On The Gracefulness of The Digraphs $n - \vec{C}_m$ *

Lingqi Zhao[1],Siqintuya[2], Jirimutu[2]

[1]College of Computer Science and Technology
[2]College of Mathematics
Inner Mongolian University for Nationalities
Tongliao 028043, P.R.China
e-mail: jrmt@sina.com

Abstract

A digraph $D(V, E)$ is said to be graceful if there exists an injection $f: V(D) \to \{0, 1, \cdots, |E|\}$ such that the induced function $f': E(D) \to \{1, 2, \cdots, |E|\}$ which is defined by $f'(u, v) = [f(v) - f(u)] \pmod{(|E| + 1)}$ for every directed edge (u, v) is a bijection. Here, f is called a graceful labeling(graceful numbering) of digraph $D(V, E)$, while f' is called the induced edge's graceful labeling of digraph $D(V, E)$. In this paper, we discuss the gracefulness of the digraph $n - \vec{C}_m$ and prove the digraph $n - \vec{C}_{17}$ is graceful for even n.

MSC: 05C65

Keywords: *Digraph, Directed Cycles , Graceful Graph, Graceful Labeling*

*This research is supported by Inner Mongolia University for Nationalities fund .

1 Introduction

A graph $G(V, E)$ is said to be graceful if there exists an injection $f : V(G) \to \{0, 1, \cdots, |E|\}$ such that the induced function $f' : E(G) \to \{1, 2, \cdots, |E|\}$ which is defined by $f'(u, v) = |f(u) - f(v)|$ for every edge (u, v) is a bijection. Here, f is called a graceful labeling (graceful numbering) of G, while f' is called the induced edge's graceful labeling of G.

A digraph $D(V, E)$ is said to be graceful if there exists an injection $f : V(D) \to \{0, 1, \cdots, |E|\}$ such that the induced function $f' : E(D) \to \{1, 2, \cdots, |E|\}$ which is defined by $f'(u, v) = [f(v) - f(u)] \pmod{|E| + 1}$ for every directed edge (u, v) is a bijection, where $[v] \pmod{n}$ denotes the least positive residue of v modulo n.

Let $\vec{C}_m$ denote the directed cycle on m vertices, $n - \vec{C}_m$ denotes the graph obtained from any n copies of $\vec{C}_m$ which have just one common edge.

As to the gracefulness of $n-\vec{C}_m$ we know the following results: Ma has showed in [3] that $n-\vec{C}_3$ is a graceful graph. Xu, Jirimutu et al. have proved that $n - \vec{C}_m$ is a graceful digraph for $m = 4, 6, 8, 10, 12$ and even n in [5], and for $m = 5, 7, 9, 11, 13$,$m = 15$ and even n in [6],[7], respectively. In [?], Wei Feng and Jirimutu put forward a conjecture and a problem as following:

Conjecture 1. For any positive even n and any integer $m \geq 14$, the digraph $n - \vec{C}_m$ is graceful .

Problem 1. For any positive odd n and any integer $m \geq 14$, whether the digraph $n - \vec{C}_m$ is graceful ?

In this paper, we discuss the gracefulness of the digraph $n - \vec{C}_m$

and prove the digraph $n-\vec{C}_{17}$ is graceful for even n.

2 Main Results

Let $\vec{C}_m^1, \vec{C}_m^2, \cdots, \vec{C}_m^n$ denote n directed cycles of digraph $n-\vec{C}_m$. Two vertices of common edge of $\vec{C}_m^i$'s are denoted by v_0 and v_{m-1}, respectively. Other $m-2$ vertices of $\vec{C}_m^i$ are denoted by v_j^i for $j = 1, 2, \ldots, m-2$ and $i = 1, 2, \ldots, n$. For convenience, we put $v_0^1 = v_0^2 = \cdots = v_0^n = v_0$, $v_{m-1}^1 = v_{m-1}^2 = \cdots = v_{m-1}^n = v_{m-1}$, and take subscripts j's modulo m. Obviously, $|E(n-\vec{C}_m)| = (m-1)n+1$.

