Chromatic Uniqueness of Certain Complete 4-partite Graphs

G.C. Lau[a,c,1], Y.H. Peng[b,c], H.H. Chu[a]

[a] *Faculty of Computer & Mathematical Sciences*
Universiti Teknologi MARA (Segamat Campus)
85200 Johor, Malaysia

[b] *Department of Mathematics, and*
[c] *Institute for Mathematical Research*
Universiti Putra Malaysia
43400 UPM Serdang, Malaysia

ABSTRACT

Let $P(G,\lambda)$ be the chromatic polynomial of a graph G. A graph G is chromatically unique if for any graph H, $P(H,\lambda) = P(G,\lambda)$ implies H is isomorphic to G. It is known that a complete tripartite graph $K(a,b,c)$ with $c \geq b \geq a \geq 2$ is chromatically unique if $c - a \leq 3$. In this paper, we proved that a complete 4-partite graph $K(a,b,c,d)$ with $d \geq c \geq b \geq a \geq 2$ is also chromatically unique if $d - a \leq 3$.

AMS Mathematical Subject Classification (2000): 05C15;05C60

Keywords: Complete 4-partite graph; Chromatic polynomial; Chromatic Uniqueness

1. Introduction

All graphs considered in this paper are finite, undirected, simple and loopless. For a graph G, we denote by $P(G;\lambda)$ (or $P(G)$), the chromatic polynomial of G. Two graphs G and H are said to be *chromatically equivalent*, or *χ-equivalent*, denoted $G \sim H$ if $P(G) = P(H)$. It is clear that the relation " $\sim$ " is an equivalence relation on the family of graphs.A graph G is said to be *chromatically unique*, or *χ-unique*, if $H \sim G$ implies that $H \cong G$. Many families of χ-unique graphs are known (see [5, 6]). In this paper, we proved that a complete 4-partite graph $K(a,b,c,d)$ is χ-unique if $d - a \leq 3$.

[1] Corresponding author. E-mail: geeclau@yahoo.com

2. Preliminary results and notations

Let $\chi(G), v(G)$, $e(G)$ and $t(G)$ be the chromatic number, the number of vertices, the number of edges and the number of triangles of G, respectively. Denote by $K(G)$ (respectively $Q(G)$) the number of subgraphs K_4 (respectively induced subgraphs C_4) in a graph G. By $\overline{G}$, we denote the complement of G. Then we let $O_p = \overline{K_p}$, where K_p denotes the complete graph with p vertices. Let S be a set of edges of G with $|S| = s$, and denote by $G - S$ (or $G - s$ if there is no confusion) the graph obtained by deleting all edges in S from G.

For a graph G and a positive integer k, a partition $\{A_1, A_2, \ldots, A_k\}$ of $V(G)$ is called a *k-independent partition* in G if each A_i is a non-empty independent set of G. Let $\alpha(G,k)$ denote the number of k-independent partitions in G. If G is of order p, then $P(G,\lambda) = \sum_{k=1}^{p} \alpha(G,k)(\lambda)_k$ where $(\lambda)_k = \lambda(\lambda-1)\cdots(\lambda-k+1)$ (see [9]). Therefore, $\alpha(G,k) = \alpha(H,k)$ for each $k = 1,2,\ldots$, if $G \sim H$.

For convenience, simply denote $G \cong H$ by $G = H$. For terms used but not defined here we refer to [1].

Lemma 1 *(Koh and Teo [5]) If $H \sim G$, then $v(G) = v(H)$, $e(G) = e(H)$, $t(G) = t(H)$ and $\chi(G) = \chi(H)$. Moreover, $\alpha(G,k) = \alpha(H,k)$ for each $k = 1,2,\ldots$.*

Lemma 2 *(Zhao [10]) Let $G = K(p_1,p_2,\ldots,p_t)$ and let $H = G - S$ for a set S of s edges of G. Then $\alpha(G,t+1) = \sum_{i=1}^{t} 2^{p_i-1} - t$. If $p_1 \geq s+1$, then*

$$s \leq \alpha_{t+1}(H) = \alpha(H,t+1) - \alpha(G,t+1) \leq 2^s - 1,$$

$\alpha_{t+1}(H) = s$ if and only if the subgraph induced by any $r \geq 2$ edges in S is not a complete multipartite graph, and $\alpha_{t+1}(H) = 2^s - 1$ if and only if all s edges in S induced a $K(1,s)$ with all end-vertices belong to the same V_i for some i.

