On the structure of graphs with exactly two near-perfect matchings *

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Abstract

A near-perfect matching is a matching saturating all but one vertex in a graph. In this note, it is proved that if a graph has a near-perfect matching then it has at least two, moreover, a concise structure construction for all graphs with exactly two near-perfect matchings is given. We also prove that every connected claw-free graph G of odd order n ($n \geq 3$) has at least $\frac{n+1}{2}$ near-perfect matchings which miss different vertices of G.

Keywords: Near-perfect matching; Maximal matching; Factor-critical graph

1 Introduction

All graphs considered in this paper are simple connected graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph induced by S, and the neighbor set of S in G, denoted by $N_G(S)$, is the set of all vertices adjacent to vertices in S. A matching of G is said to be *perfect* if it covers all vertices of G and *near-perfect* if it covers all but one vertex of G. Clearly, a graph with a perfect matching (resp. near-perfect matching) must have an even (resp. odd) number of vertices. The number of perfect matchings or near-perfect matchings in a graph G is denoted by $pm(G)$ or $npm(G)$. The *deficiency* of G, denoted by $def(G)$, is the number of vertices missed by a maximum matching of G. A bipartite graph G(A, B) is said to have *positive surplus* (as viewed from A) if $|N_G(X)| > |X|$ for all $\emptyset \neq X \subseteq A$. A graph G is said to be factor-critical if $G - v$ has a perfect matching for every vertex $v \in V(G)$. Other terminologies and notations not defined here can be found in [1] and [5].

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Enumeration of perfect matchings in graphs is an active and important subject in graph theory and combinatorial optimization since it has a wide range of applications. But the enumeration problem for perfect matchings in general graphs (even in bipartite graphs) is NP-hard [5]. Let G be a graph with a perfect matching, and let v be any vertex of G. Suppose M is any perfect matching of G. Let w be the vertex adjacent to v in M. Then $M \setminus \{vw\}$ will be a near-perfect matching of $G - v$ missing w. From this fact, a perfect matching of G is relative to a near-perfect matching of $G-v$, and hence $pm(G) \leq npm(G - v)$. So it is deserved to study the number of near-perfect matchings in graphs of odd order.

Pulleyblank [6] proved that every 2-connected factor-critical graph G contains at least $|E(G)|$ near-perfect matchings. For general factor-critical graphs, Liu [3] proved that if G is a factor-critical graph, then G has at least $|E(G)| - c + 1$ near-perfect matchings, where c is the number of blocks of G.

In this note, it is proved that if a graph has a near-perfect matching then it has at least two, moreover, a concise structure construction for all graphs with exactly two near-perfect matchings is given based on the Gallai-Edmonds decomposition theory. We also prove that every connected claw-free graph G of odd order n ($n \geq 3$) has at least $\frac{n+1}{2}$ near-perfect matchings which miss different vertices of G.

2 Main results

First we review some known results which will help to prove our main results.

Theorem 1 (Hall's Theorem [2]). *Let G be a bipartite graph with bipartition (X, Y). Then G contains a matching that saturates every vertex in X if and only if $|N_G(S)| \geq |S|$ for all $S \subseteq X$.*

The following theorem gives a method for constructing all graphs having exactly one perfect matching.

Theorem 2 ([5]). *A graph G has a unique perfect matching if and only if it can be constructed by iterating the following construction: Let G_1 and G_2 be two vertex-disjoint graphs, each with a unique perfect matching. (Either or both may be empty). Let x_1 and x_2 be two new vertices. Join at least one vertex of G_i to x_i for $i = 1$ and 2 and join x_1 to x_2.*

Some notations appeared in Gallai-Edmonds decomposition theory for graphs in terms of maximum matchings are now recalled as follows [5].

For any graph G, let $D(G)$ denote the set of vertices in G that are not saturated by at least one maximum matching, $A(G)$ the set of vertices in $V(G) - D(G)$ adjacent to at least one vertex in $D(G)$ and $C(G) = V(G) - D(G) - A(G)$.

Theorem 3 (The Gallai-Edmonds Structure Theorem [5]). *If G is a graph and $D(G)$, $A(G)$, $C(G)$ are defined as above, then*

(1) the components of the subgraph induced by $D(G)$ are factor-critical,
(2) the subgraph induced by $C(G)$ has a perfect matching,
(3) the bipartite graph obtained form G by deleting the vertices of $C(G)$ and the edges spanned by $A(G)$ and by contracting each component of $D(G)$ to a single vertex has positive surplus (as viewed from $A(G)$),
(4) if M is any maximum matching of G, it contains a near-perfect matching of each component of $D(G)$, a perfect matching of each component of $C(G)$ and matches all vertices of $A(G)$ with vertices in distinct components of $D(G)$.

Now we can state the main result of this note.