Suppose that $n-\vec{C}_m$ is graceful, f and f' are its graceful labeling and the induced edge's graceful labeling, respectively. For every i, it is easy to see that

$$\sum_{j=0}^{m-1} [f(v_j^i) - f(v_{j-1}^i)] \equiv \sum_{j=0}^{m-1} f(v_j^i) - \sum_{j=0}^{m-1} f(v_{j-1}^i) = 0 \pmod{((m-1)n+2)},$$

which means that there exists an integer k_i, such that

$$\sum_{j=0}^{m-1} [f(v_j^i) - f(v_{j-1}^i)] = k_i((m-1)n+2), \ (1 \le i \le n). \qquad (2.1)$$

It implies that there exists an integer k, such that

$$\sum_{i=1}^{n}\sum_{j=0}^{m-1} [f(v_j^i) - f(v_{j-1}^i)] = k((m-1)n+2). \qquad (2.2)$$

On the other hand, put $q = |E(n-\vec{C}_m)| = (m-1)n+1$ and $d = [f(v_0) - f(v_{m-1})]$, according to the definition

$$\sum_{i=1}^{n}\sum_{j=0}^{m-1} [f(v_j^i) - f(v_{j-1}^i)] = (n-1)d + \frac{1}{2}q(q+1) = k(q+1) (2.3)$$

We obtain the necessary condition of $n-\vec{C}_m$ is graceful as follows:

$$(n-1)d \equiv 0 \pmod{\frac{(q+1)}{2}}. \tag{2.4}$$

Considering the range of n, we often let $f(v_0)=0$ and $f(v_{m-1}) = \frac{q+1}{2}$ in following discussion. So, $d=[f(v_0)-f(v_{m-1})] = [-\frac{q+1}{2}] \equiv \frac{q+1}{2} \pmod{(q+1)}$, it satisfies the condition of (2.4).

Put $d=\frac{q+1}{2}$ into (2.3), then

$$(n-1)(\frac{q+1}{2}) + (\frac{1}{2}q(q+1)) = k(q+1),$$

and

$$\frac{n-1}{2} + \frac{n(m-1)+1}{2} = k,$$

namely, $nm \equiv 0 \pmod 2$. So, we obtain the following Lemmas.

Lemma 1. For any positive integer n, and $m \geq 3$, the necessary condition of the digraph $n-\vec{C}_m$ is graceful is $nm \equiv 0$ (mod2).

Lemma 2. If $nm \equiv 1$(mod2), then $n-\vec{C}_m$ is not graceful.

Lemma 3. If $n-\vec{C}_m$ is graceful and $f(v_0)=0$, then we have $f(v_{m-1}) = \frac{q+1}{2}$.

Theorem 1 *For any positive even n, the digraph $n-\vec{C}_{17}$ is graceful.*

Proof. We define the vertex label f of $n-\vec{C}_{17}$ as follows:

$$f(v_0)=0,\ f(v_{16})=8n+1.$$

$$f(v_j^i) = \begin{cases} \frac{j-1}{2}n+i, & j=1,3, i=1,\ 2,\ \cdots, n \\ \frac{3j}{2}n+1-i, & j=2,4, i=1,2,\cdots,n \\ (\frac{4j}{2}-1)n+\lfloor\frac{j}{2}-1\rfloor+i, & j=5,7, i=1,\ 2,\ \cdots, n \\ 4(j-5)n+\frac{j-4}{2}-i, & j=6,8, i=1,2,\cdots,n \\ 5n+1-i, & j=10, i=1,2,\cdots,n \\ 15n+1+i, & j=15, i=1,2,\cdots,n \end{cases}$$

$$f(v_j^i) = \begin{cases} 7n+1+i, & j=9, i=1,\ 2,\ \cdots,\ \frac{n}{2} \\ 12n+2+i, & j=9, i=\frac{n}{2}+1,\ \frac{n}{2}+2, \cdots,\ n \\ 12n+1+i, & j=11, i=1,\ 2,\ \cdots,\ \frac{n}{2} \\ 10n+1+i, & j=11, i=\frac{n}{2}+1,\ \frac{n}{2}+2, \cdots,\ n \\ (j-5)n+2-i, & j=12,14, i=1,\ 2,\ \cdots,\ \frac{n}{2} \\ (j-3)n+2-i, & j=12,14, i=\frac{n}{2}+1,\ \frac{n}{2}+2, \cdots,\ n \\ 6n+i, & j=13, i=1,\ 2,\ \cdots,\ \frac{n}{2} \\ 13n+2+i, & j=13, i=\frac{n}{2}+1,\ \frac{n}{2}+2, \cdots,\ n \end{cases}$$

For any integers $a \le b$, let $[a, b]$ denote the set of all consecutive integers from a to b.

Firstly, we show that f is an injective mapping from $V(n - \vec{C}_{17})$ into $[0, 16n+1]$.

