Fuzzy K-ideals of K-algebras

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Abstract

We introduce the notion of fuzzy K-ideals of K-algebras and investigate some of their properties. We characterize ascending and descending chains of K-ideals by the corresponding fuzzy K-ideals. We discuss some properties of characteristic fuzzy K-ideals of K-algebras. We construct a quotient K-algebra via fuzzy K-ideal and present the fuzzy isomorphism theorems.

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1 Introduction

The notion of a K-algebra $(G, \cdot, \odot, e)$ was first introduced by Dar and Akram [3] in 2003 and published in 2005. A K-algebra is an algebra built on a group $(G, \cdot, e)$ by adjoining an induced binary operation $\odot$ on G which is attached to an abstract K-algebra $(G, \cdot, \odot, e)$. This system is, in general non-commutative and non-associative with a right identity e, if $(G, \cdot, e)$ is non-commutative. For a given group G, the K-algebra is proper if G is not an elementary abelian 2-group. Thus, whether a K-algebra is abelian and non-abelian purely depends on the base group G. Dar and Akram further renamed *a K-algebra on a group G* as a $K(G)$-algebra [4] due to its structural basis G.

It is well known that the notion of a *fuzzy subset of a set* was first introduced by Zadeh [10] in 1965 as a method of representing uncertainty. Since then, fuzzy set theory has been developed in many directions by many scholars and has evoked great interest among mathematicians and computer scientists working in different fields of mathematics and computer science, including topological spaces, functional analysis, loops, groups, rings, semirings, hemirings, nearrings, vector spaces, differential equations, pattern recognition, robotics, computer networks, expert systems, decision making theory, and automation. In 1971, Rosenfeld [9] used the concept of a fuzzy subset of a set to introduce the notion of a fuzzy subgroup of a group. Rosenfeld's paper spearheaded the development of fuzzy abstract algebra. The fuzzy structures of K-algebras was introduced in [1]. In this paper, we introduce the notion of fuzzy K-ideals of K-algebras and investigate some of their properties. We characterize descending and ascending chains of K-ideals by the corresponding fuzzy K-ideals. Some properties of fuzzy characteristic K-ideals of K-algebras are investigated. Construction of a quotient K-algebra via fuzzy K-ideal in a K-algebra is given. The fuzzy isomorphism theorems are also established.

2 Preliminaries

In this section we cite some facts that are necessary for this paper.
Let $(G, \cdot, e)$ be a group in which each non-identity element is not of order 2. Then a *K- algebra* is a structure $\mathcal{K} = (G, \cdot, \odot, e)$ on a group G in which induced binary operation $\odot : G \times G \to G$ is defined by $\odot(x, y) = x \odot y = x.y^{-1}$ and satisfies the following axioms:

(K1) $(x \odot y) \odot (x \odot z)=(x \odot ((e \odot z) \odot (e \odot y))) \odot x$,

(K2) $x \odot (x \odot y) = (x \odot (e \odot y)) \odot x$,

(K3) $x \odot x = e$,

(K4) $x \odot e = x$,

(K5) $e \odot x = x^{-1}$

for all x, y, $z \in G$. If the group $(G, \cdot, e)$ is abelian, then the above axioms $(K1)$ and $(K2)$ can be replaced by:

$(\overline{K1})$ $(x \odot y) \odot (x \odot z) = z \odot y$.

$(\overline{K2})$ $x \odot (x \odot y) = y$.

In what follows, $\mathcal{K}$ is a K-algebra unless otherwise specified. A nonempty subset H of a K-algebra $\mathcal{K}$ is called a *subalgebra* [3] of the K-algebra $\mathcal{K}$ if $a \odot b \in H$ for all a, $b \in H$. Note that every subalgebra of a K-algebra $\mathcal{K}$ contains the identity e of the group $(G, \cdot, e)$. A mapping $f : \mathcal{K}_1 = (G_1, \cdot, \odot, e_1) \rightarrow \mathcal{K}_2 = (G_2, \cdot, \odot, e_2)$ of K-algebras is called a *homomorphism* [5] if $f(x \odot y) = f(x) \odot f(y)$ for all $x, y \in \mathcal{K}_1$. A nonempty subset A of a K-algebra $\mathcal{K}$ is called K*-ideal* of $\mathcal{K}$ if it satisfies the following conditions:

(a) $e \in A$,

(b) $(\forall x, y, z \in G)\ (x \odot (y \odot z) \in A,\ y \odot (y \odot x) \in A \Rightarrow x \odot z \in A)$.

Let μ be a *fuzzy set* on G, i.e., a map $\mu : G \rightarrow [0, 1]$.

Definition 2.1. [9] A fuzzy set μ in a group G is called a *fuzzy subgroup*' of G if it satisfies:

- $(\forall x, y \in G)\ (\mu(xy) \geq \min\{\mu(x), \mu(y)\})$.
- $(\forall x \in G)\ (\mu(x^{-1}) \geq \mu(x))$.

Definition 2.2. [9] A fuzzy subgroup μ of a group G is said to be *normal* if it satisfies:

$$(\forall x, y \in G)\,(\mu(xy) = \mu(yx)).$$

Definition 2.3. [1] A fuzzy ideal of a K-algebra $\mathcal{K}$ is a mapping $\mu : G \rightarrow [0, 1]$ such that

(c) $(\forall x \in G)\ (\mu(e) \geq \mu(x)$,

(d) $(\forall x, y \in G)\ (\mu(x) \geq \min\{\mu(x \odot y), \mu(y \odot (y \odot x))\})$.

