Generalized Padovan numbers, Perrin numbers and maximal k-independent sets in graphs

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ABSTRACT: In this paper we give generalizations of Padovan numbers and Perrin numbers. We apply this generalizations for counting of special subsets of the set of n integers. Next we give their graph representations with respect to the number of maximal k-independent sets in graphs.

1 Introduction

Let G be a simple, undirected, connected graph with the vertex set $V(G)$ and the edge set $E(G)$. By $d_G(x,y)$ we denote the distance between vertices x and y in G. Let $\mathbb{P}_n$ and $\mathbb{C}_n$ denote a path and a cycle on n vertices, respectively. Let $k \geq 2$ be integer. A subset $S \subseteq V(G)$ is a k-independent set of G if for any two distinct vertices $x, y \in S$, $d_G(x,y) \geq k$. Moreover a subset containing only one vertex and the empty set also are k-independent. If $k = 2$, then the definition reduces to the definition of an independent set in the classical sense. Let $NI_k(G)$ denote the number of k-independent sets in G and for $k = 2$, $NI_2(G) = NI(G)$. The parameter $NI(G)$ was studied by Prodinger and Tichy, see [5] and this paper gave an impetus for counting of independent sets in graphs. They proved that $NI(\mathbb{P}_n) = F_{n+1}$ and $NI(\mathbb{C}_n) = L_n$, where F_n and L_n are the Fibonacci and the Lucas numbers, respectively, defined recursively by $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$ and $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$, for $n \geq 2$.

In [3] more generalized concept was introduced, namely the generalized Fibonacci numbers and the generalized Lucas numbers which gives the numbers of all k-independent sets in special graphs. In [9] the generalizations of the Pell numbers, the Pell-Lucas numbers and the Tribonacci numbers with respect to the numbers of k-independent sets in graphs were given.

We say that the k-independent set S of the graph G is maximal if for any $x \in V(G) \setminus S$, $S \cup \{x\}$ is not k-independent. By $NMI_k(G)$ we will denote the total number of maximal k-independent sets S in the graph G, $NMI_2(G) = NMI(G)$.

The literature includes many papers dealing with the theory of counting of independent sets in graphs. In particular characterization of extremal trees with some independence properties has been considered in a number of papers for instance [2, 7, 8]. The problem of determining the maximum value of $NMI(T)$ was solved by Wilf [7]. The numbers $NMI_k(G)$ for some graph products were studied in [6].

The Padovan numbers have different applications, also in several enumeration problems, see [4], where the authors present an interesting and special connections between Baxter permutations and the Padovan numbers. Also in [1] enumeration of independent sets of some classes of regular graphs is studied. These papers motivate the study of maximal k-independent sets and their connections with the Padovan numbers and the Perrin numbers.

The Padovan numbers $PV(n)$ are defined by the recurrence relation $PV(0) = PV(1) = PV(2) = 1$ and for $n \geq 3$, $PV(n) = PV(n-2)+PV(n-3)$. The Perrin numbers $Pr(n)$ are defined by the recurrence relations $Pr(0) = 3$, $Pr(1) = 0$, $Pr(2) = 2$ and for $n \geq 3$, $Pr(n) = Pr(n-2) + Pr(n-3)$.

The Padovan numbers have the graph interpretation with respect to the number of maximal independent sets in graphs. Let $NMI_L(G)$ be the number of maximal independent sets including the set of pendant vertices. It has been proved

Theorem 1 *[8] Let T be an n-vertex tree with $n \geq 3$. Then $NMI_L(T) \leq PV(n-3)$. Moreover the equality occurs if $T = P_n$.*

In this paper we give the generalizations of the Padovan numbers, the Perrin numbers and next we give the graph interpretation with respect to the numbers of maximal k-independent sets including the set of pendant vertices as a subset.

2 Generalizations

Let $X = \{1, 2, ..., n\}$, $n \geq 3$ be the set of n integers. Let $k \geq 2$. Let $Y \subset X$ such that $\{1, n\} \subseteq Y$ and

(i) $|Y| = t$, for fixed $t \geq 2$ and

(ii) for each $i, j \in Y$, $|i - j| \geq k$ and

(iii) for each $i \notin Y$ there is $j \in Y$ such that $|i - j| < k$.

A subset Y we will call a maximal k-subset of X, for an arbitrary $t \geq 2$.