Theorem 4 *Let G be a graph with a near-perfect matchings. Then $npm(G) \geq 2$. Moreover, if G is a connected graph with exactly two near-perfect matchings, then either $|V(G)| = 3$ and $G \cong K_{1,2}$, or $|V(G)| \geq 5$ and the Gallai-Edmonds structure of G is shown in Figure 1, that is*
(a) $C(G) \neq \emptyset$, and let G_1, ..., G_k ($k \geq 1$) be the components of the subgraph induced by $C(G)$. Then for each i, $1 \leq i \leq k$, G_i contains exactly one perfect matching, and hence can be constructed by the iterating produce given in Theorem 2,
(b) $A(G)$ contains exactly one vertex d, d is adjacent to some vertices of each G_i,
(c) $D(G)$ consists of two singletons.

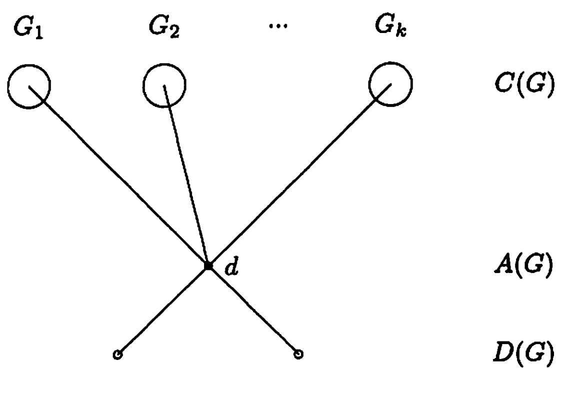

Figure 1.

Proof Let G be a graph with a near-perfect matching. Let x be a vertex missed by a near-perfect matching M of G, let y be a neighbour of x, and let yz be an edge of M. Then the matching obtained form M by removing the edge yz and adding xy is another near-perfect matching of G. This simple observation follows that $npm(G) \geq 2$.

Let G be a connected graph having exactly two near-perfect matchings. Obviously, every near-perfect matching of G is maximum and $def(G) = 1$. Let $\{D(G), A(G), C(G)\}$ be the Gallai-Edmonds decomposition of G.

If $|V(G)| = 3$, it is easily checked that $G \cong K_{1,2}$. So in the following we may assume that $|V(G)| \geq 5$. Let $D_1, ..., D_t$ be the components of the subgraph induced by $D(G)$. First we assert the following.

Claim 1. D_i is a singleton for all i.

Suppose to the contrary, there is a component D_i, without loss of generality, say D_t such that $|V(D_t)| \geq 3$. Since G has a near-perfect matching, $def(G) = 1$. On the other hand, by Theorem 3(4), $def(G) = t - |A(G)|$, thus $|A(G)| = t - 1$. Suppose $A(G) = \{u_1, ..., u_{t-1}\}$.

By Theorem 1 and Theorem 3(3), we can choose a perfect matching M_1 of the graph $G - C(G) - V(D_t)$, M_1 matches all $u_1, ..., u_{t-1}$ with vertices in distinct $D_1, ..., D_{t-1}$ and contains a near-perfect matching of each D_i for $i = 1, ..., t-1$. Let M_2 be a perfect matching of the subgraph induced by $C(G)$, and let M_3 be any near-perfect matching of D_t. Clearly, by Theorem 3(4), $M_1 \cup M_2 \cup M_3$ is a near-perfect matching of G. This observation implies that $npm(G) \geq npm(D_t)$.

Let $V(D_t) = \{x_1, ..., x_r\}$, where $r \geq 3$. Since D_t is a factor-critical graph, it follows from the definition that, for each i, $1 \leq i \leq r$, there exists a perfect matching F_i of $D_t - x_i$. Obviously, F_i is also a near-perfect matching in D_t and any two near-perfect matchings in $\{F_1, ..., F_r\}$ are distinct. So D_t has at least $r \geq 3$ near-perfect matchings. Hence $npm(G) \geq npm(D_t) \geq 3$, a contradiction.

Claim 2. $|A(G)| = 1$.

Suppose to the contrary, $|A(G)| = p \geq 2$. Since $def(G) = t - |A(G)| = 1$, $t = p + 1$. According to Claim 1, we may assume that $D(G) = \{v_1, ..., v_{p+1}\}$. By the definition of $D(G)$, there exist near-perfect matchings $M_1, ..., M_{p+1}$ of G such that M_i misses v_i, for $i = 1, ..., p+1$. Clearly, any two near-perfect matchings in $\{M_1, ..., M_{p+1}\}$ are distinct. Thus $npm(G) \geq p + 1 \geq 3$, a contradiction. This proves Claim 2 and (b).

Since $def(G) = t - |A(G)| = 1$, thus $t = 2$. Hence by Claim 1, $D(G)$ consists of two singletons, (c) is proved.

Now $|A(G) \cup D(G)| = 3$, recall that $|V(G)| \geq 5$, it follows that $C(G) \neq \emptyset$. Let $G_1, ..., G_k$ ($k \geq 1$) be the components of the subgraph induced by $C(G)$. Since $npm(G) = 2$, by Theorem 3(4), it is easily seen that for each i, $1 \leq i \leq k$, $pm(G_i) = 1$. This proves (a) and consequently, the theorem. ■

The above theorem and Theorem 2 actually gives a method for constructing all graphs with exactly two near-perfect matchings.