Proposition 2.4. [1] *Let $\mathcal{K} = (G, \cdot, \odot, e)$ be a K-algebra in which the operation "$\odot$" is induced by the group operation. Then every fuzzy subgroup of $(G, \cdot, e)$ is a fuzzy subalgebra of $\mathcal{K}$ and vice versa.*

Proposition 2.5. [1] *μ is a fuzzy ideal of a K-algebra $\mathcal{K}$ if and only if μ is a fuzzy normal subgroup of G.*

3 Fuzzy K-ideals of K-algebras

Definition 3.1. A fuzzy set μ in a K-algebra $\mathcal{K}$ is called a fuzzy K-ideal of $\mathcal{K}$ if it satisfies the following conditions:

(i) $(\forall x \in G)\ (\mu(e) \geq \mu(x))$,

(ii) $(\forall x, y, z \in G)\ (\mu(x \odot z) \geq \min\{\mu(x \odot (y \odot z)), \mu(y \odot (y \odot x))\})$.

Example 3.2. Consider the K-algebra $\mathcal{K} = (G, \cdot, \odot, e)$ on the Dihedral group $G = \{e, a, u, v, b, x, y, z\}$ where $u = a^2$, $v = a^3$, $x = ab$, $y = a^2b$, $z = a^3b$, and $\odot$ is given by the following Cayley's table:

$\odot$	e	a	u	v	b	x	y	z
e	e	v	u	a	b	x	y	z
a	a	e	v	u	x	y	z	b
u	u	a	e	v	y	z	b	x
v	v	u	a	e	z	b	x	y
b	b	x	y	z	e	v	u	a
x	x	y	z	b	a	e	v	u
y	y	z	b	x	u	a	e	v
z	z	b	x	y	v	u	a	e

Let μ be a fuzzy set in $\mathcal{K}$ defined by $\mu(e) = 0.8$, $\mu(t) = 0.06$ for all $t \neq e$. Then μ is a fuzzy K- ideal of $\mathcal{K}$.

Putting $z = e$ in Definition 3.1(ii) and using $(K4)$ induce that every fuzzy K-ideal is a fuzzy ideal.

Theorem 3.3. *Let μ be a fuzzy set in a K-algebra $\mathcal{K}$. Then μ is a fuzzy K- ideal of $\mathcal{K}$ if and only if the set $\mu_m := \{x \in G \mid \mu(x) \geq m\}$, $m \in [0, 1]$, is a K-ideal of $\mathcal{K}$ when it is nonempty.*

Let μ be a fuzzy set in a K-algebra $\mathcal{K}$. For any $w \in G$, we consider the set $\Omega_w := \{x \in G \mid \mu(x) \geq \mu(w)\}$. Obviously, $w \in \Omega_w$. If μ is a fuzzy K-ideal of $\mathcal{K}$, then $e \in \Omega_w$. The following question aries : *For a fuzzy set μ in $\mathcal{K}$ satisfying $\mu(e) \geq \mu(x \odot z)$ for all $x \in G$, is μ_w a K- ideal of $\mathcal{K}$?* The following example gives negative answer, that is, there exists $w \in G$ such that Ω_w is not a K- ideal of $\mathcal{K}$.

Example 3.4. Consider the K-algebra $\mathcal{K} = (S_3, \cdot, \odot, e)$ on the symmetric group $S_3 = \{e, a, b, x, y, z\}$ where $e = (1)$, $a = (123)$, $b = (132)$, $x = (12)$, $y = (13)$, $z = (23)$, and $\odot$ is given by the following Cayley's table:

$\odot$	e	x	y	z	a	b
e	e	x	y	z	b	a
x	x	e	a	b	z	y
y	y	b	e	a	x	z
z	z	a	b	e	y	x
a	a	z	x	y	e	b
b	b	y	z	x	a	e

Let μ be a fuzzy set in $\mathcal{K}$ defined by $\mu(e) = 0.8$, $\mu(a) = 0.6$, $\mu(b) = 0.5$, $\mu(x) = 0.3$, $\mu(y) = 0.2$, $\mu(z) = 0.1$. Then μ is not a fuzzy K-ideal of $\mathcal{K}$

because

$$\mu(b \odot y) \not\geq \min\{\mu(b \odot (x \odot y)), \mu(x \odot (x \odot b))\}.$$

Note that $\Omega_y = \{e, a, b, x, y\}$ is not a K-ideal of $\mathcal{K}$ since $b \odot (x \odot y) = b \odot a = a \in \Omega_y$ and $x \odot (x \odot b) = a \in \Omega_y$, but $b \odot y = z \notin \Omega_y$.

Theorem 3.5. Let μ be a fuzzy set in a K-algebra $\mathcal{K}$ and let $w \in G$.

(i) If Ω_w is a K- ideal of $\mathcal{K}$, then μ satisfies the following implication:

$$(\forall x, y, z \in G)(\mu(w) \leq \min\{\mu(x \odot (y \odot z)), \mu(y \odot (y \odot x))\} \Rightarrow \mu(w) \leq \mu(x \odot z)). \tag{1}$$

(ii) If μ satisfies conditions Definition 3.1(i) and (1), then Ω_w is a K-ideal of $\mathcal{K}$.

Theorem 3.6. *Let μ be a fuzzy set in G and $\mathrm{Im}(\mu) = \{\alpha_k \mid k = 0, 1, 2, \cdots, n\}$, where $\alpha_0 > \alpha_1 > \cdots > \alpha_n$. If $A_0 \subset A_1 \subset \cdots \subset A_k = G$ are K-ideals of $\mathcal{K}$ such that $\mu(A_k \setminus A_{k-1}) = \alpha_k$ for $k = 0, 1, \cdots, n$ where $A_{-1} = \emptyset$. Then μ is a fuzzy K-ideal of $\mathcal{K}$.*

Corollary 3.7. *Let μ be a fuzzy set in G and $\mathrm{Im}(\mu) = \{\alpha_k \mid k = 0, 1, 2, \cdots, n\}$, where $\alpha_0 > \alpha_1 > \cdots > \alpha_n$. If $A_0 \subset A_1 \subset \cdots \subset A_k = G$ are K-ideals of $\mathcal{K}$ such that $\mu(A_k) \geq \alpha_k$ for $k = 0, 1, \cdots, n$. Then μ is a fuzzy K-ideal of $\mathcal{K}$.*

Since $Im(\mu)$ is a bounded subset of $[0, 1]$, we can consider $Im(\mu)$ as a sequence which is either increasing or decreasing.

