem S_1. Such a design exists by assumption. We denote its block set by $\mathcal{D}_A$.

For every block $A \in \mathcal{A}_j$ $(2 \le j \le e)$, construct a $TRQS(m^{|A|} s_j^1)$ on point set $(A \times Z_m) \cup S_j$ with group set $\Gamma_A = \{G_x : x \in A\} \cup \{S_j\}$. Such a design exists by assumption and we denote its block set by $\mathcal{C}_A^j$.

For every block $A \in \mathcal{A}_T$, construct a $TRQS(m^k)$ on point set $A \times Z_m$ with group set $\Gamma_A = \{G_x : x \in A\}$. Such a design exists by assumption and we denote its block set by $\mathcal{B}_A$.

For any $1 \le i \le n$, consider the complete g-partite graph $K_{m,m,\ldots,m}$ on point set G'_i and we denote this graph by $\Gamma_{G'_i}$, which contains all the edges $\{(x, j), (y, k)\}$, $x \ne y$. Such a graph can also be considered as a graph on point set Z_{gm} which consists of edges $\{a, b\}$, where $|a-b| \in L = \{1, 2, \ldots, gm/2\} \backslash \{g, 2g, \ldots, g\lfloor m/2 \rfloor\}$ (note that gm is even). Since $g > 1$, there exists such a $1 \in L$ that $gm/gcd(gm, 1)$ is even. So by Lemma 2.8, $\Gamma_{G'_i}$ has a one-factorization $\{F_i^1, \ldots, F_i^{m(g-1)}\}$. For any $\{c, d\} \in F_i^k$ and any $\{c', d'\} \in F_j^k$, construct a block $\{c, d, c', d'\}$, where $1 \le k \le m(g-1)$, $1 \le i < j \le n$. Let $\mathcal{E} = \{\{c, d, c', d'\} : \{c, d\} \in F_i^k, \{c', d'\} \in F_j^k, 1 \le k \le m(g-1), 1 \le i < j \le n\}$.

Let $\mathcal{F} = (\bigcup_{A \in \mathcal{A}_1} \mathcal{D}_A) \bigcup (\bigcup_{2 \le j \le e} \bigcup_{A \in \mathcal{A}_j} \mathcal{C}_A^j) \bigcup (\bigcup_{A \in \mathcal{A}_T} \mathcal{B}_A) \bigcup \mathcal{E}$.

Then $(X', S, \mathcal{G}', \mathcal{F})$ is a $CQS((mg)^n : \sum_{1 \le i \le e} s_i)$. □

In the following of this section, we consider the necessary conditions for a CQS to exist. First we state two important results of $GDD(2, 3, v)$. The necessary and sufficient conditions for the existence of a $GDD(2,3,v)$ of type g^n were proved by Hanani in 1975.

Theorem 2.10 *(Hanani [5]). Let g and n be positive integers. There exists a $GDD(2,3,v)$ of type g^n if and only if $n \geq 3$ and the conditions in the following table are satisfied.*

g	n
$1,5 \pmod 6$	$1,3 \pmod 6$
$2,4 \pmod 6$	$0,1 \pmod 3$
$3 \pmod 6$	$1 \pmod 2$
$0 \pmod 6$	*No constraint*

The necessary and sufficient conditions for the existence of a $GDD(2,3,v)$ of type $g^n s^1$ were established by Colbourn, Hoffman, and Rees in 1992.

Theorem 2.11 *(Colbourn, Hoffman and Rees [2]). Let g, n and s be nonnegative integers. There exists a $GDD(2,3,v)$ of type $g^n s^1$ if and only if the following conditions are satisfied:*

1. *if $g > 0$, then $n \geq 3$, or $n = 2$ and $s = g$, or $n = 1$ and $s = 0$, or $n = 0$;*
2. *$s \geq g(n-1)$ or $gn = 0$;*
3. *$g(n-1) + s \equiv 0 \pmod 2$ or $gn = 0$;*
4. *$gn \equiv 0 \pmod 2$ or $s = 0$;*
5. *$\frac{1}{2}g^2 n(n-1) + gns \equiv 0 \pmod 3$.*

The following theorem establishes the necessary conditions for a $CQS(g^n : s)$ to exist. Note that a $CQS(g^1 : s)$ exists if we let its block set be $\emptyset$ and a $CQS(g^2 : s)$ exists if and only if g is even and $s = 0$. The construction of a $CQS(g^2 : 0)$, $g \equiv 0 \pmod 2$, can be found in [6], which is actually the standard doubling construction for Steiner quadruple systems.

Theorem 2.12 *(necessary conditions). Suppose $n \geq 3$ and $g > 0$. If a $CQS(g^n : s)$ exists, then the following hold:*

1. *$(n-1)g \equiv 0 \pmod 2$;*
2. *$ng + s \equiv 0 \pmod 2$;*
3. *If $g \equiv 0 \pmod 2$, then $s \leq (n-2)g$, and if $g \equiv 1 \pmod 2$, then $s < (n-2)g$;*
4. *$(n-1)g[(n+1)g + 2s] \equiv 0 \pmod 3$;*

5. $\frac{n(n-1)g^2}{6}[(n+1)g+3(s-1)] \equiv 0 \pmod 4$;

6. *If $s > 0$, then g and n satisfy the following conditions:*

g	n
$1,5 \pmod 6$	$1,3 \pmod 6$
$2,4 \pmod 6$	$0,1 \pmod 3$
$3 \pmod 6$	$1 \pmod 2$
$0 \pmod 6$	*No constraint*

Proof: Suppose a $CQS(g^n : s)$ $(X, S, \mathcal{G}, \mathcal{B})$ exists, where $n \geq 3$ and $g > 0$. Let $x, y \in X$ and suppose $x, y \in G \cup S$, where $G \in \mathcal{G}$. Consider the blocks which contains $\{x, y\}$, then we have $(n-1)g \equiv 0 \pmod 2$. Suppose $x \in G_1$ and $y \in G_2$, where $G_1, G_2 \in \mathcal{G}$ and $G_1 \neq G_2$. Consider the blocks which contain $\{x, y\}$, then we have $2(g-1)+(n-2)g+s \equiv 0 \pmod 2$, that is, $ng+s \equiv 0 \pmod 2$.

Suppose $x \in G$, where $G \in \mathcal{G}$. Consider the set $\{B \backslash \{x\} : x \in B, B \in \mathcal{B}\}$. It is the block set of a $GDD(2,3,v)$ of type $1^{g(n-1)}(g+s-1)^1$. Suppose $x \in S$ (that is, suppose $s > 0$). Consider the set $\{B \backslash \{x\} : x \in B, B \in \mathcal{B}\}$. It is the block set of a $GDD(2,3,v)$ of type g^n. By Theorems 2.11 and 2.10, we get the necessary conditions of 3,4 and 6.