For $j \in [0, 16]$, put $S_j = \{f(v_j^i) | i \in [1, n]\}$, and set $S_{j,1} = \{f(v_j^i) | i \in [1, \frac{n}{2}]\}$, $S_{j,2} = \{f(v_j^i) | i \in [\frac{n}{2}+1, n]\}$. Then

$$\begin{aligned}
S_0 &= \{f(v_0)\} = \{0\} \\
S_1 &= \{f(v_1^i)\} = \{1, 2, \cdots, n\} \\
S_3 &= \{f(v_3^i)\} = \{n+1, n+2, \cdots, 2n\} \\
S_2 &= \{f(v_2^i)\} = \{2n+1, \cdots, 3n-1, 3n\} \\
S_6 &= \{f(v_6^i)\} = \{3n+1, \cdots, 4n-1, 4n\} \\
S_{10} &= \{f(v_{10}^i)\} = \{4n+1, \cdots, 5n-1, 5n\} \\
S_4 &= \{f(v_4^i)\} = \{5n+1, \cdots, 6n-1, 6n\} \\
S_{13,1} &= \{f(v_{13}^i)\} = \{6n+1, \cdots, 6n+\tfrac{n}{2}\} \\
S_{12,1} &= \{f(v_{12}^i)\} = \{6n+\tfrac{n}{2}+2, \cdots, 7n+1\} \\
S_{9,1} &= \{f(v_9^i)\} = \{7n+2, \cdots, 7n+\tfrac{n}{2}+1\} \\
S_{16} &= \{f(v_{16}^i)\} = \{8n+1\} \\
S_{12,2} &= \{f(v_1^i 2)\} = \{8n+2, 8n+3, \cdots, 8n+\tfrac{n}{2}+1\} \\
S_{14,1} &= \{f(v_1^i 4)\} = \{8n+\tfrac{n}{2}+2, 8n+\tfrac{n}{2}+3, \cdots, 9n+1\} \\
S_5 &= \{f(v_5^i)\} = \{9n+2, 9n+3, \cdots, 10n+1\}
\end{aligned}$$

$$
\begin{aligned}
S_{14,2} &= \{f(v_{14}^i)\} = \{10n+2, 10n+3, \cdots, 10n+\tfrac{n}{2}+1\} \\
S_{11,2} &= \{f(v_1^i 1)\} = \{10n+\tfrac{n}{2}+2, 10n+\tfrac{n}{2}+3, \cdots, 11n+1\} \\
S_8 &= \{f(v_8^i)\} = \{11n+2, 11n+3, \cdots, 12n+1\} \\
S_{11,1} &= \{f(v_1^i 1)\} = \{12n+2, 12n+3, \cdots, 12n+\tfrac{n}{2}+1\} \\
S_{9,2} &= \{f(v_9^i)\} = \{12n+\tfrac{n}{2}+3, 12n+3, \cdots, 13n+2\} \\
S_{7,1} &= \{f(v_7^i)\} = \{13n+3, 13n+4, \cdots, 13n+\tfrac{n}{2}+2\} \\
S_{13,2} &= \{f(v_1^i 3)\} = \{13n+\tfrac{n}{2}+3, 13n+\tfrac{n}{2}+4, \cdots, 14n+2\} \\
S_{7,2} &= \{f(v_7^i)\} = \{14n+\tfrac{n}{2}+2, 14n+\tfrac{n}{2}+3, \cdots, 15n+1\} \\
S_{15} &= \{f(v_{15}^i)\} = \{15n+2, 15n+3, \cdots, 16n+1\}
\end{aligned}
$$

It is obvious that $S_i \cap S_j = \varnothing$ for $i, j \in [0, m-1]$ and $i \neq j$, which yields that f is an injection from $V(n - \vec{C}_{17})$ into $[0, 16n+1]$.

Secondly, we show the induced edges labeling f' is a bijection from $E((n - \vec{C}_{17})$ onto $[1, 16n+1]$. Set $[f(v_j^i) - f(v_{j-1}^i)] = f(v_j^i) - f(v_{j-1}^i)$ (mod $((m-1)n+2)$).

Denote $B_j = B_{j,1} \cup B_{j,2}$, where

$$
\begin{aligned}
B_{j,1} &= \{[f(v_j^i) - f(v_{j-1}^i)] | j \in [0,16],\ i \in [1, \tfrac{n}{2}]\} \\
B_{j,2} &= \{[f(v_j^i) - f(v_{j-1}^i)] | j \in [0,16],\ i \in [\tfrac{n}{2}+1, n]\},
\end{aligned}
$$

and let $B = \bigcup_{j=0}^{16} B_j$. Then, in order to prove that f' is a bijection, it suffices to show $B = [1, 16n+1]$, or $[1, 16n+1] \subseteq B$ equivalently.