Theorem 3.8. *Let $\mathcal{K}$ be a K-algebra in which every descending chain $G_1 \supset G_2 \supset \cdots$ of K-ideals of $\mathcal{K}$ terminates at finite step. If μ is a fuzzy K-ideal of $\mathcal{K}$ such that a sequence of elements of $\mathrm{Im}(\mu)$ is strictly increasing, then μ has finite number of values.*

Theorem 3.9. *Let μ be a fuzzy K-ideal of a K-algebra with the finite image. Then every descending chain of K-ideal of $\mathcal{K}$ terminates at finite step.*

Proof. Suppose that there exists a strictly descending chain $A_0 \supset A_1 \supset A_2 \supset \cdots$ of K-ideals of $\mathcal{K}$ which does not terminate at finite step. Define a fuzzy set μ in G by

$$\mu(x) := \begin{cases} \frac{n}{n+1} & \text{if } x \in A_n \setminus A_{n+1}, n = 0, 1, 2, \cdots, \\ 1 & \text{if } x \in \bigcap_{n=0}^{\infty} A_n, \end{cases}$$

where $A_0 = G$. We prove that μ is a fuzzy K-ideal of $\mathcal{K}$. Clearly $\mu(e) \geq \mu(x)$ for all $x \in \mathcal{K}$. Let $x, y, z \in \mathcal{K}$. Assume that $x \odot (y \odot z) \in A_n \setminus A_{n+1}$ and $y \odot (y \odot x) \in A_k \setminus A_{k+1}$ for $n = 0, 1, 2, \cdots$; $k = 0, 1, 2, \cdots$. Without loss of generality, we may assume that $n \leq k$. Then obviously $y \odot (y \odot x) \in A_n$, and so $x \odot z \in A_n$ because A_n is a K-ideal of $\mathcal{K}$. Thus

$$\mu(x \odot z) \geq \tfrac{n}{n+1} = \min\{\mu(x \odot (y \odot z)), \mu(y \odot (y \odot x))\}.$$

If $x \odot (y \odot z), y \odot (y \odot x) \in \bigcap_{n=0}^{\infty} A_n$, then $x \odot z \in \bigcap_{n=0}^{\infty} A_n$. Thus

$$\mu(x \odot z) = 1 = \min\{\mu(x \odot (y \odot z)), \mu(y \odot (y \odot x))\}.$$

If $x \odot (y \odot z) \notin \bigcap_{n=0}^{\infty} A_n$ and $y \odot (y \odot x) \in \bigcap_{n=0}^{\infty} A_n$, then there exists $k \in \mathbb{N}$ such that $x \odot (y \odot z) \in A_k \setminus A_{k+1}$. It follows that $x \odot z \in A_k$ so that

$$\mu(x \odot z) \geq \tfrac{k}{k+1} = \min\{\mu(x \odot (y \odot z)), \mu(y \odot (y \odot x))\}.$$

Finally, suppose that $x \odot (y \odot z) \in \bigcap_{n=0}^{\infty} A_n$ and $y \odot (y \odot x) \notin \bigcap_{n=0}^{\infty} A_n$. Then $y \odot (y \odot x) \in A_r \setminus A_{r+1}$ for some $r \in \mathbb{N}$. Hence $x \odot z \in A_r$, and so

$$\mu(x \odot z) \geq \tfrac{r}{r+1} = \min\{\mu(x \odot (y \odot z)), \mu(y \odot (y \odot x))\}.$$

Consequently, we conclude that μ is a fuzzy K-ideal of $\mathcal{K}$ and μ has infinite number of different values. This is a contradiction, and the proof is complete. □

Theorem 3.10. *Every ascending chain of K-ideals of a K-algebra $\mathcal{K}$ terminates at finite step if and only if for any K-fuzzy ideal μ of $\mathcal{K}$,* $\mathrm{Im}(\mu)$ *is well-ordered subset of* $[0,1]$.

Proof. Suppose that μ is not well-ordered subset of $[0,1]$. Then there exists a strictly decreasing sequence $\{\alpha_n\}$ such that $\alpha_n = \mu(x_n)$ for some $x_n \in \mathcal{K}$. But in this case $B_n := \{x \in \mathcal{K} \mid \mu(x) \geq \alpha_n\}$ form a strictly ascending chain of K-ideals of $\mathcal{K}$ which is not terminating. This is a contradiction. So, $\mathrm{Im}(\mu)$ must be well-ordered subset of $[0,1]$.
Conversely, suppose that there exists a strictly ascending chain $A_1 \subset A_2 \subset A_3 \subset \ldots$ of K-ideals of $\mathcal{K}$ which does not terminate at finite step. Then $A = \bigcup_{k=1}^{\infty} A_k$ is a K-ideal of $\mathcal{K}$. Define on G a fuzzy set μ by putting

$$\mu(x) := \begin{cases} \frac{1}{k} & \text{for } x \in A_k \setminus A_{k-1}, \\ 0 & \text{for } x \notin A. \end{cases}$$