At last, the number of all the admissible 3-subsets of X must be divisible by 4, so we have $\binom{n}{3}g^3 + s\binom{n}{2}g^2 + \binom{n}{1}\binom{g}{2}(n-1)g \equiv 0 \pmod 4$, that is, $\frac{n(n-1)g^2}{6}[(n+1)g+3(s-1)] \equiv 0 \pmod 4$.

The necessary conditions are concluded as above. Note that if a $CQS(g^n : s)$ exists with $g \equiv 1 \pmod 2$ and $s = (n-2)g$, then a $CQS(g^2 : 0)$ exists. It is a contradiction. So if $g \equiv 1 \pmod 2$, then $s < (n-2)g$. □

For $n = 3$, the necessary conditions can be simplified as:

g	s
$0 \pmod 6$	$0 \pmod 2$ and $s \leq g$
$1 \pmod 6$	$1 \pmod{12}$ and $s < g$
$2 \pmod 6$	$2 \pmod 6$ and $s \leq g$
$3 \pmod 6$	$1 \pmod 4$ and $s < g$
$4 \pmod 6$	$4 \pmod 6$ and $s \leq g$
$5 \pmod 6$	$5 \pmod{12}$ and $s < g$

By Lemma 1.2, we have that for the case of $n = 3$ and $g \equiv 0 \pmod 2$, the necessary conditions are also sufficient. For $n = 4, 5$ and $g \equiv 0 \pmod 2$, we will prove that the necessary conditions are also sufficient.

3 The Existence Spectrum for $CQS(g^4 : s)$

The necessary conditions for the existence of a $CQS(g^4 : s)$ can be simplified as: $g \equiv 0 \pmod 2$, $s \equiv 0 \pmod 2$ and $0 \le s \le 2g$. In this section, we will prove that the necessary conditions are also sufficient for $CQS(g^4 : s)$.

Lemma 3.1 *If a $CQS(g^4 : s)$ exists for all $g = 2, 4, 6, 12$, $s \equiv 0 \pmod 2$ and $0 \le s \le 2g$, then a $CQS(g^4 : s)$ exists for all $g \equiv 0 \pmod 2$, $s \equiv 0 \pmod 2$ and $0 \le s \le 2g$.*

Proof: Let $g \equiv 0 \pmod 2$ and $g = 2 \times 2^i 3^j \prod_k p_k^{a_k}$, where $i = 0, 1$, $j = 0, 1$, p_k is a prime, and if $p_k = 2$ or 3, then $a_k \ge 2$. Let $g_1 = 2 \times 2^i 3^j$, $g_2 = \prod p_k^{a_k}$. Then g_1 must be one of $2, 4, 6, 12$ and $g_2 \ge 1$. If $g_2 = 1$, then a $CQS(g_1^4 : s_1)$ exists by assumption, where $s_1 \equiv 0 \pmod 2$ and $0 \le s_1 \le 2g_1$.

If $g_2 > 1$, then by Lemma 2.6, a g_2-fan $GDD(3, (4, \ldots, 4), 4g_2)$ of type g_2^4 exists. We will prove that a $CQS(g^4 : s)$ exists, where $s \equiv 0 \pmod 2$ and $0 \le s \le 2g$. Let $s = m \times (2g_1) + n$, where $0 \le m \le g_2$, $0 \le n \le 2g_1$ and $n \equiv 0 \pmod 2$. Then a $CQS(g_1^4 : n)$ exists by assumption, a $TRQS(g_1^4)$ and a $TRQS(g_1^4(2g_1)^1)$ exist by Lemmas 2.1 and 2.2. Then by Theorem 2.9, let $CQS(g_1^4 : n)$, m $TRQS(g_1^4(2g_1)^1)$s and $g_2 - m$ $TRQS(g_1^4)$s be the input designs and we get a $CQS((g_1g_2)^4 : m(2g_1)+n)$. That is, a $CQS(g^4 : s)$ exists. □

By Lemma 3.1, we only need to prove that a $CQS(g^4 : s)$ exists for all $g = 2, 4, 6, 12$, $s \equiv 0 \pmod 2$ and $0 \le s \le 2g$.

Lemma 3.2 *If there exists a $TRQS(g^n s^1)$, where $g \equiv 0 \pmod 2$, then there exists a $CQS(g^n : s)$.*

Proof: By assumption, a $TRQS(g^n s^1)$ $(X, \mathcal{G} \cup S, \mathcal{B})$ exists, where $g \equiv 0 \pmod 2$ and $|S| = s$. Let $G_1, G_2 \in \mathcal{G}$, $G_1 \ne G_2$, construct a $CQS(g^2 : 0)$ on group set $\{G_1, G_2\}$ and denote its block set by $\mathcal{A}_{G_1,G_2}$. Let $\mathcal{F} = (\bigcup_{G_1,G_2 \in \mathcal{G}, G_1 \ne G_2} \mathcal{A}_{G_1,G_2}) \bigcup \mathcal{B}$. Then $(X, S, \mathcal{G}, \mathcal{F})$ is a $CQS(g^n : s)$. □

Corollary 3.3 *There exists a $CQS(g^n : (n-2)g)$, where $n(n-1)g^2 \equiv 0 \pmod 6$, $(n-1)g \equiv 0 \pmod 2$, and $(g, n) \ne (1, 7)$.*

Proof: By Lemmas 3.2 and 2.2. □

Lemma 3.4 *A $CQS(2^4 : s)$ exists for all $s \equiv 0 \pmod 2$ and $0 \le s \le 4$.*

Proof: A $CQS(2^4 : 0)$ is the same thing as a $SQS(8)$, so it exists. A $CQS(2^4 : 4)$ exists by Corollary 3.3. There exists a $SQS(10)$, take two points as set S, All blocks containing S will partition the remaining points. Removing these blocks gives a $CQS(2^4 : 2)$. □

Lemma 3.5 *A $CQS(4^4 : s)$ exists for all $s \equiv 0 \pmod 2$ and $0 \le s \le 8$.*

Proof: A $TRQS(4^4)$, a $TRQS(4^5)$ and a $TRQS(4^4 8^1)$ exist by Lemmas 2.1 and 2.2. A $TRQS(4^4 2^1)$ and a $TRQS(4^4 6^1)$ also exist [14]. Then by Lemma 3.2, the conclusion holds. □

Lemma 3.6 *A $CQS(6^4 : s)$ exists for all $s \equiv 0 \pmod 2$ and $0 \le s \le 12$.*

Proof: By Lemma 1.4, a $CQS(6^4 : s)$ exists for all $s = 0, 2, 4, 6$. There also exists a $CQS(6^4 : 12)$ by Corollary 3.3. By the known $CQS(2^4 : 4)$ we get a 4-$FG(3, (3,3,3,3,4), 8)$ of type 2^4. Apply Theorem 2.7 with the known input designs $CQS(3^3 : 1)$ in Lemma 1.3 and $TRQS(3^4)$ in Lemma 2.1. We then get a $CQS(6^4 : 10)$.

