A note on the spin-embedding of the dual polar space $DQ^-(2n+1,\mathbb{K})$

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Abstract

In [6], Cooperstein and Shult showed that the dual polar space $DQ^-(2n+1,\mathbb{K})$, $\mathbb{K}=\mathbb{F}_q$, admits a full projective embedding into the projective space $\mathrm{PG}(2^n-1,\mathbb{K}')$, $\mathbb{K}'=\mathbb{F}_{q^2}$. They also showed that this embedding is absolutely universal. The proof in [6] makes use of counting arguments and group representation theory. Because of the use of counting arguments, the proof cannot be extended automatically to the infinite case. In this note, we shall give a different proof of their results, thus showing that their conclusions remain valid for infinite fields as well. We shall also show that the above-mentioned embedding of $DQ^-(2n+1,\mathbb{K})$ into $\mathrm{PG}(2^n-1,\mathbb{K}')$ is polarized.

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1 Introduction

1.1 Basic Definitions

Let $\Gamma=(P,L,\mathrm{I})$ be a *partial linear space*, i.e. a rank 2 geometry with point-set P, line-set L and incidence relation $\mathrm{I}\subseteq P\times L$ for which every line is incident with at least two points and every two distinct points are incident with at most 1 line. A *subspace* of Γ is a set of points which contains all the points of a line as soon as it contains at least two points of it. If X is a nonempty set of points, then $\langle X\rangle_\Gamma$ denotes the smallest subspace of Γ containing the set X. The minimal number $gr(\Gamma):=\min\{|X|\,:\,X\subseteq$

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P and $\langle X\rangle_\Gamma = P\}$ of points which are necessary to generate the whole point-set P is called the *generating rank* of Γ.

A *full embedding* e of Γ into a projective space Σ is an injective mapping e from P to the point-set of Σ satisfying: (i) $\langle e(P)\rangle_\Sigma = \Sigma$; (ii) $e(L) := \{e(x) \mid x \in L\}$ is a line of Σ for every line L of Γ. The numbers $\dim(\Sigma)$ and $\dim(\Sigma)+1$ are respectively called the *projective dimension* and the *vector dimension* of the embedding e. The maximal dimension of a vector space V for which Γ has a full embedding into $PG(V)$ is called the *embedding rank* of Γ and is denoted by $er(\Gamma)$. Certainly, $er(\Gamma)$ is only defined when Γ admits a full embedding, in which case it holds that $er(\Gamma) \leq gr(\Gamma)$. If e is a full embedding of Γ into a projective space Σ, then for every hyperplane α of Σ, $e^{-1}(e(P)\cap\alpha)$ is a hyperplane of Γ. We say that the hyperplane $e^{-1}(e(P)\cap\alpha)$ *arises from the embedding* e.

Two embeddings $e_1 : \Gamma \to \Sigma_1$ and $e_2 : \Gamma \to \Sigma_2$ of Γ are called *isomorphic* ($e_1 \cong e_2$) if there exists an isomorphism $f : \Sigma_1 \to \Sigma_2$ such that $e_2 = f \circ e_1$. If $e : \Gamma \to \Sigma$ is a full embedding of Γ and if U is a subspace of Σ satisfying (C1): $\langle U, e(p)\rangle_\Sigma \neq U$ for every point p of Γ, (C2): $\langle U, e(p_1)\rangle_\Sigma \neq \langle U, e(p_2)\rangle_\Sigma$ for any two distinct points p_1 and p_2 of Γ, then there exists a full embedding e/U of Γ into the quotient space Σ/U mapping each point p of Γ to $\langle U, e(p)\rangle_\Sigma$. If $e_1 : \Gamma \to \Sigma_1$ and $e_2 : \Gamma \to \Sigma_2$ are two embeddings, then we say that $e_1 \geq e_2$ if there exists a subspace U in Σ_1 satisfying (C1), (C2) and $e_1/U \cong e_2$. If $e : \Gamma \to \Sigma$ is a full embedding of Γ, then by Ronan [11], there exists a unique (up to isomorphism) full embedding $\widetilde{e} : \Gamma \to \widetilde{\Sigma}$ satisfying (i) $\widetilde{e} \geq e$, (ii) if $e' \geq e$ for some embedding e' of Γ, then $\widetilde{e} \geq e'$. We say that $\widetilde{e}$ is *universal relative to* e. If $\widetilde{e} \cong e$ for some full embedding e of Γ, then we say that e is *relatively universal*. A full embedding e of Γ is called *absolutely universal* if it is universal relative to any full embedding of Γ defined over the same division ring as e. Kasikova and Shult [10] gave sufficient conditions for an embeddable geometry to have an absolutely universal embedding.