It is easy to see that $\mu(e) \geq \mu(x)$ for all $x \in \mathcal{K}$. Let $x, y, z \in \mathcal{K}$. We consider the case $x, y, z \in \mathcal{K}$. In this case there are m, n such that $x \odot (y \odot z) \in A_n \setminus A_{n-1}$, $y \odot (y \odot x) \in A_m \setminus A_{m-1}$. Obviously $x \odot z \in A_k \setminus A_{k-1} \subset A_p$, where $k \leq p = \max\{m, n\}$. So, $\mu(x \odot (y \odot z)) = \frac{1}{n}$, $\mu(y \odot (y \odot x)) = \frac{1}{m}$ and

$$\mu(x \odot z) = \frac{1}{k} \geq \frac{1}{p} = \min\{\mu(x \odot (y \odot z)), \mu(y \odot (y \odot x))\}.$$

Now we consider the case $x \odot (y \odot z) \notin A$, $y \odot (y \odot x) \in A$. In this case $y \odot (y \odot x) \in A_m \backslash A_{m-1}$ for some natural m. Hence $\mu(x \odot (y \odot z)) = 0$, $\mu(y \odot (y \odot x)) = \frac{1}{m}$, consequently

$$\mu(x \odot y) \geq 0 = \min\{\mu(x \odot (y \odot z)), \mu(y \odot (y \odot x))\}.$$

The case $x \odot (y \odot z) \in A$, $y \odot (y \odot x) \notin A$ is analogous. The last case $x \odot (y \odot z) \notin A$, $y \odot (y \odot x) \notin A$. is obvious. Thus μ is a fuzzy K-ideal of $\mathcal{K}$. This proves that μ is a fuzzy K-ideal. Since the chain $A_1 \subset A_2 \subset A_3 \subset \ldots$ is not terminating, μ has a strictly descending sequence of values. This contradicts that the value set of any fuzzy K-ideal is well-ordered. This completes the proof. □

We note that a set is well ordered if and only if it does not contain any infinite descending sequence.

Theorem 3.11. *Let $\mathcal{K}$ be a K-algebra and let $S = \{t_i \mid i = 1, 2, 3, \cdots\} \cup \{0\}$ where $\{t_n\}$ is a strictly descending sequence in $(0, 1)$. Then the following assertions are equivalent:*

(i) *For every ascending sequence $A_1 \subset A_2 \subset A_3 \subset \ldots$ of K-ideals of $\mathcal{K}$ there exists a natural number n such that $A_i = A_n$ for all $i \geq n$.*

(ii) *For each fuzzy K-ideal μ of $\mathcal{K}$, $Im(\mu) \subset S$ implies that there exists a natural number n_0 such that $Im(\mu) \subset \{t_i \mid i = 1, 2, 3, \cdots, n_0\} \cup \{0\}$.*

Proof. If (i) holds, then from Theorem 4.1.14 follows that $Im(\mu)$ is a well ordered subset of $[0, 1]$ and hence (ii) is valid. Suppose that (ii) is true. If the condition (i) is not valid, then there exists a strictly ascending chain $A_1 \subset A_2 \subset A_3 \subset \ldots$ of K-ideals of $\mathcal{K}$. Define a fuzzy set μ in K-algebra by

$$\mu(x) := \begin{cases} t_1 & \text{if } x \in A_1, \\ t_n & \text{if } x \in A_n \backslash A_{n-1}, n = 2, 3, 4, \cdots \\ 0 & \text{if } x \in G \backslash \bigcup_{n=1}^{\infty} A_n. \end{cases}$$

Since $e \in A_1$, $\mu(e) = t_1 \geq \mu(x)$ for all $x \in G$. Let x, y, $z \in G$. If either $x \odot (y \odot z)$ or $y \odot (y \odot x)$ belongs to $G \backslash \bigcup_{n=1}^{\infty} A_n$, then either $\mu(x \odot (y \odot z)) = 0$ or $\mu(y \odot (y \odot x)) = 0$. Thus

$$\mu(x \odot z) \geq \min\{\mu(x \odot (y \odot z)), \mu(y \odot (y \odot x))\}.$$

If $x \odot (y \odot z)$, $y \odot (y \odot x) \in A_1$, then $x \odot z \in A_1$ and so

$$\mu(x \odot z) = t_1 \geq \min\{\mu(x \odot (y \odot z)), \mu(y \odot (y \odot x))\}.$$

If $x \odot (y \odot z)$, $y \odot (y \odot x) \in A_n \backslash A_{n-1}$, then $x \odot z \in A_n$. Thus

$$\mu(x \odot z) \geq t_n = \min\{\mu(x \odot (y \odot z)), \mu(y \odot (y \odot x))\}.$$

Assume that $x \odot (y \odot z) \in A_1$ and $y \odot (y \odot x) \in A_n \backslash A_{n-1}$ for $n = 2, 3, 4, \cdots$, then $x \odot z \in A_n$ and hence

$$\mu(x \odot z) \geq t_n = \min\{t_1, t_n\} = \min\{\mu(x \odot (y \odot z)), \mu(y \odot (y \odot x))\}.$$

Similarly for $x \odot (y \odot z) \in A_n \backslash A_{n-1}$ and $y \odot (y \odot x) \in A_1$ for $n = 2, 3, 4, \cdots$, we have

$$\mu(x \odot z) \geq t_n = \min\{\mu(x \odot (y \odot z)), \mu(y \odot (y \odot x))\}.$$

Hence μ is a fuzzy K-ideal of K-algebra. □

3.1 Fuzzy Characteristic K-ideals

Definition 3.12. A K-ideal H of K-algebra is said to be *characteristic* if $f(H) = H$, for all $f \in Aut(\mathcal{K})$, where $\mathrm{Aut}(\mathcal{K})$ is the set of all automorphisms of a K-algebra $\mathcal{K}$. A fuzzy K-ideal μ of a K-algebra $\mathcal{K}$ is called a *fuzzy characteristic* if $\mu^f(x) = \mu(f(x)) = \mu(x)$ for all $x \in G$ and $f \in Aut(\mathcal{K})$.