We construct a $TRQS(6^4 8^1)$ on $Z_{24} \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7, \infty_8\}$ having groups $\{4i + j : 0 \le i \le 5\}, 0 \le j \le 3$ and $\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7, \infty_8\}$. The list of base blocks is as follows and the automorphism group: $G = \langle (0, 2, 4, \ldots, 22)(1, 3, 5, \ldots, 23)(\infty_1, \infty_5)(\infty_2, \infty_6)(\infty_3, \infty_7)(\infty_4, \infty_8) \rangle$.

0 1 2 3	0 1 6 ∞_1	0 1 7 ∞_2	0 1 10 ∞_3	0 1 11 ∞_4
0 1 14 ∞_5	0 1 15 ∞_6	0 1 18 ∞_7	0 1 19 ∞_8	0 2 5 ∞_1
0 2 7 17	0 2 9 ∞_5	0 2 11 ∞_3	0 2 13 ∞_6	0 2 15 ∞_4
0 2 19 ∞_7	0 2 21 ∞_8	0 2 23 ∞_2	0 3 5 ∞_4	0 3 6 17
0 3 9 ∞_3	0 3 10 ∞_2	0 3 13 ∞_7	0 3 14 ∞_8	0 3 18 ∞_6
0 3 21 ∞_1	0 5 6 ∞_7	0 5 7 ∞_6	0 5 10 ∞_5	0 5 11 ∞_8
0 5 14 23	0 5 18 ∞_2	0 5 19 ∞_3	0 6 13 19	0 6 15 ∞_5
0 6 21 ∞_4	0 6 23 ∞_8	0 7 9 ∞_8	0 7 10 ∞_7	0 7 14 ∞_4
0 7 21 ∞_5	0 9 11 14	0 9 19 ∞_6	0 9 23 ∞_4	0 10 23 ∞_6
0 11 13 ∞_5	0 11 17 ∞_7	0 13 15 ∞_3	0 13 23 ∞_1	0 15 17 ∞_1
0 15 21 ∞_2	0 17 19 ∞_2	0 17 23 ∞_5	0 21 23 ∞_7	

Then by Lemma 3.2, we get a $CQS(6^4 : 8)$. □

Lemma 3.7 *A $CQS(12^4 : s)$ exists for all $s \equiv 0 \pmod 2$ and $0 \le s \le 24$.*

Proof: By Lemma 1.4, a $CQS(12^4 : s)$ exists for all $s = 0, 2, 4, 6, 8, 10, 12$. There also exists a $CQS(12^4 : 24)$ by Corollary 3.3. There exists a $TRQS(12^4 18^1)$ ([13]), then a $CQS(12^4 : 18)$ exists by Lemma 3.2. Since there exist a $CQS(6^4 : 8)$ and a $CQS(6^4 : 10)$, apply Theorem 2.7 with the known input designs $CQS(2^3 : 2)$ in Lemma 1.2 and $TRQS(2^4)$ in Lemma 2.1, we then get a $CQS(12^4 : 16)$ and a $CQS(12^4 : 20)$. By the known $CQS(2^4 : 4)$ in Lemma 3.4, apply Theorem 2.7 with the known input designs $CQS(6^3 : 4)$ in Lemma 1.2 and $TRQS(6^4)$ in Lemma 2.1, we then get a $CQS(12^4 : 22)$.

By the known $CQS(2^4 : 4)$ we get a 4-$FG(3, (3,3,3,3,4), 8)$ $(X, \mathcal{G}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_T)$ of type 2^4, where $\mathcal{G} = \{G_1, G_2, G_3, G_4\}$. Let $S = \{\infty\} \times Z_{14}$, where $S \cap (X \times Z_6) = \emptyset$. We will construct a $CQS(12^4 : 14)$ on point set

$X' = (X \times Z_6) \cup S$ with group set $\mathcal{G}' = \{G'_1, G'_2, G'_3, G'_4\}$ and stem S, where $G'_i = G_i \times Z_6$ $(1 \le i \le 4)$. Block set $\mathcal{F}$ is stated blow.

If $B \in \mathcal{B}_1$, say $B = \{x_0, x_1, x_2\}$, construct a design with the following block set:

$$\mathcal{E}_B = \{\{(x_i, p), (x_i, p+1), (x_{i+1}, q), (x_{i+2}, r)\} : p, q, r \in Z_6, \\ p + q + r \equiv 2i \pmod 6, i = 0, 1, 2\}.$$

If $B \in \mathcal{B}_2$, say $B = \{x_0, x_1, x_2\}$, construct a design with the following block set:

$$\mathcal{D}_B = \{\{(x_0, p), (x_1, q), (x_2, r), \infty_0\} : p, q, r \in Z_6, p + q + r \equiv 0 \pmod 6\} \bigcup \\ \{\{(x_0, p), (x_1, q), (x_2, r), \infty_1\} : p, q, r \in Z_6, p + q + r \equiv 3 \pmod 6\} \bigcup \\ \{\{(x_i, p), (x_i, p+2), (x_{i+1}, p+3q+1), (x_{i+2}, p+3q+1)\} : p \in Z_6, i \in Z_3, q = 0, 1\} \bigcup \\ \{\{(x_i, p), (x_i, p+2), (x_{i+q}, p+3), (x_{i-q}, p+5)\} : p \in Z_6, i \in Z_3, q = 1, 2\}.$$

If $B \in \mathcal{B}_3$, say $B = \{x_0, x_1, x_2\}$, construct a $TRQS(6^4)$ with groups $\{x_i \times Z_6\}, i \in Z_3$ and $\{\infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7\}$. Denote its block set by $\mathcal{C}_B$.

If $B \in \mathcal{B}_4$, say $B = \{x_0, x_1, x_2\}$, construct a $TRQS(6^4)$ with groups $\{x_i \times Z_6\}, i \in Z_3$ and $\{\infty_8, \infty_9, \infty_{10}, \infty_{11}, \infty_{12}, \infty_{13}\}$. Denote its block set by $\mathcal{A}_B$.

If $B \in \mathcal{B}_T$, say $B = \{x_0, x_1, x_2, x_3\}$, construct a $TRQS(6^4)$ with groups $\{x_i \times Z_6\}, i \in Z_4$. Denote its block set by $\mathcal{H}_B$.

Let $\mathcal{F} = (\bigcup_{B \in \mathcal{B}_1} \mathcal{E}_B) \bigcup (\bigcup_{B \in \mathcal{B}_2} \mathcal{D}_B) \bigcup (\bigcup_{B \in \mathcal{B}_3} \mathcal{C}_B) \bigcup (\bigcup_{B \in \mathcal{B}_4} \mathcal{A}_B) \bigcup (\bigcup_{B \in \mathcal{B}_T} \mathcal{H}_B)$. Then $(X', S, \mathcal{G}', \mathcal{F})$ is a $CQS(12^4 : 14)$. □

This section can be concluded as:

Theorem 3.8 *There exists a* $CQS(g^4 : s)$ *if and only if* $g \equiv 0 \pmod 2$, $s \equiv 0 \pmod 2$ *and* $0 \le s \le 2g$.