A claw is an induced subgraph isomorphic to the complete bipartite graph $K_{1,3}$. Junger, Pulleyblank and Reinelt [4] proved the following (see also [5], Chapter 3).

Theorem 5 ([4]). *If a graph G has an odd number of vertices and is claw-free, then G contains a near-perfect matching.*

In the following, we can give a lower bound on the number of near-perfect matchings of claw-free graphs of odd order.

Theorem 6 *Let n be an odd integer with $n \geq 3$. Then every connected claw-free graph of order n has at least $\frac{n+1}{2}$ near-perfect matchings which miss different vertices of G.*

Proof Let G be a connected claw-free graph of order n. Let $\{D(G), A(G), C(G)\}$ be the Gallai-Edmonds decomposition of G, and let $D_1, \ldots, D_t$ be the components of the subgraph induced by $D(G)$. First we assert the following.

Claim 1. $C(G) = \emptyset$.

Suppose $C(G) \neq \emptyset$ and thus there is an edge xy in G with $x \in C(G)$ and $y \in A(G)$. By Theorem 3(3), there exist two vertices z_1 and z_2 in distinct components of the subgraph induced by $D(G)$ such that y is adjacent to z_1 and z_2. But $G[\{x, y, z_1, z_2\}]$ is isomorphic to a claw, a contradiction. This proves Claim 1.

Claim 2. G has at least $|D(G)|$ near-perfect matchings which miss different vertices of $D(G)$.

By Theorem 5, G has a near-perfect matching. So $def(G) = 1$. By Theorem 3(4), $def(G) = t - |A(G)|$, thus $|A(G)| = t - 1$. Suppose $A(G) = \{u_1, \ldots, u_{t-1}\}$. Let $x \in D(G)$. Without loss of generality, we may assume $x \in V(D_t)$. By Theorem 3(1), there is a near-perfect matching M_1 of D_t which miss x. By Theorem 3(3) and Theorem 1, there is a perfect matching M_2 of the graph $G - V(D_t)$, M_2 matches all $u_1, \ldots, u_{t-1}$ with vertices in distinct $D_1, \ldots, D_{t-1}$ and contains a near-perfect matching of each D_i for $i = 1, \ldots, t-1$. Then $F_x = M_1 \cup M_2$ is a near-perfect matching of G which miss x. Now $\{F_x : x \in D(G)\}$ is a set of $|D(G)|$ near-perfect matchings which miss different vertices of $D(G)$. Claim 2 thus follows.

By Claim 1, $C(G) = \emptyset$. Thus $|A(G)| + |D(G)| = n$. On the other hand, By Theorem 3(3), $|D(G)| > |A(G)|$. It follows that $|D(G)| \geq \frac{n+1}{2}$. So by Claim 2, G has at least $\frac{n+1}{2}$ near-perfect matchings which miss different vertices of G. ■

Obviously, for each odd integer n with $n \geq 3$, the path P_n on n vertices serves to show the bound in Theorem 6 is sharp. But the extremal graphs realizing this bound are not unique.

For example, if $n = 7$, the graph H shown in Figure 2 and the path P_7 are two connected claw-free graphs with exactly $\frac{n+1}{2} = 4$ near-perfect matchings.

Figure 2.

Based on Theorem 3, some structural properties of these extremal graphs can be deduced. For a vertex $v \in V(G)$, let $deg_G(v)$ denote its degree in the graph G.

Theorem 7 *Let n be an odd integer with $n \geq 3$. Let G be a connected claw-free graph of order n with exactly $\frac{n+1}{2}$ near-perfect matchings. If $\{D(G), A(G), C(G)\}$ is the Gallai-Edmonds decomposition of G, then*
(a) $C(G) = \emptyset$,
(b) the subgraph induced by $D(G)$ consists of $\frac{n+1}{2}$ singletons,
(c) Suppose G' is the bipartite graph obtained form G by deleting the edges spanned by $A(G)$, then for each $v \in A(G)$, $deg_{G'}(v) = 2$.

Proof (a) follows directly from the proof of Theorem 6.

The proof of Theorem 6 also gives that $npm(G) \geq |D(G)| \geq \frac{n+1}{2}$. It is evident that equality in above relations will hold if and only if $D(G)$ consists of $\frac{n+1}{2}$ singletons. This proves (b).

Suppose v is an arbitrary vertex in $A(G)$. By Theorem 3(3), $deg_{G'}(v) \geq 2$. Suppose $deg_{G'}(v) = r \geq 3$. Let $u_1,...,u_r$ be the neighbors of v in G'. Since the subgraph induced by $D(G)$ consists of $\frac{n+1}{2}$ singletons, G[$\{v,u_1,u_2, u_3\}$] is isomorphic to a claw, a contradiction. Hence $deg_{G'}(v) = 2$. This proves (c).

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