Lemma 3.13. *If $\{\mu_i \,|\, i \in I\}$ is a family of fuzzy fully invariant K-ideals of $\mathcal{K}$, then $\bigwedge_{i\in I} \mu_{A_i}$ is a fuzzy characteristic K-ideal of $\mathcal{K}$, where*

$$\bigwedge_{i\in I} \mu_{A_i}(x) = \inf\{\mu_{A_i}(x) \,|\, i \in I,\, x \in G\}.$$

Theorem 3.14. *Let H be a nonempty subset of a K-algebra $\mathcal{K}$ and let μ be a fuzzy set defined by*

$$\mu(x) = \begin{cases} \alpha_2 & \text{if } x \in H, \\ \alpha_1 & \text{otherwise,} \end{cases}$$

where $0 \leq \alpha_1 < \alpha_2 \leq 1$. If H is a fuzzy characteristic K-ideal of $\mathcal{K}$, then μ is a fuzzy characteristic K-ideal of $\mathcal{K}$.

Proof. Clearly, $\mu(e) \geq \mu(x)$ for all $x \in G$. Let $x, y, z \in G$. If $x \odot (y \odot z) \notin H$, then $\mu(x \odot (y \odot z)) = t_1$ and so

$$\mu(x \odot z) \geq \alpha_1 = \min\{\mu(x \odot (y \odot z)), \mu(y \odot (y \odot x))\}.$$

Assume that $x \odot (y \odot z) \in H$. If $x \odot z \in H$ then $y \odot (y \odot x)$ may or may not be belong to H. In any case,

$$\mu(x \odot z) = \alpha_2 \geq \min\{\mu(x \odot (y \odot z)), \mu(y \odot (y \odot x))\}.$$

If $x \odot z \notin H$ then $y \odot (y \odot x) \notin H$ because H is a K-ideal. Thus

$$\mu(x \odot z) = \alpha_1 = \min\{\mu(x \odot (y \odot z)), \mu(y \odot (y \odot x))\}.$$

Hence μ is a fuzzy K-ideal of $\mathcal{K}$.
Let $x \in G$ and $f \in Aut(\mathcal{K})$. If $x \in H$, then $f(x) \in f(H) \subseteq H$. Thus

$$\mu^f(x) = \mu(f(x)) = \alpha_2 = \mu(x),$$

Otherwise,

$$\mu^f(x) = \mu(f(x)) = \alpha_1 = \mu(x).$$

Hence μ is a fuzzy characteristic K-ideal of $\mathcal{K}$. □

Lemma 3.15. *Let μ be a fuzzy K-ideal of a K-algebra $\mathcal{K}$ and let $x \in G$. Then $\mu(x) = t$ if and only if $x \in \mu_t$ and $x \notin \mu_s$, for all $s > t$.*

Proof. Straightforward. □

Theorem 3.16. *A fuzzy K-ideal is characteristic if and only if each its level set is a characteristic K-ideal.*

Proof. Suppose that μ is fuzzy characteristic and let $t \in Im(\mu)$, $f \in$Aut$(\mathcal{K})$ and $x \in \mu_t$. Then

$$\mu^f(x) = \mu(x) \geq t \Longrightarrow \mu(f(x)) \geq t \Longrightarrow f(x) \in \mu_t.$$

Thus $f(\mu_t) \subseteq \mu_t$. Let $x \in \mu_t$ and $y \in G$ such that $f(y) = x$. Then

$$\mu(y) = \mu^f(y) = \mu(f(y)) = \mu(x) \geq t \Longrightarrow y \in \mu_t$$

so that $x = f(y) \in \mu_t$. Consequently, $\mu_t \subseteq f(\mu_t)$. Hence $f(\mu_t) = \mu_t$, i.e., μ_t is characteristic.
Conversely, suppose that each level K-ideal of μ is characteristic and let $x \in G$, $f \in Aut(\mathcal{K})$ and $\mu(x) = t$. Then, by virtue of Lemma 4.1.19, $x \in \mu_t$ and $x \notin \mu_s$, for all $s > t$. It follows from the assumption that $f(x) \in f(\mu_t) = \mu_t$, so that $\mu^f(x) = \mu(f(x))) \geq t$. Let $s = \mu^f(x)$ and assume that $s > t$. Then $f(x) \in \mu_s = f(\mu_s)$, which implies from the injectivity of f that $x \in \mu_s$, a contradiction. Hence $\mu^f(x) = \mu(f(x)) = t = \mu(x)$ showing that μ is fuzzy characteristic. □

Definition 3.17. Let μ and λ be fuzzy K-ideals of $\mathcal{K}$. Then μ is said to be a *fuzzy same type* with λ if there exists $f \in Aut(\mathcal{K})$ such that $\mu(x) = \lambda(f(x))$ for all $x \in G$.

Theorem 3.18. *Let μ and λ be fuzzy K-ideals of $\mathcal{K}$. Then the following are equivalent:*

(i) *μ is fuzzy same type with λ,*

(ii) *$\mu \circ f = \lambda$ for some $f \in Aut(\mathcal{K})$,*

(iii) $g(\mu) = \lambda$ *for some* $g \in Aut(\mathcal{K})$,

(iv) $h(\lambda) = \mu$ *for some* $h \in Aut(\mathcal{K})$,

(v) *there exist* $h \in Aut(\mathcal{K})$ *such that* $U(\mu; t) = h(U(\lambda; t))$ *for all* $t \in [0, 1]$.