4 The Existence of $CQS(g^5 : s)$, $g \equiv 0 \pmod 6$

The necessary conditions for the existence of a $CQS(g^5 : s)$ can be simplified as:

g	s
$0 \pmod 6$	$0 \pmod 2$ and $s \le 3g$
$3 \pmod 6$	$1 \pmod 2$ and $s < 3g$
$2, 4 \pmod 6$	0

For $g \equiv 2, 4 \pmod 6$ and $s = 0$, we know that the necessary conditions are also sufficient by Lemma 1.6. In the following we discuss the case of $g \equiv 0 \pmod 6$, $s \equiv 0 \pmod 2$ and $0 \le s \le 3g$.

Lemma 4.1 *If a $CQS(g^5 : s)$ exists for all $g = 6, 12, 18, 36$, $s \equiv 0 \pmod 2$ and $0 \le s \le 3g$, then a $CQS(g^5 : s)$ exists for all $g \equiv 0 \pmod 6$, $s \equiv 0 \pmod 2$ and $0 \le s \le 3g$.*

Proof: Let $g \equiv 0 \pmod 6$ and $g = 6 \times 2^i 3^j \prod_k p_k^{a_k}$, where $i = 0, 1$, $j = 0, 1$, p_k is a prime, and if $p_k = 2$ or 3, then $a_k \ge 2$. Let $g_1 = 6 \times 2^i 3^j$, $g_2 = \prod p_k^{a_k}$. Then g_1 must be one of $6, 12, 18, 36$ and $g_2 \ge 1$. If $g_2 = 1$, then a $CQS(g_1^5 : s_1)$ exist by assumption, where $s_1 \equiv 0 \pmod 2$ and $0 \le s_1 \le 3g_1$.

If $g_2 > 1$, then by Lemma 2.6, a g_2-fan $GDD(3, (5, \ldots, 5), 5g_2)$ of type g_2^5 exists. We will prove that a $CQS(g^5 : s)$ exists, where $s \equiv 0 \pmod 2$ and $0 \le s \le 3g$. Let $s = m \times (3g_1) + n$, where $0 \le m \le g_2$, $0 \le n \le 3g_1$ and $n \equiv 0 \pmod 2$. Then a $CQS(g_1^5 : n)$ exists by assumption, a $TRQS(g_1^5)$ and a $TRQS(g_1^5(3g_1)^1)$ exist by Lemmas 2.1 and 2.2. Then by Theorem 2.9, let $CQS(g_1^5 : n)$, m $TRQS(g_1^5(3g_1)^1)$s and $g_2 - m$ $TRQS(g_1^5)$s be the input designs and we get a $CQS((g_1g_2)^5 : m(3g_1)+n)$. That is, a $CQS(g^5 : s)$ exists. □

By Lemma 4.1, we only need to prove that a $CQS(g^5 : s)$ exists for all $g = 6, 12, 18, 36$, $s \equiv 0 \pmod 2$ and $0 \le s \le 3g$.

For convenience, an $(s+1)$-$FG(3, (3, \ldots, 3, 4, 4), gn)$ of type g^n is shortly denoted by $CQS^*(g^n : s)$, where CQS stands for candelabra quadruple system and the star "$*$" stands for the fan in which all blocks have size 4.

Lemma 4.2 *There exists a $CQS^*(3^5 : 5)$.*

Proof: We construct a $CQS^*(3^5 : 5)$ on $Z_{15} \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$ having groups $\{5i+j : 0 \le i \le 2\}$, $0 \le j \le 4$, and a stem $S = \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}$. The list of base blocks is as follows, developing them $(+5 \pmod{15})$ gives all the blocks. The underlined base blocks generate the set of a $GDD(2, 4, 15)$ with the same groups.

<u>0 1 2 3</u>	<u>0 4 6 12</u>	<u>0 7 9 13</u>	<u>0 8 11 14</u>	<u>1 8 9 12</u>
0 1 4 ∞_1	0 2 6 ∞_1	0 3 12 ∞_1	0 7 14 ∞_1	0 8 9 ∞_1
0 11 13 ∞_1	1 2 13 ∞_1	1 7 9 ∞_1	1 8 14 ∞_1	2 3 14 ∞_1
0 1 7 ∞_2	0 2 13 ∞_2	0 3 9 ∞_2	0 4 11 ∞_2	0 6 8 ∞_2
0 12 14 ∞_2	1 2 14 ∞_2	1 4 8 ∞_2	1 12 13 ∞_2	2 8 9 ∞_2
0 1 8 ∞_3	0 2 11 ∞_3	0 3 14 ∞_3	0 4 7 ∞_3	0 6 9 ∞_3
0 12 13 ∞_3	1 2 9 ∞_3	1 3 12 ∞_3	1 13 14 ∞_3	2 4 13 ∞_3
0 1 12 ∞_4	0 2 4 ∞_4	0 3 6 ∞_4	0 7 8 ∞_4	0 9 11 ∞_4
0 13 14 ∞_4	1 2 8 ∞_4	1 3 9 ∞_4	1 4 7 ∞_4	2 9 13 ∞_4
0 1 13 ∞_5	0 2 9 ∞_5	0 3 7 ∞_5	0 4 8 ∞_5	0 6 14 ∞_5
0 11 12 ∞_5	1 3 4 ∞_5	1 7 8 ∞_5	1 12 14 ∞_5	2 8 14 ∞_5
0 1 5 6	0 1 9 14	0 2 5 14	0 2 7 10	0 2 8 12
0 3 4 13	0 3 5 11	0 3 8 10	0 4 5 9	0 6 7 11
1 2 4 12	1 2 7 11	1 3 6 14	1 3 7 13	1 3 8 11

1 4 6 9 2 3 4 9 2 3 7 8 2 4 7 14 3 4 8 14

□

Lemma 4.3 *There exists a* $TRQS(3^5 7^1)$.

Proof: We construct a $TRQS(3^5 7^1)$ on $Z_{15} \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7\}$ having groups $\{5i+j : 0 \le i \le 2\}, 0 \le j \le 4$, and $\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7\}$. The list of base blocks is as follows, developing them $(+5 \pmod{15})$ gives all the blocks.