The problem of determining generating sets of small size for a given geometry Γ is very important for embedding problems. Suppose X is a finite generating set of a geometry Γ such that there exists a full embedding e of Γ into a projective space $PG(V)$ with $\dim(V) = |X|$. Then since $|X| = \dim(V) \leq er(\Gamma) \leq gr(\Gamma) \leq |X|$, we necessarily have $er(\Gamma) = gr(\Gamma) = |X|$. It follows that e is a relatively universal embedding. If moreover the conditions of Kasikova and Shult are satisfied, then we can conclude that e is absolutely universal.

1.2 Dual polar spaces

Let Π be a non-degenerate polar space of rank $n \geq 2$. With Π there is associated a point-line geometry Δ whose points are the maximal singular subspaces of Π, whose lines are the next-to-maximal singular subspaces of Π and whose incidence relation is reverse containment. We call Δ a *dual polar space* (Cameron [2]).

If x and y are two points of Δ, then $\mathrm{d}(x, y)$ denotes the distance between x and y in the point or collinearity graph of Δ. Every convex subspace of Δ consists of the maximal singular subspaces through a given subspace of Π. The maximal distance between two points of a convex subspace A of Δ is called the *diameter* of A. The convex subspaces of diameter 2, respectively $n-1$, are called the *quads*, respectively *maxes*, of Δ. Every dual polar space is an example of a *near polygon* (Shult and Yanushka [14]; De Bruyn [7]). This means that for every point x and every line L, there exists a unique point $\pi_L(x)$ on L nearest to x. More generally, the following property holds in every dual polar space Δ: if x is a point and A is a convex subspace, then A contains a unique point $\pi_A(x)$ nearest to x and $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_A(x)) + \mathrm{d}(\pi_A(x), y)$ for every point y of A. We call $\pi_A(x)$ *the projection of* x *onto* A. If M is a max of Δ, then $\mathrm{d}(x, M) \leq 1$ for every point x of Δ.

Since a dual polar space Δ is a near polygon, the set H_x of points of Δ at non-maximal distance from a given point x is a hyperplane of Δ. We call H_x the *singular hyperplane of* Δ *with deepest point* x. By Shult [13, Lemma 6.1(ii)], every hyperplane of a thick dual polar space Δ is a maximal subspace.

A full embedding of a dual polar space is called *polarized* if every singular hyperplane arises from it. If e is a full polarized embedding of a thick dual polar space Δ into a projective space Σ, then for every point x of Δ, $\langle e(H_x)\rangle_\Sigma$ is a hyperplane of Σ (recall that H_x is a maximal subspace of Δ). If e is a full embedding of a thick generalized quadrangle Q into a projective space Σ, then the underlying division ring of Σ is uniquely determined by Q by Tits [15, 8.6]. In view of the existence of quads in dual polar spaces, a similar conclusion holds for full embeddings of thick dual polar spaces of rank at least 2. By Kasikova and Shult [10, 4.6], every full embedding of a thick dual polar space admits the absolutely universal embedding. By the above we know that the underlying division ring of this absolutely universal embedding space is uniquely determined by Δ; in other words: Δ admits essentially only one absolutely universal embedding.