By $pv(n, k, t)$ we denote the number of all maximal k-subset Y having exactly t elements and further let $PV(n, k) = \sum_{t \geq 2} pv(n, k, t)$. For $k = 2$ we put $pv(n, k, t) = pv(n, t)$ and $PV(n, 2) = PV(n)$.

Theorem 2 *Let $n \geq 3$, $k \geq 2$, $t \geq 2$ be integers.*
If $n \leq k$ or $n > t + (t-1)(2k-2)$ or $n < 1 + (t-1)k$, then for $k \geq 2$ $pv(n, k, t) = 0$.
If $k + 1 \leq n \leq 2k$, then $pv(n, k, 2) = 1$. For $t \geq 3$, $pv(2k+1, k, 3) = 1$ and for $n \geq 2k + 2$ we have $pv(n, k, t) = pv(n-k, k, t-1) + pv(n-(k+1), k, t-1) + ... + pv(n-(2k-1), k, t-1)$.

P R O O F: The initial conditions are obvious. It remains to consider the case that $n \geq 2k+2$ and $t \geq 3$. Let $Y \subset X$. We recall that $\{1, n\} \subset Y$ and Y is a maximal k-subset having t elements. Because $n \in Y$, then by $|i-j| \geq k$ it is obvious that $n - r \notin Y$, for $r = 1, 2, \ldots, k-1$. This means that others $(t-1)$ integers (different from n) belonging to Y must be chosen among $n - k$ integers from X. From the fact that Y satisfies (iii) immediately follows that exactly one vertex from the set $\{n - s; s = k, \ldots, 2k-1\}$ belongs to Y. Hence $pv(n, k, t) = pv(n-k, k, t-1) + pv(n-(k+1), k, t-1) + ... + pv(n-(2k-1), k, t-1)$, which ends the proof. □

Theorem 3 *Let $n \geq 3$, $k \geq 2$ be integers. If $n < k+1$, then $PV(n, k) = 0$. If $k + 1 \leq n \leq 2k$, then $PV(n, k) = 1$ and for $n \geq 2k+1$ we have the recurrence relation*
$PV(n, k) = PV(n-k, k) + PV(n-(k-1), k) + ... + PV(n-(2k-1), k)$.

P R O O F: From Theorem 2 we have that if $k + 1 \leq n \leq 2k$, then $PV(n, k) = \sum_{t \geq 2} pv(n, k, t) = pv(n, k, 2) = 1$. If $n = 2k+1$ then $Pv(n, k) = \sum_{t \geq 2} pv(n, k, t) = pv(n, k, 2) + pv(n, k, 3) = 1$.
Let $n \geq 2k+2$. Then Theorem 2 gives $pv(n, k, 2) = 0$. Hence $PV(n, k) = \sum_{t \geq 2} pv(n, k, t) = \sum_{t \geq 3} pv(n, k, t) = \sum_{t \geq 3} (pv(n-k, k, t-1) + pv(n-(k+1), k, t-1) + ... + pv(n-(2k-1), k, t-1)) = \sum_{t \geq 2} (pv(n-k, k, t) + pv(n-(k+1), k, t) + ... + pv(n-(2k-1), k, t)) = PV(n-k, k) + PV(n-(k+1), k) + ... + PV(n-(2k-1), k)$.
Thus the theorem is proved. □

The numbers $PV(n, k)$ we will called the generalized Padovan numbers. If $k = 2$ and $n \geq 3$, then numbers $PV(n, 2) = PV(n)$ create the Padovan sequence $a_n = a_{n-2} + a_{n-3}$ for $n \geq 6$ with the initial conditions $a_3 = a_4 = a_5 = 1$.

The generalized Padovan numbers have the graph representations with respect to the number of maximal k-independent sets including the set of pendant vertices in graphs.

The set X can be regarded as the vertex set of the graph P_n where vertices from $V(P_n)$ are labeled by integers belonging to X. However

$\{i,j\} \in E(P_n)$ if i and j are consecutive. Thus the number $PV(n,k)$ for $n \geq 3$, $k \geq 2$ is equal to the total number of subset $S \subset V(G)$ such that S is a maximal k-independent set of P_n including end vertices as a subset.

Now we have applied the numbers $PV(n,k)$ for counting of others special subsets of the set X.

Let $X = \{1,2,\ldots,n\}$, $n \geq 3$ be the set of n integers. Let $k \geq 2$. Let $I \subset X$ such that

(i) $|I| = t$, for fixed $t \geq 1$

(ii) for each $i,j \in I$, $k \leq |i-j| \leq n-k$ and

(iii) I is a maximal with respect to the set inclusion.