0 1 2 3	0 4 6 12	0 7 9 13	0 8 11 14	1 8 9 12
0 1 4 ∞_1	0 1 7 ∞_2	0 1 8 ∞_3	0 1 9 ∞_4	0 1 12 ∞_5
0 1 13 ∞_6	0 1 14 ∞_7	0 2 4 ∞_2	0 2 6 ∞_1	0 2 8 ∞_5
0 2 9 ∞_6	0 2 11 ∞_4	0 2 13 ∞_7	0 2 14 ∞_3	0 3 4 ∞_6
0 3 6 ∞_7	0 3 7 ∞_1	0 3 9 ∞_5	0 3 11 ∞_2	0 3 12 ∞_3
0 3 14 ∞_4	0 4 7 ∞_4	0 4 8 ∞_7	0 4 11 ∞_5	0 4 13 ∞_3
0 6 7 ∞_3	0 6 8 ∞_4	0 6 9 ∞_2	0 6 13 ∞_5	0 6 14 ∞_6
0 7 8 ∞_6	0 7 11 ∞_7	0 7 14 ∞_5	0 8 9 ∞_1	0 8 12 ∞_2
0 9 11 ∞_3	0 9 12 ∞_7	0 11 12 ∞_6	0 11 13 ∞_1	0 12 13 ∞_4
0 12 14 ∞_1	0 13 14 ∞_2	1 2 4 ∞_7	1 2 8 ∞_4	1 2 9 ∞_1
1 2 13 ∞_5	1 2 14 ∞_2	1 3 4 ∞_3	1 3 7 ∞_6	1 3 9 ∞_7
1 3 12 ∞_2	1 3 14 ∞_5	1 4 7 ∞_5	1 4 8 ∞_6	1 4 12 ∞_4
1 7 8 ∞_7	1 7 9 ∞_3	1 7 13 ∞_1	1 8 14 ∞_1	1 9 13 ∞_2
1 12 13 ∞_3	1 12 14 ∞_6	1 13 14 ∞_4	2 3 4 ∞_5	2 3 9 ∞_2
2 3 14 ∞_1	2 4 13 ∞_4	2 8 9 ∞_7	2 8 14 ∞_6	2 9 13 ∞_3

□

Lemma 4.4 *A* $CQS(6^5 : s)$ *exists for all* $s \equiv 0 \pmod{2}$ *and* $0 \le s \le 18$.

Proof: There exist a $TRQS(6^5)$, a $TRQS(6^6)$ and a $TRQS(6^5 18^1)$ by Lemmas 2.1 and 2.2, then by Lemma 3.2, we get a $CQS(6^5 : 0)$, a $CQS(6^5 : 6)$ and a $CQS(6^5 : 18)$.

It is well known that there exists a 1-$FG(3,(4,4),16)$ $(X, \mathcal{G}, \mathcal{B}_1, \mathcal{B}_T)$ of type 1^{16} [1]. For a given point $x \in X$, let $X' = X \backslash \{x\}$, $\mathcal{G}' = \{B \backslash \{x\} : x \in B, B \in \mathcal{B}_1\}$, $\mathcal{A}_1 = \{B \backslash \{x\} : x \in B, B \in \mathcal{B}_T\}$, $\mathcal{A}_2 = \{B : B \in \mathcal{B}_1, x \notin B\}$, $\mathcal{A}_T = \{B : B \in \mathcal{B}_T, x \notin B\}$. Then $(X', \mathcal{G}', \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_T)$ is a 2-$FG(3,(3,4,4),15)$ of type 3^5. Note that it can also be viewed as a 1-$FG(3,(3,4),15)$ of type 3^5 (it is also a $CQS(3^5 : 1)$). Beginning with the 1-$FG(3,(3,4),15)$ of type 3^5, apply Theorem 2.7 with the known input designs $CQS(2^3 : 2)$ in Lemma 1.2 and $TRQS(2^4)$ in Lemma 2.1. We then get a $CQS(6^5 : 2)$. As well, beginning with the 2-$FG(3,(3,4,4),15)$ $(X', \mathcal{G}', \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_T)$ of type 3^5 apply Theorem 2.7: for every block in $\mathcal{A}_2$ construct a $CQS(2^4 : 2)$ (which is shown in Lemma 1.4) and for every other block of the 2-$FG(3,(3,4,4),15)$ construct a $TRQS(2^4)$

(which is shown in Lemma 2.1, note that a $TRQS(2^4)$ can also be viewed as a $TRQS(2^3 2^1)$). We then get a $CQS(6^5 : 4)$.

There exists a $CQS^*(3^5 : 5)$ by Lemma 4.2. Apply Theorem 2.7 with the known input designs $CQS(2^3 : 2)$ in Lemma 1.2 and $TRQS(2^4)$ in Lemma 2.1, we then get a $CQS(6^5 : 10)$. As well, if we apply Theorem 2.7 with the known input designs $CQS(2^4 : 2)$ in Lemma 1.4 and $TRQS(2^4)$ in Lemma 2.1, we then get a $CQS(6^5 : 12)$.

There exists a $TRQS(3^5 7^1)$ by Lemma 4.3, we then get a 8-fan $GDD(3, (3, 3, 3, 3, 3, 3, 3, 4), 15)$ of type 3^5 by deleting $\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7$. Apply Theorem 2.9. Arbitrarily choose a $GDD(2, 3, 15)$ of type 3^5 from the 8-fan $GDD(3, (3, 3, 3, 3, 3, 3, 3, 4), 15)$ of type 3^5, and then for every block of this $GDD(2, 3, 15)$, construct a $CQS(2^3 : 2)$ (which is shown in Lemma 1.2) and for every other block of the 8-fan $GDD(3, (3, 3, 3, 3, 3, 3, 3, 4), 15)$, construct a $TRQS(2^4)$ (which is shown in Lemma 2.1, note that a $TRQS(2^4)$ can also be viewed as a $TRQS(2^3 2^1)$). We then get a $CQS(6^5 : 14)$. As well, apply Theorem 2.9. For every block of the $GDD(2, 4, 15)$ in the 8-fan $GDD(3, (3, 3, 3, 3, 3, 3, 3, 4), 15)$, construct a $CQS(2^4 : 2)$ (which is shown in Lemma 1.4) and for every other block of the 8-fan $GDD(3, (3, 3, 3, 3, 3, 3, 3, 4), 15)$, construct a $TRQS(2^4)$ (which is shown in Lemma 2.1). We then get a $CQS(6^5 : 16)$.

We now construct a $CQS(6^5 : 8)$ on $Z_{30} \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7, \infty_8\}$ having groups $\{5i + j : 0 \leq i \leq 5\}$, $0 \leq j \leq 4$, and a stem $S = \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7, \infty_8\}$. The list of base blocks is as follows, developing them $(+1 \pmod{30})$ gives all the blocks.