1.3 The results

Let $\mathbb{K}$ and $\mathbb{K}'$ be fields such that $\mathbb{K}'$ is a quadratic Galois extension of $\mathbb{K}$. Let θ denote the unique nontrivial element in the Galois group $Gal(\mathbb{K}'/\mathbb{K})$ and let $n \in \mathbb{N} \setminus \{0, 1\}$. For all $i, j \in \{0, \ldots, 2n+1\}$ with $i \leq j$, let $a_{ij} \in \mathbb{K}$ such that $q(\overline{X}) = \sum_{0 \leq i \leq j \leq 2n+1} a_{ij} X_i X_j$ is a quadratic form defining a quadric $Q^-(2n+1, \mathbb{K})$ of Witt-index n in $\mathrm{PG}(2n+1, \mathbb{K})$ and a quadric

$Q^+(2n+1, \mathbb{K}')$ of Witt-index $n+1$ in $PG(2n+1, \mathbb{K}')$. Let $\mathcal{M}^+$ and $\mathcal{M}^-$ denote the two families of maximal subspaces of $Q^+(2n+1, \mathbb{K}')$. Recall that two maximal subspaces of $Q^+(2n+1, \mathbb{K}')$ belong to the same family if and only if they intersect in a subspace of even co-dimension.

Let $DQ^-(2n+1, \mathbb{K})$ denote the dual polar space associated with the quadric $Q^-(2n+1, \mathbb{K})$ (regarded as a polar space of rank n). For every $\epsilon \in \{+, -\}$, let $\mathcal{S}^\epsilon$ denote the following point-line geometry:

- the points of $\mathcal{S}^\epsilon$ are the elements of $\mathcal{M}^\epsilon$;
- the lines of $\mathcal{S}^\epsilon$ are the $(n-2)$-dimensional subspaces of $Q^+(2n+1, \mathbb{K}')$;
- incidence is reverse containment.

The geometries $\mathcal{S}^+$ and $\mathcal{S}^-$ are isomorphic and are called the *half-spin geometries* for $Q^+(2n+1, \mathbb{K}')$. The geometry $\mathcal{S}^+$ admits a full embedding e' into the projective space $PG(2^n-1, \mathbb{K}')$ which is called the *spin-embedding* of $\mathcal{S}^+$. We refer to Chevalley [5] or Buekenhout & Cameron [1] for more details about the construction of this embedding. For every maximal subspace α of $Q^-(2n+1, \mathbb{K})$, there exists a unique element of $\mathcal{M}^+$ containing all points of α. We denote this element by $\phi(\alpha)$. Then $e := e' \circ \phi$ is a map from the set of points of $DQ^-(2n+1, \mathbb{K})$ to the set of points of $PG(2^n-1, \mathbb{K}')$.

Cooperstein and Shult [6, Theorem 2.4] showed that if $\mathbb{K}$ and $\mathbb{K}'$ are finite (so $\mathbb{K} = \mathbb{F}_q$ and $\mathbb{K}' = \mathbb{F}_{q^2}$ for some prime power q), then e realizes a full projective embedding of $DQ^-(2n+1, \mathbb{K})$ into $PG(2^n-1, \mathbb{K}')$. The proof in [6] makes use of counting arguments and group representation theory. Because of the use of counting arguments, the proof cannot be extended automatically to the infinite case. In Section 2 we shall give a different proof of the fact that e is a full projective embedding. This proof does not make use of counting arguments nor of group representation theory and is valid for infinite fields as well.

Theorem 1.1 (Section 2) *The map e realizes a full projective embedding of $DQ^-(2n+1, \mathbb{K})$ into $PG(2^n-1, \mathbb{K}')$, regardless of whether $\mathbb{K}, \mathbb{K}'$ are finite or not.*

Definition. The full embedding e is called the *spin-embedding* of the dual polar space $DQ^-(2n+1, \mathbb{K})$.

Without the knowledge that the spin-embedding of $DQ^-(2n+1, \mathbb{K})$ also exists when $\mathbb{K}$ is infinite, one is a priori obliged to state any result regarding this embedding for the finite case only, while it might very good be possible that that particular result also holds in the infinite case (maybe even with the same proof). So, future results might benefit from our discussion here, as it might be possible that these results can be stated in a more general form.

Almost no results are known which describe the structure of general full embeddings of dual polar spaces. This situation drastically changes if one assumes that the embeddings under consideration have to be polarized, see e.g. the papers by Cardinali & De Bruyn [3], Cardinali, De Bruyn & Pasini [4] and De Bruyn & Pasini [9]. In Section 2, we show that the spin-embedding of $DQ^-(2n+1,\mathbb{K})$ is polarized.

