The Generalized k-Fibonacci and k-Lucas Numbers

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Abstract

In this paper we give the generalization $\{G_{k,n}\}_{n\in\mathbb{N}}$ of $k-$Fibonacci and $k-$Lucas numbers. After that, by using this generalization, it has been obtained some new algebraic properties on these numbers.

Keywords :Generalized $k-$Fibonacci numbers, G*eneralized* $k-$Lucas numbers

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1 Introduction

In the last years, there is huge interest of positive science in the application of Fibonacci and Lucas numbers. The well-known Fibonacci $\{F_n\}$, Lucas $\{L_n\}$ and Generalized Fibonacci $\{G_n\}$ sequences are defined for $n \geq 2$ recurrence relations $F_n = F_{n-1} + F_{n-2}$, $(F_0 = 0,\ F_1 = 1)$, $L_n = L_{n-1} + L_{n-2}$, $(L_0 = 2,\ L_1 = 1)$, $G_n = G_{n-1} + G_{n-2}$, $(G_0 = a,\ G_1 = b,\ a, b \in \mathbb{R})$, respectively. In the literature, we can see generalizations of the Fibonacci and Lucas sequence [1-10].

For rich applications of these numbers in science and nature, one can see the citations in [11-13]. For instance, the ratio of two consecutive of these numbers converges to the Golden section $\alpha = \frac{1+\sqrt{5}}{2}$. (The applications of Golden ratio appears in many research areas, particularly in Physics,

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Engineering, Architecture, Nature and Art. Physicists Naschie and Marek-Crnjac gived some examples of the Golden ratio in Theorotical Physics and Physics of High Energy Particles. In [14,15], some new properties of Fibonacci and Lucas numbers with binomial coefficients have been obtained to write Fibonacci and Lucas sequences in a new direct way.

Falcon and Plaza, in [16-19], introduced $k-$Fibonacci sequence $\{F_{k,n}\}_{n=0}^{\infty}$ by using Fibonacci and Pell sequences. Many properties of these numbers were deduced directly from elementary matrix algebra. Furthermore the 3-dimensional $k-$Fibonacci spirals were studied from a geometric point of view.

In this paper, we have defined a generalization $\{G_{k,n}\}_{n\in\mathbb{N}}$ of $k-$Fibonacci and $k-$Lucas numbers given in [6,16]. For these numbers, we obtained generalized Binet formula. In addition to this definition, we have investigated the some new algebraic properties of generalized $k-$Fibonacci, $k-$Lucas numbers.

2 Main result

In this section, we define a generalization $\{G_{k,n}\}_{n\in\mathbb{N}}$ of $k-$Fibonacci and $k-$Lucas numbers. Also, we obtain some equalities related with this generalization. Now, we can give following generalization.

Definition 1 *For any positive real number k, Generalized k-Fibonacci sequence $\{G_{k,n}\}_{n\in\mathbb{N}}$ is defined recurrently by*

$$G_{k,n+1} = kG_{k,n} + G_{k,n-1}, \quad n \geq 1 \tag{1}$$

with initial conditions $G_{k,0} = a, \ G_{k,1} = b \quad (a, b \in \mathbb{R})$.

Generalized k-Fibonacci number is called to each element of Generalized k-Fibonacci sequence. For $a = 0, \ b = 1$ and $a = 2, \ b = 1$, it has been obtained k-Fibonacci sequence and k-Lucas sequence, respectively. Also, Generalized k-Fibonacci sequence have turned into integer number sequence for some special values of k. For example, in Generalized k-Fibonacci sequence $\{G_{k,n}\}_{n\in\mathbb{N}}$;

i. If $k = 1$, then we have Generalized Fibonacci sequence

$$\{G_{1,n}\} = \{a, \ b, \ a+b, \ a+2b, \ 2a+3b, \ \cdots\}.$$

- For $a = 0, \ b = 1$, it has been obtained Fibonacci sequence known as $F_n = \{0, 1, 1, 2, 3, 5, \cdots\}$.
- For $a = 2, \ b = 1$, it has been obtained Lucas sequence known as $L_n = \{2, 1, 3, 4, 7, 11, \cdots\}$.

ii. If $k = 2$, then we have Generalized Pell sequence

$$\{G_{2,n}\} = \{a,\ b,\ a+2b,\ 2a+5b,\ 5a+12b,\ 12a+29b,\ \cdots\}.$$

- For $a = 0,\ b = 1$, it has been obtained Pell sequence known as $P_n = \{0, 1, 2, 5, 12, 29, \cdots\}$.
- For $a = 2,\ b = 2$, it has been obtained Pell-Lucas sequence known as $P_n = \{2, 2, 6, 14, 34, 82, \cdots\}$.