Proof. (i) $\Rightarrow$ (ii): Proof follows immediately from Definition 4.1.21.
(ii) $\Rightarrow$ (iii): Suppose that $\mu \circ f = \lambda$ for some $f \in Aut(\mathcal{K})$. Then $\mu(f(x)) = \lambda(x)$ for all $x \in \mathcal{K}$. It follows that

$$f^{-1}(\mu)(x) = \sup_{y \in f(x)} \mu(y) = \mu(f(x)) = \lambda(x) \quad \text{for all x, y } \in \text{G}.$$

If $g = f^{-1}$, then $g \in Aut(\mathcal{K})$ and $g(\mu) = \lambda$.
(iii) $\Rightarrow$ (iv): Suppose that $g(\mu) = \lambda$ for some $g \in Aut(\mathcal{K})$ holds. Then

$$\lambda(x) = g(\mu) = \sup_{y \in g^{-1}(x)} \mu(y) = \mu(g^{-1}(x)) \quad \text{for all x, y } \in \text{G}.$$

Hence

$$g^{-1}(x) = \sup_{y \in g(x)} \lambda(y) = \lambda(g(y)) = \mu(g^{-1}(g(x))) = \mu(x), \text{ for all x, y } \in \text{G}.$$

If $h = g^{-1}$, then $h \in Aut(\mathcal{K})$ and $h(\lambda) = \mu$.
(iv) $\Rightarrow$ (v): If there exists $h \in$ Aut($\mathcal{K}$) such that $h(\lambda) = \mu$, then

$$\mu(x) = h(\lambda)(x) = \sup_{y \in h^{-1}(x)} \lambda(y) = \lambda(h^{-1}(x)), \text{ for all x, y } \in \text{G}.$$

Let $t \in [0, 1]$. We need to show that $U(\mu; t) = h(U(\lambda; t))$. If $x \in U(\mu; t)$, then $\lambda(h^{-1}(x)) = \mu \geq t$ which implies that $h^{-1}(x) \in U(\lambda; t)$, i.e. $x \in h(U(\lambda; t))$. This shows that $U(\mu; t) \subseteq h(U(\lambda; t))$. On the other hand, let $x \in h(U(\lambda; t))$. Then $h^{-1}(x) \in U(\lambda; t)$ and so $\mu(x) = \lambda(h^{-1}(x)) \geq t$. It follows that $x \in U(\mu; t)$. Hence $h(U(\lambda; t)) \subseteq U(\mu; t)$ and (v) holds.
(v) $\Rightarrow$ (i): Suppose that there exists $h \in$ Aut($\mathcal{K}$) such that $U(\mu; t) = h(U(\lambda; t))$ for all $t \in [0, 1]$. Let $x \in \mathcal{K}$ and $\mu(x)$. Putting $\lambda(h^{-1}(x)) = s$, then $h^{-1}(x) \in U(\lambda; s)$ and hence $x \in h(U(\lambda; s)) = U(\mu; s)$. It follows that $\mu(x) \geq s = \lambda(h^{-1}(x))$. Hence $\mu(x) = \lambda(h^{-1}(x))$ for all $x \in \mathcal{K}$. Indeed $h^{-1} \in$ Aut($\mathcal{K}$), then μ is fuzzy same type with λ. This completes the proof. □

Theorem 3.19. *Let* μ *and* λ *be* K*-fuzzy ideals of* $\mathcal{K}$. *Then* μ *is a fuzzy* K*-ideal having the same type of* λ *if and only if* μ *is isomorphic to* λ.

Proof. We only need to prove the necessity part because the sufficiency part is trivial. Since μ is a fuzzy same type with λ, there exists $\phi \in Aut(\mathcal{K})$ such that

$$\mu(x) = \lambda(\phi(x)) \text{ for all x, y } \in \text{G}.$$

Let $f : \mu(\mathcal{K}) \to \lambda(\mathcal{K})$ be a mapping defined by $f(\mu(x)) = \lambda(\phi(x))$ for all $x \in \mathcal{K}$. Clearly f is surjective. Next, f is injective because if $f(\mu(x)) = f(\mu(y))$ for all $x, y \in \mathcal{K}$, then $\lambda(\phi(x)) = \lambda(\phi(y))$ and hence $\mu(x) = \lambda(y)$ for all $x, y \in \mathcal{K}$. Finally, f is homomorphism because for $x, y \in \mathcal{K}$,

$$f(\mu(x \odot y)) = \lambda(\phi(x \odot y)) = \lambda(\phi(x) \odot \phi(y)).$$

Hence μ is isomorphic to λ. □

3.2 Quotient K-algebras via fuzzy K-ideals

Definition 3.20. A quotient K-algebra is a K-algebra that is the quotient of a K-algebra $\mathcal{K}$ and one of its ideal I, denoted $\mathcal{K}/I$. Let I be a K-idea of $\mathcal{K}$, then for all $x, y \in \mathcal{K}$ and $x \odot I, y \odot I \in \mathcal{K}/I$, we define $(x \odot I) \odot (y \odot I) = (x \odot y) \odot I$.

Theorem 3.21. *Let I be a K- ideal of a K-algebra $\mathcal{K}$. If μ is a fuzzy K-ideal of $\mathcal{K}$, then the fuzzy set μ^* of $\mathcal{K}/I$ defined by*

$$\mu^*(a \odot I) = \sup_{x \in I} \mu(a \odot x)$$

is a fuzzy K-ideal of the quotient algebra $\mathcal{K}/I$ of $\mathcal{K}$ with respect to I.