Lemma 3 *(Zhao [10]) Let $G = K(p_1,p_2,\ldots,p_t)$ with $2 \leq p_1 \leq p_2 \leq \cdots \leq p_t$. If $H \sim G$, then*

(i) $H \in [G] \subset \{K(x_1,x_2,\ldots,x_t) - S | 1 \leq x_1 \leq x_2 \leq \cdots \leq x_t \leq p_t, \sum_{i=1}^{t} x_i = \sum_{i=1}^{t} p_i, S \subset E(K(x_1,x_2,\ldots,x_t))\}$;

(ii) there exists an integer $b \geq 2$ such that $x_1 \leq x_2 \leq \cdots \leq x_b \leq p_b - 1$ and K_{p_i} is a component of $\overline{H}$ for any $i \geq b+1$;

(iii) if $p_1 \geq 2$ and $x_i = p_i$ for any $i \geq 3$, then $H = G$.

A complete tripartite graph tripartite graph $K(a,b,c)$ with $c \geq b \geq a \geq 2$ is χ-unique if $c - a \leq 3$ (see [3]) or if $c - b \leq 1$ (see [7]). Very few results on the chromaticity of complete 4-partite graphs $K(a,b,c,d)$ with $d \geq c \geq b \geq a \geq 2$ are available.

Lemma 4 *(i) $K(1,b,c,d)$ is χ-unique if and only if $d \leq 2$ [8];*

(ii) $K(a-1,a,a,a+1)$ is χ-unique if $a \geq 3$ [4];

(iii) $K(a,b,c,d)$ is χ-unique if $d - b \leq 1$ [2, 7].

3. Chromaticity of $K(a,b,c,d)$

We now present our main theorem.

Theorem 1 *The complete 4-partite graph $K(a,b,c,d)$ is χ-unique for $d \geq c \geq b \geq a \geq 2$ and $d - a \leq 3$.*

Proof. Suppose $G = K(a,b,c,d)$ with $d \geq c \geq b \geq a \geq 2$. Suppose $H \sim G$. By Lemma 3, there exists $F = K(x,y,z,w)$ and $S \subset E(F)$ such that $H = F - S$ with $1 \leq x \leq y \leq z \leq w \leq d$ where $|S| = s = e(F) - e(G) = xy + (x+y)(z+w) + zw - ab - (a+b)(c+d) - cd \geq 0$ and $x + y + z + w = a + b + c + d$.

By Lemma 1, $t(G) = t(H)$. Hence, we shall consider the number of triangles in G and H. Let $S = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_s\} \subset E(F)$. Denote by $t(\epsilon_i)$ the number of triangles containing the edge ϵ_i in F. It is not hard to see that $t(\epsilon_i) \leq (z+w)$. Then

$$t(H) \geq t(F) - s(z+w), \tag{1}$$

and the equality holds only if $t(\epsilon_i) = z + w$ for all $\epsilon_i \in S$.

Let $\beta = t(F) - t(G)$. It is obvious that $t(F) = xyz + xyw + (x+y)zw$, $t(G) = abc + abd + (a+b)cd$ and $\beta = xyz + xyw + xzw + yzw - abc - abd - (a+b)cd$. So, we have

$$t(G) = t(F) - \beta. \tag{2}$$

Since $t(G) = t(H)$, from (1) and (2) it follows that

$$\beta \leq s(z+w). \tag{3}$$

Let $f(z,w) = \beta - s(z+w)$. Recalling that $x + y = a + b + c + d - z - w$. By calculation, we have

$$\begin{aligned} f(z,w) &= (z-a)(z-b)(z-c) + (w-a)(w-c)(w-d) + \\ &\quad (z+w-a-c)(z-b)(w-d) \leq 0. \end{aligned} \tag{4}$$

We first consider $d-a=2$. By Lemma 4, G is χ-unique if $d-b \leq 1$. Hence, we only need to consider $G \in \{K(p-2,p-2,p-2,p), K(p-2,p-2,p-1,p), K(p-2,p-2,p,p)\}$ with $p \geq 4$ and $w \leq p$. We have the following cases.