0 1 3 ∞_1	0 4 12 ∞_1	0 6 17 ∞_1	0 7 16 ∞_1	0 1 7 ∞_2
0 2 11 ∞_2	0 3 16 ∞_2	0 4 22 ∞_2	0 1 8 ∞_3	0 2 13 ∞_3
0 3 9 ∞_3	0 4 16 ∞_3	0 1 9 ∞_4	0 2 6 ∞_4	0 3 17 ∞_4
0 7 18 ∞_4	0 1 13 ∞_5	0 2 21 ∞_5	0 3 7 ∞_5	0 6 14 ∞_5
0 1 14 ∞_6	0 2 8 ∞_6	0 3 21 ∞_6	0 4 11 ∞_6	0 1 22 ∞_7
0 2 14 ∞_7	0 3 26 ∞_7	0 6 19 ∞_7	0 1 24 ∞_8	0 2 18 ∞_8
0 3 22 ∞_8	0 4 21 ∞_8	0 1 2 5	0 1 6 10	0 1 11 25
0 1 12 20	0 1 15 26	0 1 16 19	0 1 17 27	0 1 18 21
0 1 23 28	0 2 4 9	0 2 10 17	0 2 12 23	0 2 15 19
0 2 16 24	0 2 20 27	0 2 22 26	0 3 6 11	0 3 10 18
0 3 19 24	0 4 13 18	0 4 14 23	0 5 12 18	0 5 13 22
0 6 12 21	0 6 13 20			

□

Lemma 4.5 *There exists a* $CQS(3^5 : 3)$.

Proof: We construct a $CQS(3^5 : 3)$ on $Z_{15} \cup \{\infty_1, \infty_2, \infty_3\}$ having groups $\{5i + j : 0 \leq i \leq 2\}$, $0 \leq j \leq 4$, and a stem $S = \{\infty_1, \infty_2, \infty_3\}$. The list of base blocks is as follows, developing them $(+5 \pmod{15})$ gives all the blocks.

0 1 2 3   0 1 4 5   0 1 6 9   0 1 7 10   0 1 8 11

0 1 12 ∞_1	0 1 13 ∞_2	0 1 14 ∞_3	0 2 4 ∞_2	0 2 5 7
0 2 6 11	0 2 8 13	0 2 9 ∞_3	0 2 14 ∞_1	0 3 4 ∞_3
0 3 5 8	0 3 6 ∞_1	0 3 7 ∞_2	0 3 9 11	0 3 12 14
0 4 6 7	0 4 8 14	0 4 9 10	0 4 11 ∞_1	0 4 12 13
0 5 11 13	0 6 13 ∞_3	0 6 14 ∞_2	0 7 8 ∞_1	0 7 9 12
0 7 11 ∞_3	0 7 13 14	0 8 9 ∞_2	0 8 12 ∞_3	0 9 13 ∞_1
0 11 12 ∞_2	1 2 4 ∞_3	1 2 6 7	1 2 8 ∞_1	1 2 9 13
1 3 4 ∞_1	1 3 6 8	1 3 7 12	1 3 9 ∞_3	1 3 14 ∞_2
1 4 6 14	1 4 7 9	1 4 8 13	1 4 12 ∞_2	1 7 8 ∞_2
1 7 13 ∞_3	1 7 14 ∞_1	1 8 9 12	1 12 13 14	2 3 7 8
2 3 14 ∞_3	2 4 9 12	2 4 13 ∞_1	2 8 14 ∞_2	3 4 8 9

□

Lemma 4.6 *There exists a* $CQS(3^5 : 7)$.

Proof: We construct a $CQS(3^5 : 7)$ on $Z_{15} \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7\}$ having groups $\{5i + j : 0 \le i \le 2\}, 0 \le j \le 4$, and a stem $S = \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6, \infty_7\}$. The list of base blocks is as follows, developing them (+5 (mod 15)) gives all the blocks.

0 1 2 ∞_1	0 3 4 ∞_1	0 6 8 ∞_1	0 7 9 ∞_1	0 11 14 ∞_1
0 12 13 ∞_1	1 7 13 ∞_1	1 8 14 ∞_1	1 9 12 ∞_1	2 9 13 ∞_1
0 1 3 ∞_2	0 2 6 ∞_2	0 4 7 ∞_2	0 8 12 ∞_2	0 9 11 ∞_2
0 13 14 ∞_2	1 2 9 ∞_2	1 4 13 ∞_2	1 7 8 ∞_2	2 4 8 ∞_2
0 1 7 ∞_3	0 2 4 ∞_3	0 3 12 ∞_3	0 6 14 ∞_3	0 8 11 ∞_3
0 9 13 ∞_3	1 2 14 ∞_3	1 3 4 ∞_3	1 8 12 ∞_3	2 3 9 ∞_3
0 1 8 ∞_4	0 2 13 ∞_4	0 3 9 ∞_4	0 4 6 ∞_4	0 7 14 ∞_4
0 11 12 ∞_4	1 3 12 ∞_4	1 4 7 ∞_4	1 9 13 ∞_4	2 3 4 ∞_4
0 1 9 ∞_5	0 2 8 ∞_5	0 3 11 ∞_5	0 4 13 ∞_5	0 6 7 ∞_5
0 12 14 ∞_5	1 3 7 ∞_5	1 4 12 ∞_5	1 13 14 ∞_5	2 3 14 ∞_5
0 1 12 ∞_6	0 2 14 ∞_6	0 3 6 ∞_6	0 4 11 ∞_6	0 7 13 ∞_6
0 8 9 ∞_6	1 2 3 ∞_6	1 4 8 ∞_6	1 7 14 ∞_6	2 4 13 ∞_6
0 1 13 ∞_7	0 2 3 ∞_7	0 4 12 ∞_7	0 6 9 ∞_7	0 7 11 ∞_7
0 8 14 ∞_7	1 2 8 ∞_7	1 3 14 ∞_7	1 7 9 ∞_7	2 13 14 ∞_7
0 1 4 14	0 1 5 6	0 2 5 11	0 2 7 10	0 2 9 12
0 3 5 13	0 3 7 8	0 3 10 14	0 4 9 10	0 6 11 13
1 2 4 6	1 2 7 11	1 2 12 13	1 3 6 13	1 3 8 9
1 4 9 11	2 3 8 12	2 4 7 9	2 8 9 14	3 4 8 14

□

Note that by the proof of Lemma 4.4, beginning with the known 1-$FG(3, (4, 4), 16)$ of type 1^{16} [1] we get a 2-$FG(3, (3, 4, 4), 15)$ of type 3^5. It can also be viewed as a 1-$FG(3, (3, 4), 15)$ of type 3^5, we then get a $CQS(3^5 : 1)$. And by Lemmas 4.5, 4.2 and 4.6, we get that a $CQS(3^5 : s)$ exists for all $s = 1, 3, 5, 7$.

Lemma 4.7 *A $CQS(12^5 : s)$ exists for all $s \equiv 0 \pmod 2$ and $0 \le s \le 36$.*

Proof: Firstly, a $CQS(12^5 : 36)$ exists by Corollary 3.3.