Proof. Clearly, μ^* is well-defined. It is easy to see that $\mu^*(e) \geq \mu^*(x \odot I)$ for all $x \odot I \in \mathcal{K}$. Let $x \odot I, y \odot I, z \odot I \in \mathcal{K}/I$, then

$$\begin{aligned}
\mu^*((x \odot I) \odot (z \odot I)) &= \mu_A^*((x \odot z) \odot I) \\
&= \sup_{u \in I} \mu((x \odot z) \odot u) \\
&= \sup_{u = s \odot t \in I} \mu(x \odot z) \odot (s \odot t) \\
&\geq \sup_{s,\, t \in I} \min\{\mu(x \odot (y \odot z)) \odot s), \mu((y \odot (y \odot x)) \odot t)\} \\
&= \min\{\sup_{s \in I} \mu(x \odot (y \odot z)) \odot s, \sup_{t \in I} \mu((y \odot (y \odot x)) \odot t)\} \\
&= \min\{\mu^*(x \odot (y \odot z)) \odot I, \mu^*((y \odot (y \odot x)) \odot I)\}.
\end{aligned}$$

Hence μ^* is a fuzzy K-ideal of $\mathcal{K}/I$. □

Theorem 3.22. *Let I be a K-ideal of a K-algebra $\mathcal{K}$. Then there is a one-to-one correspondence between the set of fuzzy K-ideals μ of $\mathcal{K}$ such that $\mu(e) = \mu(s)$ for all $s \in I$ and the set of all fuzzy K-ideals μ^* of $\mathcal{K}/I$.*

Proof. Let μ be a fuzzy K-ideal of $\mathcal{K}$. Using Theorem 3.21, we prove that μ^* defined by

$$\mu^*(a \odot I) = \sup_{x \in I} \mu(a \odot x)$$

is a fuzzy K-ideal of $\mathcal{K}/I$. Since $\mu(e) = \mu(s)$ for all $s \in I$, by straightforward verification, we have $\mu(a \odot s) = \mu(a)$ for all $s \in I$, that is, $\mu^*(a \odot I) = \mu(a)$. Hence the correspondence $\mu \mapsto \mu^*$ is one-to-one. Let μ^* be a fuzzy K-ideal of $\mathcal{K}/I$ and define fuzzy set μ in $\mathcal{K}$ by $\mu(a) = \mu^*(a \odot I)$ for all $a \in I$.
For x, y, $z \in \mathcal{K}$, we have

$$\begin{aligned} \mu(x \odot z) &= \mu^*((x \odot z) \odot I) \\ &= \mu^*((x \odot I) \odot (z \odot I)) \\ &\geq \min\{\mu^*(x \odot (y \odot z)) \odot I, \mu^*((y \odot (y \odot x)) \odot I)\} \\ &= \min\{\mu(x \odot (y \odot z)), \mu(y \odot (y \odot x))\}. \end{aligned}$$

Thus μ is a fuzzy K-ideal of $\mathcal{K}$. Note that $\mu(z) = \mu^*(z \odot I) = \mu^*(I)$ for all $z \in I$, which shows that $\mu(z) = \mu(e)$ for all $z \in I$. This ends the proof. □

Theorem 3.23. *Let μ be a fuzzy K-ideal of a K-algebra $\mathcal{K}$ and let $\mu(e) = t$. Then the fuzzy subset μ^* of $\mathcal{K}/\overline{U}(\mu;t)$ defined by $\mu^*(x \odot \overline{U}(\mu;t)) = \mu(x)$ for all $x \in \mathcal{K}$ is a fuzzy K-ideal of $\mathcal{K}/\overline{U}(\mu;t)$.*

Proof. μ^* is well-defined because

$$\begin{aligned} x \odot \overline{U}(\mu;t) &= y \odot \overline{U}(\mu;t) \ \forall\, x, y \in \mathcal{G} \\ &\Rightarrow x \odot y \in \overline{U}(\mu;t) \\ &\Rightarrow \mu(x \odot y) = \mu(e) \\ &\Rightarrow \mu(x) = \mu(y) \\ &\Rightarrow \mu^*(x \odot \overline{U}(\mu(x);t)) = \mu^*(y \odot \overline{U}(\mu(x);t)). \end{aligned}$$

Next we show that μ^* is a fuzzy K-ideal of $\mathcal{K}$. Clearly, $\mu^*(e) \geq \mu^*(x \odot \overline{U}(\mu(x);t))$ for all $x \in \mathcal{K}$. For x, y, $z \in \mathcal{K}$,

$$\begin{aligned} U^*((x \odot z) \odot \overline{U}(\mu,t)) = \mu(x \odot z) &\geq \min\{\mu(x \odot (y \odot z)), \mu(y \odot (y \odot x))\} \\ &= \min\{U^*(x \odot (y \odot z) \odot \overline{U}(\mu,t)) \\ &\quad , U^*(y \odot (y \odot x) + \overline{U}(\mu,t))\}. \end{aligned}$$

This completes the proof. □

Theorem 3.24. *Let μ be a fuzzy K-ideal of a K-algebra $\mathcal{K}$ and let f be a fuzzy ideal of $\mathcal{K}/I$ such that $f(x \odot I) = f(I)$, then $x \in I$ there exists a fuzzy K-ideal μ of L such that $\overline{U}(\mu;t) = I$, where $\mu(e) = t$ and $f = \mu^*$.*

Proof. Define a fuzzy K-ideal μ of $\mathcal{K}$ by $\mu(x) = f(x \odot I)$ for all $x \in \mathcal{K}$. It is easy to see that μ is fuzzy K-ideal of $\mathcal{K}$ such that $U(\mu; t) = I$ because

$$\begin{aligned} &\Leftrightarrow \quad \mu(x) = t = \mu(e) \\ &\Leftrightarrow \quad f(x \odot I) = f(I) \\ &\Leftrightarrow \quad x \in I. \end{aligned}$$

We conclude that $\mu^* = f$ because

$$\mu^*(x \odot I) = \mu^*(x \odot \overline{U}(\mu; t)) = \mu(x) = f(x \odot I).$$

This ends the proof. □

Theorem 3.25. *(Fuzzy correspondence theorem) Let $f : \mathcal{K}_1 \to \mathcal{K}_2$ be a homomorphism of K-algebras $\mathcal{K}_1$ onto $\mathcal{K}_2$. Then the following hold:*