Theorem 1.2 (Section 2) *The spin-embedding of $DQ^-(2n+1,\mathbb{K})$ is polarized.*

Cooperstein and Shult [6, Theorem 2.3] showed that if $\mathbb{K}$ and $\mathbb{K}'$ are finite, then the dual polar space $DQ^-(2n+1,\mathbb{K})$ can be generated by 2^n points. Again the proof in [6] makes use of counting arguments. In Section 3, we shall give a different proof which shows that this result also holds for infinite fields.

Theorem 1.3 (Section 3) *The dual polar space $DQ^-(2n+1,\mathbb{K})$ can be generated by 2^n points, regardless of whether $\mathbb{K}$ is finite or not.*

Recall that $er(DQ^-(2n+1,\mathbb{K})) \leq gr(DQ^-(2n+1,\mathbb{K}))$. Now, $er(DQ^-(2n+1,\mathbb{K})) \geq 2^n$ by Theorem 1.1 and $gr(DQ^-(2n+1,\mathbb{K})) \leq 2^n$ by Theorem 1.3. Hence, we can say the following:

Corollary 1.4 (1) *The generating and embedding ranks of $DQ^-(2n+1,\mathbb{K})$ are equal to 2^n.*

(2) *The spin-embedding of $DQ^-(2n+1,\mathbb{K})$ is the absolutely universal embedding of $DQ^-(2n+1,\mathbb{K})$.*

De Bruyn and Pasini [9] showed that any full polarized embedding of a dual polar space of rank n has vector dimension at least 2^n. Since the absolutely universal embedding of $DQ^-(2n+1,\mathbb{K})$ has vector dimension 2^n, we can say the following:

Corollary 1.5 *Up to isomorphism, the spin-embedding of $DQ^-(2n+1,\mathbb{K})$ is the unique full polarized embedding of $DQ^-(2n+1,\mathbb{K})$.*

2 The spin-embedding of $DQ^-(2n+1,\mathbb{K})$

Let $\mathbb{K}$ and $\mathbb{K}'$ be fields such that $\mathbb{K}'$ is a quadratic Galois extension of $\mathbb{K}$. Let θ denote the unique nontrivial element in $Gal(\mathbb{K}'/\mathbb{K})$ and let $n \in \mathbb{N}\setminus\{0,1\}$.

Since $\mathbb{K} \subseteq \mathbb{K}'$, every point of the projective space $\mathrm{PG}(2n+1,\mathbb{K})$ can be regarded as a point of $\mathrm{PG}(2n+1,\mathbb{K}')$. Every subspace α of $\mathrm{PG}(2n+1,\mathbb{K})$ then generates a subspace α' of $\mathrm{PG}(2n+1,\mathbb{K}')$ with the same dimension as α. The map $\theta : (X_0,\ldots,X_{2n+1}) \mapsto (X_0^\theta, X_1^\theta,\ldots,X_{2n+1}^\theta)$ is an automorphism of $\mathrm{PG}(2n+1,\mathbb{K}')$. For every subspace α of $\mathrm{PG}(2n+1,\mathbb{K}')$, we define $\alpha^\theta := \{p^\theta \,|\, p \in \alpha\}$.

Lemma 2.1 *If α is a subspace of $\mathrm{PG}(2n+1,\mathbb{K}')$, then there exists a subspace β of $\mathrm{PG}(2n+1,\mathbb{K})$ such that $\alpha \cap \alpha^\theta = \beta'$.*