It is well known that there exists a $S(3,5,17)$ $(X, \mathcal{B})$ [4]. For two given points $x, y \in X$, let $X' = X \backslash \{x, y\}$, $\mathcal{G} = \{B \backslash \{x,y\} : x, y \in B, B \in \mathcal{B}\}$, $\mathcal{A}_1 = \{B \backslash \{x\} : x \in B, B \in \mathcal{B}\}$, $\mathcal{A}_2 = \{B \backslash \{y\} : y \in B, B \in \mathcal{B}\}$, $\mathcal{A}_T = \{B : B \in \mathcal{B}, x \notin B \text{ and } y \notin B\}$. Then $(X', \mathcal{G}, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_T)$ is a 2-$FG(3,(4,4,5),15)$ of type 3^5. Apply Theorem 2.7 with the known input designs $CQS(4^4 : s')$ in Lemma 3.5, $TRQS(4^4)$ and $TRQS(4^5)$ in Lemma 2.1, where $s' = 0,2,4,6,8$. We then get a $CQS(12^5 : s)$, where $s = 0,2,4,6,8$. And if we apply Theorem 2.7 with the known input designs $CQS(4^4 : s')$ in Lemma 3.5, $TRQS(4^4 8^1)$ in Lemma 2.2 and $TRQS(4^5)$ in Lemma 2.1, where $s' = 0,2,4,6,8$, we then get a $CQS(12^5 : s)$, where $s = 8,10,12,14,16$.

There exists a 4-fan $GDD(3,(5,5,5,5),20)$ of type 4^5 by Lemma 2.6. Apply Theorem 2.9 with the known input designs $CQS(3^5 : t)$ and $TRQS(3^5 u^1)$, where $t = 1,3,5,7$ and $u = 3,7,9$ (by Lemmas 2.1, 4.3 and 2.2 respectively). We then get a $CQS(12^5 : s)$ for all $s \equiv 0 \pmod 2$ and $10 \le s \le 34$. □

Lemma 4.8 *A $CQS(18^5 : s)$ exists for all $s \equiv 0 \pmod 2$ and $0 \le s \le 54$.*

Proof: There exists a 2-$FG(3,(4,4,5),15)$ of type 3^5 by Lemma 4.7. Apply Theorem 2.7 with the known input designs $CQS(6^4 : s')$ in Lemma 3.6, $TRQS(6^4)$ and $TRQS(6^5)$ in Lemma 2.1, where $s' = 0,2,4,6,8,10,12$. We then get a $CQS(18^5 : s)$, where $s = 0,2,4,6,8,10,12$. And if we apply Theorem 2.7 with the known input designs $CQS(6^4 : s')$ in Lemma 3.6, $TRQS(6^4 12^1)$ in Lemma 2.2 and $TRQS(6^5)$ in Lemma 2.1, where $s' = 0,2,4,6,8,10,12$, we then get a $CQS(18^5 : s)$, where $s = 12,14,16,18,20,24$.

There exists a $CQS^*(3^5 : 5)$ by Lemma 4.2. Apply Theorem 2.7 with the known input designs $CQS(6^3 : s')$ in Lemma 1.2, $TRQS(6^4)$ in Lemma 2.1, where $s' = 0,2,4,6$. We then get a $CQS(18^5 : s)$, where $s = 24,26,28,30$. And if we apply Theorem 2.7 with the known input designs $CQS(6^4 : s')$ in Lemma 3.6, $TRQS(6^4)$ in Lemma 2.1, where $s' = 0,2,4,6,8,10,12$, we then get a $CQS(18^5 : s)$, where $s = 30,32,\ldots,42$.

There exists a 8-fan $GDD(3,(3,3,3,3,3,3,3,4),15)$ of type 3^5 by Lemma 4.3. Apply Theorem 2.9 with the known input designs $CQS(6^4 : s')$ in Lemma 3.6, $TRQS(6^4)$ in Lemma 2.1, where $s' = 0,2,4,6,8,10,12$. We then get a $CQS(18^5 : s)$, where $s = 42,44,\ldots,54$. □

Lemma 4.9 *A $CQS(36^5 : s)$ exists for all $s \equiv 0 \pmod 2$ and $0 \le s \le 108$.*

Proof: There exists a 2-$FG(3,(4,4,5),15)$ of type 3^5 by Lemma 4.7. Apply Theorem 2.7 with the known input designs $CQS(12^4 : s')$ in Lemma 3.7, $TRQS(12^4)$ and $TRQS(12^5)$ in Lemma 2.1, where $s' = 0,2,\ldots,24$. We then get a $CQS(36^5 : s)$, where $s = 0, 2, \dots, 24$. And if we apply Theorem 2.7 with the known input designs $CQS(12^4 : s')$ in Lemma 3.7, $TRQS(12^4 24^1)$ in Lemma 2.2 and $TRQS(12^5)$ in Lemma 2.1, where $s' = 0, 2, \dots, 24$, we then get a $CQS(36^5 : s)$, where $s = 24, 26, \dots, 48$.

There exists a $CQS^*(3^5 : 5)$ by Lemma 4.2. Apply Theorem 2.7 with the known input designs $CQS(12^3 : s')$ in Lemma 1.2, $TRQS(12^4)$ in Lemma 2.1, where $s' = 0, 2, \dots, 12$. We then get a $CQS(36^5 : s)$, where $s = 48, 50, \dots, 60$. And if we apply Theorem 2.7 with the known input designs $CQS(12^4 : s')$ in Lemma 3.7, $TRQS(12^4)$ in Lemma 2.1, where $s' = 0, 2, \dots, 24$, we then get a $CQS(36^5 : s)$, where $s = 60, 62, \dots, 84$.

There exists a 8-fan $GDD(3, (3, 3, 3, 3, 3, 3, 3, 4), 15)$ of type 3^5 by Lemma 4.3. Apply Theorem 2.9 with the known input designs $CQS(12^4 : s')$ in Lemma 3.7, $TRQS(12^4)$ in Lemma 2.1, where $s' = 0, 2, \dots, 24$. We then get a $CQS(36^5 : s)$, where $s = 84, 88, \dots, 108$. □

This section can be concluded as:

Theorem 4.10 *A $CQS(g^5 : s)$ exists for all $g \equiv 0 \pmod 6$, $s \equiv 0 \pmod 2$ and $0 \le s \le 3g$.*

5 $CQS(g^n : s)$ with $g \equiv 0 \pmod 6$ and $s \le g$

Theorem 5.1 *For any $n \in \{n \ge 3 : n \not\equiv 2, 6 \pmod{12}$ and $n \ne 8\}$, there exists a $CQS(g^n : s)$ for all $g \equiv 0 \pmod 6$, $s \equiv 0 \pmod 2$ and $0 \le s \le g$.*

Proof: For $n = 3, 4, 5$, it is known that such a $CQS(g^n : s)$ exists by the above discussion.