Now, we can write the characteristic equation of (1) as

$$\alpha^2 = k\alpha + 1 \tag{2}$$

Let α_1 and α_2 be the roots of (2) for $\alpha_1 > \alpha_2$. Then, the following identities are hold:

1.
$$\alpha_1 = \frac{k + \sqrt{k^2+4}}{2},\ \alpha_2 = \frac{k - \sqrt{k^2+4}}{2} \tag{3}$$

2. $\alpha_1 < 0 < \alpha_2,\ |\alpha_2| < \alpha_1$

3. $\alpha_1 + \alpha_2 = k,\ \ \alpha_1\alpha_2 = -1,\ \ \alpha_1 - \alpha_2 = \sqrt{k^2+4}.$

During this study, we will firstly denote $G_{k,n}$ as Generalized k-Fibonacci number and define the Generalized Binet Formula. Then we will new formulas and properties related to Generalized k-Fibonacci sequence by using (1).

Lemma 2 *For* $\forall n \in \mathbb{N}$, *the relations hold* $\alpha_1^{n+2} = k\alpha_1^{n+1} + \alpha_1^n$ *and* $\alpha_2^{n+2} = k\alpha_2^{n+1} + \alpha_2^n$ *[2,17].*

Theorem 3 *For* $X = \frac{a+b\alpha_1}{\alpha_1}$ *and* $Y = \frac{a+b\alpha_2}{\alpha_2}$, *we have the Generalized Binet Formula*

$$G_{k,n} = \frac{X\ \alpha_1^n - Y\ \alpha_2^n}{\alpha_1 - \alpha_2}. \tag{4}$$

Proof. Let us use the principle of mathematical induction on n.
For $n = 0$, it is easy to see that

$$\begin{aligned} \frac{1}{\alpha_1 - \alpha_2}\left(X\ \alpha_1^0 - Y\ \alpha_2^0\right) &= \frac{1}{\alpha_1 - \alpha_2}\left[\frac{a+b\alpha_1}{\alpha_1} - \frac{a+b\alpha_2}{\alpha_2}\right] \\ &= \frac{1}{\alpha_1 - \alpha_2}\left[\frac{a\alpha_2 + b\alpha_1\alpha_2 - a\alpha_1 - b\alpha_1\alpha_2}{\alpha_1 r_2}\right] \\ &= \frac{a(\alpha_1 - \alpha_2)}{(\alpha_1 - \alpha_2)} = a = G_{k,0}\ . \end{aligned}$$

Assume that it is true for all positive integers i. That is, $G_{k,i} = \frac{X\ \alpha_1^i - Y\ \alpha_2^i}{\alpha_1 - \alpha_2}$. Therefore, we have to show that it is true for $n = i+1$. From the definition of Generalized k-Fibonacci number, (4) and Lemma 2, we can write

$$\begin{aligned} G_{k,i+1} &= kG_{k,i} + G_{k,i-1} \\ &= \frac{k}{\alpha_1 - \alpha_2}\left[X\ \alpha_1^i - Y\ \alpha_2^i\right] + \frac{1}{\alpha_1 - \alpha_2}\left[X\ \alpha_1^{i-1} - Y\ \alpha_2^{i-1}\right] \\ &= \frac{X\ \alpha_1^{i-1}\left[k\alpha_1 + 1\right] - Y\ \alpha_2^{i-1}\left[k\alpha_2 + 1\right]}{\alpha_1 - \alpha_2} \\ &= \frac{X\ \alpha_1^{i+1} - Y\ \alpha_2^{i+1}}{\alpha_1 - \alpha_2} \end{aligned}$$

which ends up the induction. Therefore we have the required formulate on $G_{k,n} = \frac{X\ \alpha_1^n - Y\ \alpha_2^n}{\alpha_1 - \alpha_2}$. ■

In the following theorem, it is given the ratio of two consecutive of Generalized k-Fibonacci numbers which is positive root α_1 of characteristic equation of (1). It is interesting that the well-known golden ratio $\alpha = \frac{1+\sqrt{5}}{2}$ is clear for $k = 1$ in (1).

Theorem 4 *We have the relation* $\lim\limits_{n\to\infty} \frac{G_{k,n}}{G_{k,n-1}} = \alpha_1$.

Proof. From $|\alpha_2| < \alpha_1$, we have $\lim\limits_{n\to\infty}\left(\frac{\alpha_2}{\alpha_1}\right)^n = 0$. By considering (4), we write

$$\begin{aligned} \lim_{n\to\infty} \frac{G_{k,n}}{G_{k,n-1}} &= \lim_{n\to\infty} \frac{\left(\frac{X\alpha_1^n - Y\alpha_2^n}{\alpha_1 - \alpha_2}\right)}{\left(\frac{X\alpha_1^{n-1} - Y\alpha_2^{n-1}}{\alpha_1 - \alpha_2}\right)} = \lim_{n\to\infty} \frac{\alpha_1^n\left[X - Y\left(\frac{\alpha_2}{\alpha_1}\right)^n\right]}{\alpha_1^{n-1}\left[X - Y\left(\frac{\alpha_2}{\alpha_1}\right)^{n-1}\right]} \\ &= \lim_{n\to\infty} \frac{X - Y\left(\frac{\alpha_2}{\alpha_1}\right)^n}{\frac{X}{\alpha_1} - Y\left(\frac{\alpha_2}{\alpha_1}\right)^n \frac{1}{\alpha_1}} = \alpha_1. \end{aligned}$$

■

In the following theorem, it has been given the sum of Generalized k-Fibonacci numbers.