Proof. Suppose α is the subspace of $\mathrm{PG}(2n+1,\mathbb{K}')$ described by the $k \geq 0$ equations $a_0^{(i)}X_0 + a_1^{(i)}X_1 + \cdots + a_{2n+1}^{(i)}X_{2n+1} = 0$, $1 \leq i \leq k$. Let $(1,\epsilon)$ be a basis of $\mathbb{K}'$ regarded as a two-dimensional vector space over $\mathbb{K}$ (so, $\epsilon^\theta \neq \epsilon$) and let $b_j^{(i)}$ and $c_j^{(i)}$ be elements of $\mathbb{K}$ such that $a_j^{(i)} = b_j^{(i)} + \epsilon c_j^{(i)}$ for all $i \in \{1,\ldots,k\}$ and all $j \in \{0,\ldots,2n+1\}$. Then the subspace $\alpha \cap \alpha^\theta$ is described by the following equations

$$\left\{\begin{array}{lcll} b_0^{(i)}X_0 + \cdots + b_{2n+1}^{(i)}X_{2n+1} & = & 0 & (1 \leq i \leq k), \\ c_0^{(i)}X_0 + \cdots + c_{2n+1}^{(i)}X_{2n+1} & = & 0 & (1 \leq i \leq k). \end{array}\right. \tag{1}$$

The system (1) also determines a subspace β of $\mathrm{PG}(2n+1,\mathbb{K})$. Obviously, $\alpha \cap \alpha^\theta = \beta'$. ∎

For all $i,j \in \{0,\ldots,2n+1\}$ with $i \leq j$, let $a_{ij} \in \mathbb{K}$ such that $q(\overline{X}) = \sum_{0\leq i\leq j\leq 2n+1} a_{ij}X_iX_j$ is a quadratic form defining a quadric $Q^-(2n+1,\mathbb{K})$ of Witt-index n in $\mathrm{PG}(2n+1,\mathbb{K})$ and a quadric $Q^+(2n+1,\mathbb{K}')$ of Witt-index $n+1$ in $\mathrm{PG}(2n+1,\mathbb{K}')$. Let $\mathcal{M}^+$ and $\mathcal{M}^-$ denote the two families of maximal subspaces of $Q^+(2n+1,\mathbb{K}')$. The automorphism θ of $\mathrm{PG}(2n+1,\mathbb{K}')$ fixes $Q^+(2n+1,\mathbb{K}')$ setwise. So, either $\mathcal{M}^{+\theta} = \mathcal{M}^+$, $\mathcal{M}^{-\theta} = \mathcal{M}^-$ or $\mathcal{M}^{+\theta} = \mathcal{M}^-$, $\mathcal{M}^{-\theta} = \mathcal{M}^+$.

Lemma 2.2 *We have $\mathcal{M}^{+\theta} = \mathcal{M}^-$ and $\mathcal{M}^{-\theta} = \mathcal{M}^+$.*

Proof. With respect to a certain reference system in $\mathrm{PG}(2n+1,\mathbb{K})$, $Q^-(2n+1,\mathbb{K})$ has equation:

$$Y_0^2 + (\delta + \delta^\theta)Y_0Y_1 + \delta^{\theta+1}Y_1^2 + Y_2Y_3 + \cdots + Y_{2n}Y_{2n+1} = 0$$

for some $\delta \in \mathbb{K}'\setminus\mathbb{K}$. Now, let M be the maximal subspace of $Q^+(2n+1,\mathbb{K}')$ with equation $Y_0 + \delta Y_1 = Y_2 = Y_4 = \cdots = Y_{2n} = 0$. Then M^θ has equation

$Y_0 + \delta^\theta Y_1 = Y_2 = Y_4 = \cdots = Y_{2n}$. So, $M \cap M^\theta$ has co-dimension 1 in M. Hence, M and M^θ belong to different families. The lemma now readily follows. ∎

In the sequel, we will denote by $HS(2n+1,\mathbb{K}')$ the half-spin geometry for $Q^+(2n+1,\mathbb{K}')$ defined on the set $\mathcal{M}^+$. Define the following map ϕ from the points and lines of the dual polar space $DQ^-(2n+1,\mathbb{K})$ to the points and lines of $HS(2n+1,\mathbb{K}')$.

- If α is a maximal subspace of $Q^-(2n+1,\mathbb{K})$, then $\phi(\alpha)$ denotes the unique element of $\mathcal{M}^+$ through α'.
- If α is an $(n-2)$-dimensional subspace of $Q^-(2n+1,\mathbb{K})$, then $\phi(\alpha) := \alpha'$.