Case 1. $G = K(p-2,p-2,p-2,p)$. Clearly, $z = p-2, w = p$ or $w \geq z \geq p-1$ so that $s \geq 0$. Since $f(z,w) \leq 0$, by Lemma 3, we have $H \in \{G, K(p-3,p-1,p-1,p-1), K(p-2,p-2,p-1,p-1)-1\}$. If $H = K(p-3,p-1,p-1,p-1)$, Lemma 2 implies that $\alpha(G,5) > \alpha(H,5)$. If $H = K(p-2,p-2,p-1,p-1)-1$, we have $\alpha(G,5)-\alpha(H,5) = 2^{p-3}-1 > 0$. Both contradicting Lemma 1. Hence, $H = G$.

Case 2. $G = K(p-2,p-2,p-1,p)$. Clearly, $w \geq z \geq p-1$. Since $s \geq 0$ and $f(z,w) \leq 0$, we have $H \in \{G, K(p-2,p-1,p-1,p-1)-1\}$. By an argument similar to that in Case 1, we have $H = G$.

Case 3. $G = K(p-2,p-2,p,p)$. Clearly, $w \geq z \geq p-1$. Since $f(z,w) \leq 0$, we have $H \in \{G, K(p-1,p-1,p-1,p-1)-2, K(p-2,p-1,p-1,p)-1\}$. By an argument similar to that in Case 1, we have $H = G$.

We now consider $d-a=3$. By Lemma 4, G is χ-unique if $d-b \leq 1$. Hence, we only need to consider $G \in \{K(p-3,p-3,p-3,p), K(p-3,p-3,p-2,p), K(p-3,p-3,p-1,p), K(p-3,p-3,p,p), K(p-3,p-2,p-2,p), K(p-3,p-2,p-1,p), K(p-3,p-2,p,p)\}$ with $p \geq 5$ and $w \leq p$. We have the following cases.

Case 4. $G = K(p-3,p-3,p-3,p)$. Clearly, $w = p-3, z = p$ or $w \geq z \geq p-2$ so that $s \geq 0$. Since $f(z,w) \leq 0$, we have $H \in \{G, K(p-3,p-2,p-2,p-2)-3, K(p-4,p-2,p-2,p-1)-1, K(p-3,p-3,p-2,p-1)-2, K(p-4,p-3,p-1,p-1)\}$.

Let $H = K(p-3,p-2,p-2,p-2)-3$, we consider the following subcases.

Subcase 4.1. $G = K(2,2,2,5)$ and $H = K(2,3,3,3)-3$. If $\langle S \rangle = K(1,3)$ with all the end-vertices in the same partite set $V_i, i = 2,3,4$ or $\langle S \rangle$ has a $K(1,2)$ subgraph with all the end-vertices in the partite set V_1, then $\alpha(H,4) > \alpha(G,4)$, a contradiction. Otherwise, $\alpha_5(H) \leq 5$ and $\alpha(G,5) - \alpha(H,5) > 0$, also a contradiction.

Subcase 4.2. $G = K(3,3,3,6)$ and $H = K(3,4,4,4)-3$. If $\langle S \rangle = K(1,3)$ with all the end-vertices in the partite set V_1, then $\alpha(H,4) > \alpha(G,4)$, a contradiction. Otherwise, $\alpha_5(H) \leq 7$ and $\alpha(G,5) - \alpha(H,5) > 0$, also a contradiction.

Subcase 4.3. If $p \geq 7$, then $\alpha(G,5)-\alpha(H,5) \geq 2^{p-2}-7 > 0$, a contradiction.

Let $H = K(p-4,p-2,p-2,p-1)-1$. If $H = K(1,3,3,4)-1$ and the deleted edge has an end-vertex in V_1, then $\alpha(H,4) > \alpha(G,4)$, a contradiction. Otherwise, $\alpha(G,5) - \alpha(H,5) = 2^{p-3} + 2^{p-5} - 1 > 0$, also a contradiction.

Let $H = K(p-3,p-3,p-2,p-1)-2$. If $H = K(2,2,3,4)-2$ and $\langle S\rangle = K(1,2)$ with all the end-vertices in partite set V_1 or V_2, then $\alpha(H,4) > \alpha(G,4)$, a contradiction. Otherwise, $\alpha(G,5)-\alpha(H,5) \geq 2^{p-3}+2^{p-4}-3 > 0$, also a contradiction.

Let $H = K(p-4,p-3,p-1,p-1)$, then $\alpha(G,5) > \alpha(H,5)$, a contradiction. Hence, $H = G$.