(i) *If μ is a fuzzy K-ideal of $\mathcal{K}_1$, then $f(\mu)$ is a fuzzy K-ideal of $\mathcal{K}_2$,*

(ii) *If λ is a fuzzy K-ideal of $\mathcal{K}_2$, then $f^{-1}(\lambda)$ is a fuzzy K-ideal of $\mathcal{K}_1$.*

Proof. Straightforward. □

Let μ be a fuzzy K-ideal of a K-algebra $\mathcal{K}$. For any x, $y \in \mathcal{K}$, define a binary relation $\sim$ on $\mathcal{K}$ by $x \sim y$ if and only if $\mu(x \odot y) = \mu(e)$. Then $\sim$ is a congruence relation of $\mathcal{K}$. We denote $\mu[x]$ the equivalence class containing x, and $\mathcal{K}/\mu = \{\mu[x] \mid x \in \mathcal{K}\}$ the set of all equivalence classes of $\mathcal{K}$. Then $\mathcal{K}/\mu$ is a K-algebra under the following operation:

$$\mu[x] \odot \mu[y] = \mu[x \odot y] \quad \text{for all x, y } \in \text{G}.$$

Theorem 3.26. *(First fuzzy isomorphism theorem) Let $f : \mathcal{K}_1 \to \mathcal{K}_2$ be an epimorphism of K-algebras and let μ be a fuzzy K-ideal of $\mathcal{K}_2$. Then $\frac{\mathcal{K}_1}{f^{-1}(\mu)} \cong \frac{\mathcal{K}_2}{\mu}$.*

Proof. Define a map $\theta : \frac{\mathcal{K}_1}{f^{-1}(\mu)} \to \frac{\mathcal{K}_2}{\mu}$ by $\theta(f^{-1}(\mu)[x]) = \mu[f(x)]$. θ is well-defined since

$$\begin{aligned} & f^{-1}(\mu)[x] = f^{-1}(\mu)[y] \\ \Rightarrow\ & f^{-1}(\mu)(x \odot y) = f^{-1}(\mu)(e) \\ \Rightarrow\ & \mu(f(x) \odot f(y)) = \mu(f(e)) \\ \Rightarrow\ & \mu(f(x) \odot f(y)) = \mu(e), \\ & i.e., \mu[f(x)] = \mu[f(y)]. \end{aligned}$$

θ is one to one because

$$\begin{aligned}
&\mu[f(x)] = \mu[f(y)] \\
&\Rightarrow \mu(f(x) \odot f(y)) = \mu(e) \\
&\Rightarrow \mu(f(x) \odot f(y)) = \mu(f(e)) \\
&\Rightarrow f^{-1}(\mu)(x \odot y) = f^{-1}(\mu)(e) \\
&\Rightarrow f^{-1}(\mu)[x] = f^{-1}(\mu)[y].
\end{aligned}$$

Since f is an onto, θ is an onto. Finally, θ is a homomorphism because

$$\begin{aligned}
\theta(f^{-1}(\mu)[x] \odot f^{-1}(\mu)[y]) &= \theta(f^{-1}(\mu)[x \odot y]) \\
&= \mu[f(x \odot y)] \\
&= \mu[f(x) \odot f(y)] \\
&= \mu[f(x)] \odot \mu[f(y)] \\
&= \theta(f^{-1}(\mu)[x]) \odot \theta(f^{-1}(\mu)[y]).
\end{aligned}$$

Hence

$$\frac{\mathcal{K}_1}{f^{-1}(\mu)} \cong \frac{\mathcal{K}_2}{\mu}.$$

□

We state the following fuzzy isomorphism Theorems without proofs.

Theorem 3.27. *(Second fuzzy isomorphism theorem) Let μ be a fuzzy subalgebra of K-algebra and let λ be a fuzzy K-ideal of a K-algebra. Then*

(i) *λ is a fuzzy K-ideal of $\mu \odot \lambda$,*

(ii) *$\mu \cap \lambda$ is a fuzzy K-ideal of μ,*

(iii) $\frac{\mu \odot \lambda}{\lambda} \cong \frac{\mu}{\mu \cap \lambda}$.

Theorem 3.28. *(Third fuzzy isomorphism theorem) Let $\mathcal{K}_1$ be a K-algebra having fuzzy K-ideals μ and λ with $\mu \leq \lambda$. Then*

(i) *$\frac{\lambda}{\mu}$ is fuzzy K-ideal of $\frac{\mathcal{K}_1}{\mu}$,*

(ii) $\frac{\mathcal{K}_1}{\mu} / \frac{\lambda}{\mu} \cong \frac{\mathcal{K}_1}{\lambda}$.

Lemma 3.29. *(Fuzzy Zassenhaus lemma) Let μ and λ be fuzzy subalgebras of a K-algebra $\mathcal{K}$ and let μ_1 and λ_1 be fuzzy K-ideals of μ and λ, respectively. Then*

(a) *$\mu_1 \odot (\mu \cap \lambda_1)$ is a fuzzy K-ideal of $\mu_1 \odot (\mu \cap \lambda)$,*

(b) *$\lambda_1 \odot (\mu_1 \cap \lambda)$ is a fuzzy K-ideal of $\lambda_1 \odot (\mu \cap \lambda)$,*

(c) $\frac{\mu_1 \odot (\mu \cap \lambda)}{\mu_1 \odot (\mu \cap \lambda_1)} \simeq \frac{\lambda_1 \odot (\mu \cap \lambda)}{\lambda_1 \odot (\mu_1 \cap \lambda)}$.

4 Conclusions

In the present paper, we have presented some properties of fuzzy K-ideals of K-algebras. The obtained results can be used in various fields such as artificial intelligence, signal processing, multiagent systems, pattern recognition, robotics, computer networks, genetic algorithm, neural networks, expert systems, decision making, automata theory and medical diagnosis.

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