Lemma 2.3 *The map ϕ defines an injection from the set of points of the dual polar space $DQ^-(2n+1,\mathbb{K})$ to the set of points of the half-spin geometry $HS(2n+1,\mathbb{K}')$.*

Proof. Suppose $\alpha = \phi(\beta)$, where β is some point of $DQ^-(2n+1,\mathbb{K})$. Then $\beta' \subseteq \alpha \cap \alpha^\theta$ since $\beta' \subseteq \alpha$ and $\beta' = \beta'^\theta \subseteq \alpha^\theta$. If $\alpha = \alpha^\theta$, then by Lemma 2.1 $\alpha = \alpha \cap \alpha^\theta = \gamma'$ for some subspace γ of PG$(2n+1,\mathbb{K})$. This would imply that γ is a singular subspace of $Q^-(2n+1,\mathbb{K})$ of dimension n which is impossible. Hence, $\dim(\alpha \cap \alpha^\theta) \leq n-1$. Together with $\beta' \subseteq \alpha \cap \alpha^\theta$ and $\dim(\beta') = n-1$, this implies that $\beta' = \alpha \cap \alpha^\theta$. Hence, β is completely determined by α, proving the lemma. ∎

Lemma 2.4 *ϕ maps lines of $DQ^-(2n+1,\mathbb{K})$ to full lines of $HS(2n+1,\mathbb{K}')$.*

Proof. Let β be an $(n-2)$-dimensional subspace of $Q^-(2n+1,\mathbb{K})$ and let $\phi(\beta) = \beta'$ denote the associated line of $HS(2n+1,\mathbb{K}')$. Let M denote an arbitrary element of $\mathcal{M}^+$ through β'. We must show that $M = \phi(\alpha)$ for some maximal subspace α of $Q^-(2n+1,\mathbb{K})$ through β. Since $\beta' \subseteq M$, $\beta'^\theta = \beta' \subseteq M^\theta$ and hence $\beta' \subseteq M \cap M^\theta$. By Lemma 2.1, $M \cap M^\theta = \alpha'$ for some subspace α of PG$(2n+1,\mathbb{K})$. Obviously, α is a subspace of $Q^-(2n+1,\mathbb{K})$. Since $M \in \mathcal{M}^+$ and $M^\theta \in \mathcal{M}^-$ (recall Lemma 2.2), $\alpha' = M \cap M^\theta$ has odd co-dimension in M. Since $\beta' \subseteq \alpha'$ and $\dim(\beta') = n-2$, $\dim(\alpha) = n-1$, i.e., α is a maximal subspace of $Q^-(2n+1,\mathbb{K})$. Obviously, $\phi(\alpha) = M$. ∎

Let e' denote the spin-embedding of $HS(2n+1,\mathbb{K}')$ into the projective space PG$(2^n-1,\mathbb{K}')$. By Shult [12] (see also De Bruyn [8] for an alternative proof), every hyperplane of $HS(2n+1,\mathbb{K}')$ arises from the embedding e'. By Lemmas 2.3 and 2.4, the map $e := e' \circ \phi$ defines a full projective embedding of the dual polar space $DQ^-(2n+1,\mathbb{K})$ into a subspace Σ of PG$(2^n-1,\mathbb{K}')$. This embedding is called the spin-embedding of $\Delta := DQ^-(2n+1,\mathbb{K})$.

Lemma 2.5 *Let $\alpha \in \mathcal{M}^+$ if n is odd and $\alpha \in \mathcal{M}^-$ if n is even. Then the set H_α of elements of $\mathcal{M}^+$ meeting α is a hyperplane of $HS(2n+1, \mathbb{K}')$.*

Proof. Notice first that independent of whether n is odd or even, every element of $\mathcal{M}^+$ meets α in a subspace of odd dimension and every element of $\mathcal{M}^-$ meets α in a subspace of even dimension.

Now, let β be an arbitrary line of $HS(2n+1, \mathbb{K}')$. So, β is an $(n-2)$-dimensional subspace of $Q^+(2n+1, \mathbb{K}')$. If β meets α, then every point of $HS(2n+1, \mathbb{K}')$ incident with the line β belongs to H_α.