Ji [11] has proved that there exists a $S(3, \{4, 5, 6\}, v)$ for any $v \in \{v > 0 : v \equiv 0, 1, 2 \pmod 4$ *and* $v \ne 9, 13\}$. Then there exists a $1\text{-}FG(3, (\{3, 4, 5\}, \{4, 5, 6\}), n)$ of type 1^n for any $n \in \{n \ge 3 : n \equiv 0, 1, 3 \pmod 4$ *and* $n \ne 8, 12\}$. Apply Theorem 2.7 with the known input designs $CQS(g^3 : s)$, $CQS(g^4 : s)$, $CQS(g^5 : s)$ and $TRQS(g^4)$, $TRQS(g^5)$, $TRQS(g^6)$ in Lemma 2.1, where $g \equiv 0 \pmod 6$, $s \equiv 0 \pmod 2$ and $0 \le s \le g$. Then we get a $CQS(g^n : s)$ for any $n \in \{n \ge 3 : n \equiv 0, 1, 3 \pmod 4$ *and* $n \ne 8, 12\}$, where $g \equiv 0 \pmod 6$, $s \equiv 0 \pmod 2$ and $0 \le s \le g$.

For $n = 12$, since $g \equiv 0 \pmod 6$, $s \equiv 0 \pmod 2$ and $0 \le s \le g$, by Lemma 1.2 we know that there exists a $CQS((4g)^3 : s)(X, S, \mathcal{G}, \mathcal{A})$, where $\mathcal{G} = \{G_1, G_2, G_3\}$. For each G_i, $1 \le i \le 3$, split it into four groups $G_{i1}, G_{i2}, G_{i3}, G_{i4}$ with $|G_{i1}| = |G_{i2}| = |G_{i3}| = |G_{i4}|$, then we can construct a $CQS(g^4 : s)(G_i \cup S, S, \{G_{i1}, G_{i2}, G_{i3}, G_{i4}\}, \mathcal{B}_i)$, it exists by Theorem 3.8. Let $\mathcal{G}' = \{G_{i1}, G_{i2}, G_{i3}, G_{i4} : 1 \le i \le 3\}$, $\mathcal{T} = \mathcal{A} \cup (\cup_{i=1}^{3} \mathcal{B}_i)$, then $(X, S, \mathcal{G}', \mathcal{T})$ is a $CQS(g^{12} : s)$, where $g \equiv 0 \pmod 6$, $s \equiv 0 \pmod 2$ and $0 \le s \le g$.

For $n \equiv 10 \pmod{12}$, let $n = 12k + 10$. Hanani [8] proved that there exists an $S(3, \{4, 6\}, v)$ for all $v \equiv 0 \pmod 2$. Then we can get a $1\text{-}FG(3, (\{3, 5\},$

$\{4,6\}), 4k+3)$ of type 1^{4k+3}. Apply Theorem 2.7 with the known input designs $CQS((3g)^i : g+s)$, $i \in \{3,5\}$, and $TRQS((3g)^4)$, $TRQS((3g)^6)$, we get a $CQS((3g)^{4k+3} : g+s)(X, S, \mathcal{G}, \mathcal{A})$, where $\mathcal{G} = \{G_1, G_2, \cdots, G_{4k+3}\}$. For each G_i, $1 \le i \le 4k+3$, split it into three groups G_{i1}, G_{i2}, G_{i3} with $|G_{i1}| = |G_{i2}| = |G_{i3}|$. Let $G_0 \subset S$ and $|G_0| = g$, let $S' = S \backslash G_0$, then $|S'| = s$. Now we construct a $CQS(g^4 : s)(G_{i1} \cup G_{i2} \cup G_{i3} \cup G_0 \cup S', S', \{G_{i1}, G_{i2}, G_{i3}, G_0\}, \mathcal{B}_i)$, $1 \le i \le 4k+3$. Let $\mathcal{G}' = \{G_{i1}, G_{i2}, G_{i3}, G_0 : 1 \le i \le 4k+3\}$, $\mathcal{T} = \mathcal{A} \cup (\cup_{i=1}^{4k+3} \mathcal{B}_i)$, then $(X, S', \mathcal{G}', \mathcal{T})$ is a $CQS(g^{12k+10} : s)$, where $g \equiv 0 \pmod 6$, $s \equiv 0 \pmod 2$ and $0 \le s \le g$. □

Acknowledgements

The author thank Professor L. Zhu and the referee for their very helpful suggestions on this paper.

References

[1] R. D. Baker, Partitioning the planes of $AG(2m, 2)$ into 2-designs, Discrete Math. 15 (1976), 205-211.

[2] C. J. Colbourn, D. G. Hoffman, and R. Rees, A new class of group divisible designs with block size three, J. Combin. Theory A 59 (1992), 73-89.

[3] A. Granville, A. Hartman, Subdesigns in Steiner quadruple systems. J. Combin. Theory A 56 (1991), 239-270.

[4] H. Hanani, A class of three designs, J. Combin. Theory A 26 (1979), 1-19.

[5] H. Hanani, Balanced incomplete block designs and related designs, Discrete Math. 11 (1975), 255-369.

[6] H. Hanani, On quadruple systems, Canad. J. Math. 12 (1960), 145-157.

[7] H. Hanani, On some tactical configurations, Canad. J. Math. 15 (1963), 702-722.

[8] H. Hanani, Truncated finite planes, in "Combinatorics", 115-120, Proceedings of Symposia in Pure Mathematics, Vol. XIX, Amer. Math. Soc., Providence, R.I., 1971.

[9] A. Hartman, The fundamental construction for 3-designs, Discrete Math. 124 (1994), 107-131.

[10] L. Ji, An improvement on H design, J. Combin. Designs 17 (2009), 25-35.

[11] L. Ji, On the 3BD-closed set $B_3(\{4,5,6\})$, J. Combin. Designs 12 (2004), 92-102.

[12] L. Ji, J. Yin, Constructions of new orthogonal arrays and covering arrays of strength three, J. Combin. Theory A, 117 (2010), 236-247.

[13] M. Keranen, D. Kreher, Transverse quadruple systems with five holes, J. Combin. Designs 15 (2007), 315-340.

[14] K. Lauinger, D. Kreher, R. Rees, and D. Stinson, Computing transverse t-designs, J. Combin. Math. Combin. Comput. 54 (2005), 33-56.

[15] H. Lenz, Tripling quadruple systems, Ars Combin. 20 (1985), 193-202.

[16] W. H. Mills, On the covering of triples by quadruples, Congr. Numer. 10 (1974), 563-581.

[17] W. H. Mills, On the existence of H designs, Congr. Number. 79 (1990), 129-141.

[18] K. T. Phelps, Rotational Steiner quadruple systems, Ars Combin. 4 (1977), 177-185.

[19] G. Stern, H. Lenz, Steiner triple systems with given subspaces, another proof of the Doyen-Wilson Theorem, Bull Un Mal Ital 5 (1980), 109-114.

[20] A. A. Zhuralev, M. S. Keranen, D. L. Kreher, Small group divisible Steiner quadruple systems, Electron. J. Combin. 15 (2008), no. 1, Research paper 40, 14 pp.