By $pr(n,k,t)$ we denote the number of all subsets I having t elements and $Pr(n,k) = \sum_{t\geq 1} pr(n,k,t)$. For $k=2$ we put $Pr(n,2) = Pr(n)$.

Theorem 4 *Let $n \geq 3$, $k \geq 2$ be integers. If $n < tk$ for $t > 1$ or $n > t(2k+1)$ for $t \geq 2$, then $pr(n,k,t) = 0$. If $3 \leq n \leq 2k-1$, then $pr(n,k,1) = n$. Let $n \geq 2k$. Then $pr(2k,k,2) = k$ and for $n \geq 2k+1$ $pr(n,k,t) = pv(n-(2k-2),k,t) + (2k-2)pv(n+1,k,t+1) + (k-2)pv(n-(k-1),k,t) + (k-3)pv(n-k,k,t) + \ldots + pv(n-(2k-4),k,t)$.*

P R O O F: The statement is easy to verified for $n \leq 2k$. Let $n \geq 2k+1$. Assume that $I \subset X$ be a t elements subset of X which satisfies conditions (ii) and (iii).

Let $X \supset I^* = \{i; 1,\ldots,2k-2\}$. We consider two cases:

1. $I \cap I^* = \emptyset$.

Then $n \in I$ and $2k-1 \in I$, in otherwise I does not satisfy (iii). Moreover for each $p,q \in X \setminus I^*$, $|p-q| \leq n-k$. This implies that I is t elements k-subset of the set $X^* = X \setminus \{1,\ldots,(2k-2)\}$. Hence there are exactly $pv(n-(2k-2),k,t)$ subsets I such that $I \cap I^* = \emptyset$.

2. $I \cap I^* \neq \emptyset$.

Clearly $|I \cap I^*| \leq 2$. Hence we distinguish two possibilities

2.1. $|I \cap I^*| = 1$.

Let $i \in I$, $1 \leq i \leq 2k-2$. Then counting of subsets I is equivalent to counting of k-subsets of the set $X' = \{1,\ldots,n+1\}$ having $t+1$ elements. By previous considerations the number of k-subsets of $X' = \{1,\ldots,n+1\}$ having $t+1$ elements is equal to $pv(n+1,k,t+1)$. Since the integer i we can choose on $2k-2$ ways, so we have $(2k-2)pv(n+1,k,t+1)$ subsets I in this subcase.

2.2. $|I \cap I^*| = 2$, $k \geq 3$.

Let $i,j \in I$, where $j \neq i$ and $1 \leq i,j \leq 2k-2$. If $i=1$, then j can be chosen from the set $\{k+1,\ldots,2k-2\}$. Consequently we have $pv(n-(k-$

$1), k, t) + pv(n-k, k, t) + ... + pv(n-(2k-4), k, t)$ subsets I with t elements. If $i = 2$, then j can be chosen from the set $\{k+2, ..., 2k-2\}$ and we have $pv(n-(k-1), k, t) + ... + pv(n-(k+(2k-3), k, t)$ subsets I. Considering analogously step by step we obtain that if $i = k-2$, then $j = 2k-2$ and we have $pv(n-(k-1), k, t)$ subsets I.

Hence this case gives

$(k-2)pv(n-(k-1), k, t) + (k-3)pv(n-k, k, t) + ... + pv(n-(2k-4), k, t)$.

Finally from the above cases we obtain that $pr(n, k, t) = pv(n-(2k-2), k, t) + (2k-2)pv(n+1, k, t+1) + (k-2)pv(n-(k-1), k, t) + (k-3)pv(n-k, k, t) + ... + pv(n-(2k-4), k, t)$, which ends the proof. □

Using the same method as in Theorem 3 we can prove:

Theorem 5 *Let $n \geq 3$, $k \geq 2$ be integers. If $n \leq 2k-1$, then $Pr(n, k) = n$. If $n = 2k$, then $Pr(2k, k) = k$ and for $n \geq 2k+1$ we have* $Pr(n, k) = PV(n-(2k-2), k) + (k-2)PV(n+1, k) + (k-2)PV(n-(k-1), k) + (k-3)PV(n-k, k) + ... + PV(n-(2k-4), k)$.

The numbers $Pr(n, k)$ we will called the generalized Perrin numbers. If $k = 2$ and $n \geq 3$, then numbers $Pr(n, 2)$ create the Perrin sequence.

The generalized Perrin numbers have the graph representations with respect to the number of maximal k-independent sets in a graph C_n.

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