Suppose that β does not meet α. Then there exists a unique maximal subspace γ of $Q^+(2n+1, \mathbb{K}')$ through β meeting α in a line L. This maximal subspace belongs to $\mathcal{M}^+$ and every other maximal subspace of $\mathcal{M}^+$ through β is disjoint from α. Hence, precisely one point of $HS(2n+1, \mathbb{K}')$ incident with the line β belongs to H_α. ∎

The following proposition is precisely Theorem 1.2.

Proposition 2.6 *The embedding $e : \Delta \to \Sigma$ is polarized.*

Proof. Let α denote an arbitrary maximal subspace of $Q^-(2n+1, \mathbb{K})$. Let α^ϵ, $\epsilon \in \{+, -\}$, denote the unique element of $\mathcal{M}^\epsilon$ through α.

Suppose n is odd. Then by Lemma 2.5 the set of elements of $\mathcal{M}^+$ meeting α^+ is a hyperplane of $HS(2n+1, \mathbb{K}')$ which arises from a hyperplane π of $\mathrm{PG}(2^n - 1, \mathbb{K}')$. Now, a maximal subspace $\bar{\alpha}$ of $Q^-(2n+1, \mathbb{K})$ meets α if and only if $\phi(\bar{\alpha}) = \bar{\alpha}^+$ meets α^+ (two elements of $\mathcal{M}^+$ meet in a subspace of odd dimension). It readily follows that the singular hyperplane of $DQ^-(2n+1, \mathbb{K})$ with deepest point α arises from the hyperplane $\pi \cap \Sigma$ of Σ.

Suppose n is even. Then by Lemma 2.5 the set of elements of $\mathcal{M}^+$ meeting α^- is a hyperplane of $HS(2n+1, \mathbb{K}')$ which arises from a hyperplane π of $\mathrm{PG}(2^n - 1, \mathbb{K}')$. Now, a maximal subspace $\bar{\alpha}$ of $Q^-(2n+1, \mathbb{K})$ meets α if and only if $\phi(\bar{\alpha}) = \bar{\alpha}^+$ meets α^- (an element of $\mathcal{M}^+$ meets an element of $\mathcal{M}^-$ in a subspace of odd dimension). It readily follows that the singular hyperplane of $DQ^-(2n+1, \mathbb{K})$ with deepest point α arises from a hyperplane $\pi \cap \Sigma$ of Σ. ∎

The following proposition completes the proof of Theorem 1.1.

Proposition 2.7 *We have $\Sigma = \mathrm{PG}(2^n - 1, \mathbb{K}')$.*

Proof. By De Bruyn and Pasini [9], every full polarized embedding of a thick dual polar space of rank n has vector dimension at least 2^n. Hence, $\dim(\Sigma) \geq 2^n - 1$. On the other hand, we know that Σ is a subspace of $\mathrm{PG}(2^n - 1, \mathbb{K}')$. The proposition follows. ∎

3 The generating rank of $DQ^-(2n+1,\mathbb{K})$

Lemma 3.1 *Let H denote a Hermitian variety of Witt-index 2 in $\mathrm{PG}(3,\mathbb{K}')$ with associated involutory automorphism θ. Let $H(3,\mathbb{K}',\theta)$ denote the associated Hermitian generalized quadrangle. Then $H(3,\mathbb{K}',\theta)$ can be generated by 4 points.*

Proof. Let V be a 4-dimensional vector space over $\mathbb{K}'$ equipped with a skew-θ-hermitian form $(\cdot,\cdot)$ (which is linear in the first and semi-linear in the second argument) giving rise to the Hermitian variety H of $\mathrm{PG}(3,\mathbb{K}')$.