Theorem 5 *We have the relation* $k\sum\limits_{i=1}^{n} G_{k,i} = G_{k,n+1} + G_{k,n} - (a+b)$.

Proof. Using the summation symbol in (1), we can write $\sum\limits_{i=1}^{n} G_{k,i+1} = k\sum\limits_{i=1}^{n} G_{k,i} + \sum\limits_{i=1}^{n} G_{k,i-1}$. If the last equation is arranged, then we obtain

$$G_{k,2} + \cdots + G_{k,n} + G_{k,n+1} = k\sum_{i=1}^{n} G_{k,i} + \left(G_{k,0} + G_{k,1} \cdots + G_{k,n-2} + G_{k,n-1}\right).$$

Consequently, if we make the sufficient simplifications, we have the formula $k\sum_{i=1}^{n} G_{k,i} = G_{k,n} + G_{k,n+1} - (a+b)$. ■

We can give the following generalized Cassini's or Simson's identity for generalized k-Fibonacci and k-Lucas numbers.

Theorem 6 *For $G_{k,0} = a$, $G_{k,1} = b$, we have the relation $G_{k,n-1}G_{k,n+1} - G_{k,n}^2 = (-1)^{n-1}\left[abk + a^2 - b^2\right]$.*

Proof. Let us consider the system

$$\begin{aligned} G_{k,n}x + G_{k,n-1}y &= G_{k,n+1} \\ G_{k,n+1}x + G_{k,n}y &= G_{k,n+2}. \end{aligned}$$

If we denote

$$A = \begin{pmatrix} G_{k,n} & G_{k,n-1} \\ G_{k,n+1} & G_{k,n} \end{pmatrix}, \quad u = \begin{pmatrix} x \\ y \end{pmatrix}, \quad b = \begin{pmatrix} G_{k,n+1} \\ G_{k,n+2} \end{pmatrix},$$

then this system can be written in matrix form $Au = b$. It is obvious that the solution of this system is $\binom{x}{y} = \binom{k}{1}$. Also, the determinant of matrix A is

$$|A| = \begin{vmatrix} G_{k,n} & G_{k,n-1} \\ G_{k,n+1} & G_{k,n} \end{vmatrix} = G_{k,n}^2 - G_{k,n-1}G_{k,n+1} \neq 0$$

By using the cramer method, we have the following solution

$$y = \frac{\begin{vmatrix} G_{k,n} & G_{k,n+1} \\ G_{k,n+1} & G_{k,n+2} \end{vmatrix}}{\begin{vmatrix} G_{k,n} & G_{k,n-1} \\ G_{k,n+1} & G_{k,n} \end{vmatrix}} = 1.$$

If this solution is arranged, then we can write the identity $G_{k,n+2}G_{k,n} - G_{k,n+1}^2 = -\left(G_{k,n-1}G_{k,n+1} - G_{k,n}^2\right)$. Let us take

$$P_{k,n} = G_{k,n+2}G_{k,n} - G_{k,n+1}^2. \tag{5}$$

Therefore, we have

$$P_{k,n} = -P_{k,n-1}. \tag{6}$$

To prove the correction of (6), we use the iteration method. For $n = 1$, it is clearly seen that

$$\begin{aligned} P_{k,1} &= -P_{k,0} = -\left(G_{k,2}G_{k,0} - G_{k,1}^2\right) \\ &= -\left((bk+a)\,a - b^2\right). \end{aligned}$$

Hence, by iterating this procedure, we can write the formula $P_{k,n} = (-1)^n \left[abk + a^2 - b^2\right]$ or using the equation (5), we write $G_{k,n-1}G_{k,n+1} - G_{k,n}^2 = (-1)^{n-1}\left[abk + a^2 - b^2\right]$ as required. ■

The following theorem gives the sum of even (and odd) Generalized k-Fibonacci numbers.