Case 5. $G = K(p-3,p-3,p-2,p)$. Clearly, $w \geq z \geq p-2$. Since $s \geq 0$ and $f(z,w) \leq 0$, we have $H \in \{G, K(p-2,p-2,p-2,p-2)-3, K(p-3,p-2,p-2,p-1)-2, K(p-4,p-2,p-1,p-1), K(p-3,p-3,p-1,p-1)-1\}$. By an argument similar to that in Case 4, we have $H = G$.

Case 6. $G = K(p-3,p-3,p-1,p)$. Clearly, $z \geq p-2, w \geq p-1$. Since $s \geq 0$ and $f(z,w) \leq 0$, we have $H \in \{G, K(p-2,p-2,p-2,p-1)-3, K(p-3,p-2,p-2,p)-1, K(p-3,p-2,p-1,p-1)-2, K(p-4,p-1,p-1,p-1)\}$. By an argument similar to that in Case 4, we have $H = G$.

Case 7. $G = K(p-3,p-3,p,p)$. Clearly, $z = p-2, w = p$ or $w \geq z \geq p-1$. Since $s \geq 0$ and $f(z,w) \leq 0$, we have $H \in \{G, K(p-2,p-2,p-2,p)-3, K(p-3,p-1,p-1,p-1)-3, K(p-2,p-2,p-1,p-1)-4, K(p-4,p-1,p-1,p), K(p-3,p-2,p-1,p)-2\}$. By an argument similar to that in Case 4, we have $H = G$.

Case 8. $G = K(p-3,p-2,p-2,p)$. Clearly, $z \geq p-2, w \geq p-1$. Since $s \geq 0$ and $f(z,w) \leq 0$, we have $H \in \{G, K(p-2,p-2,p-2,p-1)-2, K(p-3,p-2,p-1,p-1)-1\}$. By an argument similar to that in Case 4, we have $H = G$.

Case 9. $G = K(p-3,p-2,p-1,p)$. Clearly, $z = p-2, w = p$ or $w \geq z \geq p-1$. Since $s \geq 0$ and $f(z,w) \leq 0$, we have $H \in \{G, K(p-2,p-2,p-2,p)-1, K(p-3,p-1,p-1,p-1)-1, K(p-2,p-2,p-1,p-1)-2\}$. By an argument similar to that in Case 4, we have $H = G$.

Case 10. $G = K(p-3,p-2,p,p)$. Clearly, $w \geq z \geq p-1$. Since $s \geq 0$ and $f(z,w) \leq 0$, we have $H \in \{G, K(p-2,p-1,p-1,p-1)-3, K(p-3,p-1,p-1,p)-1, K(p-2,p-2,p-1,p)-2\}$. By an argument similar to that in Case 4, we have $H = G$.

Thus the proof is completed. □

Acknowledgement

The authors would like to thank the referee for the valuable comments.

References

1. M. Behzad, G. Chartrand and L. Lesniak-Foster, Graphs and Digraphs (Wadsworth, Belmont, CA 1979).

2. C.Y. Chao and Novacky Jr., On maximally saturated graphs, *Discrete Math.* **41** (1982) 139 – 143.

3. C.L. Chia and C.K. Ho, Chromatic equivalence classes of complete tripartite graphs, *Discrete Math.* **309** (1) (2009) 134–143.

4. R.E. Giudici and M.A. López, Chromatic uniqueness of sKn, Report No. 85 – 03, Dpto. de Mat. y Ciencia de la Comp. Univ. Simón Bolivar, 1985.

5. K.M. Koh and K.L. Teo, The search for chromatically unique graphs, *Graphs and Combinatorics* **6** (1990) 259 – 285.

6. K.M. Koh and K.L. Teo, The search for chromatically unique graphs - II, *Discrete Math.* **172** (1997) 59 – 78.

7. G.C. Lau and Y.H. Peng, Chromatic uniqueness of certain complete t-partite graphs, *Ars Comb.* **92** (2009) 353–376.

8. N.Z. Li and R.Y. Liu, The chromaticity of the complete t-partite graph $K(1, p_2, \ldots, p_t)$, *J. Xinjiang Univ. Natur. Sci.* **7** (1990), No.3, 95 – 96.

9. R.C. Read and W.T. Tutte, Chromatic polynomials, In: *Selected Topics in Graph Theory 3*, Academic Press, 1988, 15 – 42.

10. H.X. Zhao, Chromaticity and adjoint polynomials of graphs, Ph.D. *thesis* (2005), University of Twente, Netherland.