Choose four points x_1, x_2, x_3 and x_4 in $H(3,\mathbb{K}',\theta)$ such that $x_1 \sim x_2 \sim x_3 \sim x_4 \sim x_1$ and $x_1 \not\sim x_3$. Choose vectors $\bar{a}_i$, $i \in \{1,2,3,4\}$, in V such that $x_i = \langle \bar{a}_i \rangle$, $(\bar{a}_1,\bar{a}_2) = (\bar{a}_2,\bar{a}_3) = (\bar{a}_3,\bar{a}_4) = (\bar{a}_4,\bar{a}_1) = 0$ and $(\bar{a}_1,\bar{a}_3) = (\bar{a}_2,\bar{a}_4) = 1$. Consider the following lines:

$$
\begin{aligned}
M_1 &:= x_1x_2\\
M_2 &:= x_3x_4\\
L^* &:= x_1x_4\\
L_\lambda &:= \langle \lambda\bar{a}_1+\bar{a}_2, \bar{a}_3-\lambda^\theta\bar{a}_4\rangle, \qquad \lambda \in \mathbb{K}'.
\end{aligned}
$$

Notice that the points $\langle \lambda\bar{a}_1+\bar{a}_2\rangle \in M_1$ and $\langle \bar{a}_3-\lambda^\theta\bar{a}_4\rangle \in M_2$ are collinear in $H(3,\mathbb{K}',\theta)$. Now, let S denote the smallest subspace of $H(3,\mathbb{K}',\theta)$ containing the points x_1, x_2, x_3 and x_4. Then the lines M_1, M_2, L^* and L_λ ($\lambda \in \mathbb{K}'$) are contained in S.

We now show that every line L through $x^* := \langle \bar{a}_1+\bar{a}_4\rangle \in L^*$ is contained in S. We may suppose that $L \neq L^*$. Then there exists a $k \in \mathbb{K}$ such that

$$L = \langle \bar{a}_1+\bar{a}_4, \bar{a}_2+\bar{a}_3+k\bar{a}_4\rangle.$$

In order to show that L is contained in S, it suffices to show that L meets at least one line L_λ, $\lambda \in \mathbb{K}'$. Now, L meets L_λ if and only if

$$\begin{vmatrix} 1 & 0 & 0 & 1\\ 0 & 1 & 1 & k\\ \lambda & 1 & 0 & 0\\ 0 & 0 & 1 & -\lambda^\theta \end{vmatrix} = 0,$$

i.e., if and only if

$$\lambda+\lambda^\theta+k=0.$$

This equation is satisfied if we take λ equal to $\frac{ku}{u^\theta-u}$, where u is an arbitrary element of $\mathbb{K}'\setminus\mathbb{K}$. This proves that $L \subseteq S$.

So, S contains the singular hyperplane of $H(3,\mathbb{K}',\theta)$ with deepest point x^*. This singular hyperplane is a maximal subspace of $H(3,\mathbb{K}',\theta)$ (recall

Shult [13, Lemma 6.1]). Since also $x_2 \in S$, it follows that S coincides with the whole point-set of $H(3, \mathbb{K}', \theta)$. ∎

The following proposition is precisely Theorem 1.3.

Proposition 3.2 *The dual polar space $DQ^-(2n+1, \mathbb{K})$, $n \geq 2$, can be generated by 2^n points.*

Proof. We will prove the proposition by induction on n. The case $n = 2$ has been treated in Lemma 3.1. Suppose therefore that $n \geq 3$.

Let M_1 and M_2 be two disjoint maxes of $DQ^-(2n+1, \mathbb{K})$, Since $M_i \cong DQ^-(2n-1, \mathbb{K})$, there exists a set X_i of 2^{n-1} points in M_i which generates M_i. Let S denote the smallest subspace of $DQ^-(2n+1, \mathbb{K})$ containing $X_1 \cup X_2$.

Now, let x be an arbitrary point of $DQ^-(2n+1, \mathbb{K})$. If x is on a line connecting a point of M_1 with a point of M_2, then $x \in S$. Suppose x is not on such a line. Let x_1 be the unique point of M_1 collinear with x and let x_2 be the unique point of M_2 collinear with x_2. Since x, x_1 and x_2 are not on a line, they are contained in a unique quad Q which intersects M_1 and M_2 in the respective lines L_1 and L_2. Since $L_1, L_2 \subseteq S$ and $Q \cong DQ^-(5, \mathbb{K}) \cong H(3, \mathbb{K}', \theta)$, it follows from Lemma 3.1 that $Q \subseteq S$. In particular, $x \in S$.

Hence, the dual polar space $DQ^-(2n+1, \mathbb{K})$ can be generated by the set $X_1 \cup X_2$ of size 2^n. ∎

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