Theorem 7 *The following equalities hold for Generalized k-Fibonacci even (and odd) numbers:*

i) $\sum\limits_{i=1}^{n} G_{k,2i} = \frac{1}{k}\left[G_{k,2n+1} - b\right].$

ii) $\sum\limits_{i=1}^{n} G_{k,2i+1} = \frac{1}{k}\left[G_{k,2n+2} - a - bk\right].$

Proof.

i) By using Generalized Binet Formula given in (4), we write

$$\begin{aligned}
\sum_{i=1}^{n} G_{k,2i} &= \sum_{i=1}^{n} \frac{X\alpha_1^{2i} - Y\alpha_2^{2i}}{\alpha_1 - \alpha_2} = \frac{1}{\alpha_1 - \alpha_2}\left[X\sum_{i=1}^{n}\alpha_1^{2i} - Y\sum_{i=1}^{n}\alpha_2^{2i}\right] \\
&= \frac{1}{\alpha_1 - \alpha_2}\left[X\alpha_1^2\left(\frac{\alpha_1^{2n} - 1}{\alpha_1^2 - 1}\right) - Y\alpha_2^2\left(\frac{\alpha_2^{2n} - 1}{\alpha_2^2 - 1}\right)\right] \\
&= \frac{1}{\alpha_1 - \alpha_2}\left[\frac{\left(X\,\alpha_1^{2n+2} - X\alpha_1^2\right)\left(\alpha_2^2 - 1\right)}{\left(\alpha_1^2 - 1\right)\left(\alpha_2^2 - 1\right)}\right] \\
&\quad + \frac{1}{\alpha_1 - \alpha_2}\left[\frac{\left(-Y\,\alpha_2^{2n+2} + Y\alpha_2^2\right)\left(\alpha_1^2 - 1\right)}{\left(\alpha_1^2 - 1\right)\left(\alpha_2^2 - 1\right)}\right].
\end{aligned}$$

Hence, by rearranging the last equality, we can write the formula

$$\begin{aligned}
\sum_{i=1}^{n} G_{k,2i} &= \frac{-1}{k^2}\left[-\left(\frac{X - Y}{\alpha_1 - \alpha_2}\right) - \left(\frac{X\alpha_1^{2n+2} - Y\alpha_2^{2n+2}}{\alpha_1 - \alpha_2}\right)\right] + \\
&\quad \frac{-1}{k^2}\left[\left(\frac{X\alpha_1^2 - Y\alpha_2^2}{\alpha_1 - \alpha_2}\right) + (\alpha_1\alpha_2)^2\left(\frac{X\alpha_1^{2n} - Y\alpha_2^{2n}}{\alpha_1 - \alpha_2}\right)\right] \\
&= \frac{-1}{k^2}\left[-G_{k,0} - G_{k,2n+2} + G_{k,2} + G_{k,2n}\right] \\
&= \frac{1}{k^2}\left[-bk + kG_{k,2n+1}\right] = \frac{1}{k}\left[G_{k,2n+1} - b\right].
\end{aligned}$$

ii) By applying the same method as in the proof of *i*), the proof can be seen easily.

■

By using the above theorems, the following results which are well-known in literature can be written.

Corollary 8 *(a) For $a = 0$, $b = 1$, we can write $G_{k,n} = F_{k,n}$ and it is obtained the k-Fibonacci identities in [17-19] as given:*

- $F_{k,n} = \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}$, $\lim_{n\to\infty} \frac{F_{k,n}}{F_{k,n-1}} = \alpha_1$, $F_{k,n-1}F_{k,n+1} - F_{k,n}^2 = (-1)^n$,
- $\sum_{i=1}^{n} F_{k,i} = \frac{1}{k}\left(F_{k,n} + F_{k,n+1} - 1\right)$,
- $\sum_{i=1}^{n} F_{k,2i} = \frac{1}{k}\left(F_{k,2n+1} - 1\right)$, $\sum_{i=1}^{n} F_{k,2i+1} = \frac{1}{k}(F_{k,2n+2} - k)$.

(b) For $a = 2$ and $b = 1$, we can write $G_{k,n} = L_{k,n}$ and it is obtained the k-Lucas identities in [6] as given:

- $\lim_{n\to\infty} \frac{L_{k,n}}{L_{k,n-1}} = \alpha_1$, $L_{k,n-1}L_{k,n+1} - L_{k,n}^2 = (-1)^{n-1}(2k+3)$,
- $L_{k,n} = \frac{X\alpha_1^n - Y\alpha_2^n}{\alpha_1 - \alpha_2}$, *where* $X = \frac{2+\alpha_1}{\alpha_1}$ *and* $Y = \frac{2+\alpha_2}{\alpha_2}$,
- $\sum_{i=1}^{n} L_{k,i} = \frac{1}{k}\left(L_{k,n} + L_{k,n+1} - 3\right)$,
- $\sum_{i=1}^{n} L_{k,2i} = \frac{1}{k}\left(L_{k,2n+1} - 1\right)$, $\sum_{i=1}^{n} L_{k,2i+1} = \frac{1}{k}\left(L_{k,2n+2} - (k+2)\right)$.

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