(1)For $j = 1, i \in [1, n]$, $\quad B_1 = \{1, 2, \cdots, n\} = [1, n]$.

(2)For $j = 2, i \in [1, n]$ and $j = 9,\ i \in [\frac{n}{2}+1, n]]$ and $j = 14,\ i \in [1, \frac{n}{2}]$, we have

$$
B_2 \cup B_{9,2} \cup B_{14,1} = \{n+1, n+2, \cdots, 3n\} = [n+1, 3n],
$$

which and (1) imply $[1, 3n] \subseteq B$.

(3) For $j = 4,\ i \in [1, n]$ and $j = 5,\ i \in [1, n]$, we have

$$
B_4 \cup B_5 = \{3n+1, 3n+2, \cdots, 5n\} = [3n+1, 5n],
$$

which and (2) imply $[1,5n] \subseteq B$.

(4) For $j=15,\ i \in [1,n]$ and $j=11,13,\ i \in [\frac{n}{2}+1,n]$, we have

$$B_{15} \cup B_{11,2} \cup B_{13,2} = \{5n+1, 5n+2, \cdots, 7n\} = [5n+1, 7n],$$

which and (3) imply $[1,7n] \subseteq B$.

(5) For $j=10,\ i \in [1,\frac{n}{2}]$ and $j=11,\ i \in [1,\frac{n}{2}]]$ and $j=0,\ i \in [1,n]$ and $j=16,\ i \in [1,n]$, we have

$$\begin{aligned} B_0 \cup B_{10,2} \cup B_{11,1} \cup B_{16} &= \{7n+1, 7n+2, \cdots, 9n+1\} \\ &= [7n+1, 9n+1], \end{aligned}$$

which and (4) imply $[1,9n+1] \subseteq B$.

(6) For $j=6,\ i \in [1,n]$ and $j=7,12,\ i \in [1,\frac{n}{2}]$, we have

$$\begin{aligned} B_6 \cup B_{7,1} \cup B_{12,1} &= \{9n+2, 9n+3, \cdots, 11n+1\} \\ &= [9n+2, 11n+1], \end{aligned}$$

which and (5) imply $[1,11n+1] \subseteq B$.

(7) For $j=7,8,14,\ i \in [\frac{n}{2}+1,n]$; $j=9,\ i \in [1,\frac{n}{2}]$, we have

$$\begin{aligned} B_{7,1} \cup B_{8,2} \cup B_{14,2} \cup B_{9,1} &= \{11n+2, 11n+3, \cdots, 13n+1\} \\ &= [11n+2, 13n+1], \end{aligned}$$

which and (6) imply $[1,13n+1] \subseteq B$.

(8) For $j=10,13,\ i \in [1,\frac{n}{2}]$; $j=3,\ i \in [1,n]$and $j=12,\ i \in [\frac{n}{2}+1,n]$, we have

$$\begin{aligned} B_{10,1} \cup B_{13,1} \cup B_3 \cup B_{12,2} &= \{13n+2, 13n+3, \cdots, 16n+1\} \\ &= [13n+2, 16n+1], \end{aligned}$$

which and (7) imply $[1,16n+1] \subseteq B$. So f' is an bijection, which completes the proof of $n-\vec{C}_{17}$ is graceful for even n. □

Acknowledgement

This research is supported by the project from Inner Mongolian University for the nationalities(MDX2009043).We wish to sincerely thank the referees for their valuable suggestions and improvements to the paper.

References

[1] J. A. Bondy, U S R Murty. Graph Theory with Applications, Macmillan, London and Elsevier, New York, 1976.

[2] J. A. Gallian, A Dynamic Survey of Graph Labeling. The Electronic Journal Of Combinatorics, # DS6(OCT)(2008 VERSION).

[3] Kejie Ma . Graceful graphs: The first edition Beijing: Beijing University press, 1991.

[4] S. A. Choudum, S.Pitchai Muthu Kishore. Graceful labeling of the union of paths and cycles. Discrete Math., 206(1999):105-117.

[5] Xirong Xu, Jirimutu. On the gracefulness of the digraph $n-\vec{C}_m$ for m=4,6,8,10. Ars Combinatoria,92:419-428 JUL 2009.

[6] Lingqi Zhao, Jirimutu. On the gracefulness of the digraph $n-\vec{C}_m$ for m=5,7,9,11,13. Journal of Prime Research and in Mathematics, 12(4)(2008):118-126.

[7] Lingqi Zhao, Jirimutu. On the gracefulness of the digraph $n-\vec{C}_m$ for m=15. Ars Combinatoria,to appear.