Generalized Frobenius partitions and mock-theta functions

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Abstract

Four new combinatorial identities involving certain generalized F-partition functions and n-colour partition functions are proved bijectively. This leads to new combinatorial interpretations of four mock theta functions of S. Ramanujan.

1 Introduction, Definitions and the Main Results

Recently in [1], the first author gave n-colour partition theoretic interpretations of the following mock theta functions of S. Ramanujan:

$$\psi(q) = \sum_{m=1}^{\infty} \frac{q^{m^2}}{(q;q^2)_m}, \tag{1.1.1}$$

$$F_0(q) = \sum_{m=0}^{\infty} \frac{q^{2m^2}}{(q;q^2)_m}, \tag{1.1.2}$$

$$\phi_0(q) = \sum_{m=0}^{\infty} q^{m^2}(-q;q^2)_m, \tag{1.1.3}$$

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and

$$\phi_1(q) = \sum_{m=0}^{\infty} q^{(m+1)^2}(-q;q^2)_m, \qquad (1.1.4)$$

where

$$(a;q)_n = \prod_{i=0}^{\infty} \frac{(1-aq^i)}{(1-aq^{n+i})}\text{ , for any constant } a.$$

We remark that $\psi(q)$ is of order 3 while the remaining three are of order 5. For the definitions of the mock theta functions and their order the reader is referred to [4].

In this paper we shall prove bijectively four new combinatorial identities involving certain F-partition functions and n-colour partition functions. This leads to new combinatorial interpretations of (1.1)-(1.4) in terms of F-partitions. Before we state our main results we recall the following definitions:

Definition 1 [3]. A two-rowed array of non-negative integers

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}$$

with each row arranged in non increasing order is called a generalized Frobenius partition or more simply an F-partition of ν if

$$\nu = r + \sum_{i=1}^{r} a_i + \sum_{i=1}^{r} b_i.$$

Definition 2 [2]. An n-colour partition (also called a partition with "n copies of n") of a positive integer ν is a partition in which a part of size n can come in n different colours denoted by subscripts : $n_1, n_2, \cdots, n_n$ and the parts satisfy the order

$1_1 < 2_1 < 2_2 < 3_1 < 3_2 < 3_3 < 4_1 < 4_2 < 4_3 < 4_4 < 5_1 < 5_2 \cdots$

Thus for example, the n-colour partitions of 3 are $3_1, 3_2, 3_3, 2_1 1_1, 2_2 1_1, 1_1 1_1 1_1$.

Definition 3 [2]. The weighted difference of two parts m_i, n_j, $m \geq n$ is defined by $m - n - i - j$ and is denoted by $((m_i - n_j))$.

In our next section we shall prove the following combinatorial identities:

Theorem 1. For $\nu \geq 1$, let $B_1(\nu)$ denote the number of F-partitions of ν such that

(1.a) $a_i \geq b_i$, $1 \leq i \leq r$,

(1.b) $b_i = a_{i+1} + 1$, $1 \leq i \leq r - 1$,

(1.c) $b_r = 0$.

And let $A_1(\nu)$ denote the number of n-colour partitions of ν such that

(1.d) even parts appear with even subscripts and odd with odd,

(1.e) the weighted difference of any two consecutive parts is 0, and
(1.f) for some k, k_k is a part.
Then $B_1(\nu) = A_1(\nu)$, for all ν.

Example. $B_1(8) = 3$, since the relevant F-partitions are $\begin{pmatrix} 7 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 5 & 0 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}$; also $A_1(8) = 3$, since the relevant n-colour partitions are 8_8, $7_5 + 1_1$, $6_2 + 2_2$.

Theorem 2. For $\nu \geq 0$, let $B_2(\nu)$ denote the number of F-partitions of ν such that
(2.a) $a_i \geq b_i$, $1 \leq i \leq r$,
(2.b) $b_i = a_{i+1} + 1$, $1 \leq i \leq r - 1$,
(2.c) $b_r = 0$,
(2.d) $a_r \neq 0$.
And let $A_2(\nu)$ denote the number of n-colour partitions of ν such that
(2.e) even parts appear with even subscripts and odd with odd greater than 1,
(2.f) the weighted difference of any two consecutive parts is 0,
(2.g) for some k, k_k is a part.
Then $B_2(\nu) = A_2(\nu)$, for all ν.

Theorem 3. For $\nu \geq 0$, let $B_3(\nu)$ denote the number of F-partitions of ν such that
(3.a) $a_i = b_i$ or $a_i = b_i + 1$, $1 \leq i \leq r$,
(3.b) $b_i = a_{i+1} + 1$, $1 \leq i \leq r - 1$,
(3.c) $b_r = 0$.
And let $A_3(\nu)$ denote the number of n-colour partitions of ν such that
(3.d) the parts are of the type $(2k+1)_1$ or $(2k)_2$,
(3.e) the minimum part is 1_1 or 2_2,
(3.f) the weighted difference of any two consecutive parts is 0,
Then $B_3(\nu) = A_3(\nu)$, for all ν.

Example. $B_3(17) = 2$, since the relevant F-partitions are $\begin{pmatrix} 4 & 3 & 1 \\ 4 & 2 & 0 \end{pmatrix}$ and $\begin{pmatrix} 4 & 2 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix}$; also $A_3(17) = 2$, since the relevant n-colour partitions are $9_1 + 6_2 + 2_2$ and $8_2 + 5_1 + 3_1 + 1_1$.

Theorem 4. For $\nu \geq 1$, let $B_4(\nu)$ denote the number of F-partitions of ν such that
(4.a) $a_i = b_i$ or $a_i = b_i + 1$, $1 \leq i \leq r$,
(4.b) $b_i = a_{i+1} + 1$, $1 \leq i \leq r - 1$,
(4.c) $a_r = b_r = 0$.
And let $A_4(\nu)$ denote the number of n-colour partitions of ν such that
(4.d) the parts are of the type $(2k+1)_1$ or $(2k)_2$,

(4.e) the minimum part is 1_1, and
(4.f) the weighted difference of any two consecutive parts is 0,
Then $B_4(\nu) = A_4(\nu)$, for all ν.

It was proved in [1] that

$$\psi(q) = \sum_{\nu=1}^{\infty} A_1(\nu) q^{\nu}, \tag{1.1.5}$$

$$F_0(q) = \sum_{\nu=0}^{\infty} A_2(\nu) q^{\nu}, \tag{1.1.6}$$

$$\phi_0(q) = \sum_{\nu=0}^{\infty} A_3(\nu) q^{\nu}, \tag{1.1.7}$$

and

$$\phi_1(q) = \sum_{\nu=1}^{\infty} A_4(\nu) q^{\nu}. \tag{1.1.8}$$

Theorems 1-4 and Equations (1.5)-(1.8) lead to the following new combinatorial interpretations of the mock theta functions (1.1)-(1.4):

$$\psi(q) = \sum_{\nu=1}^{\infty} B_1(\nu) q^{\nu}, \tag{1.1.9}$$

$$F_0(q) = \sum_{\nu=0}^{\infty} B_2(\nu) q^{\nu}, \tag{1.1.10}$$

$$\phi_0(q) = \sum_{\nu=0}^{\infty} B_3(\nu) q^{\nu}, \tag{1.1.11}$$

and

$$\phi_1(q) = \sum_{\nu=1}^{\infty} B_4(\nu) q^{\nu}. \tag{1.1.12}$$

2 Proofs

Proof of Theorem 1. We establish a 1-1 correspondence between the F-partitions enumerated by $B_1(\nu)$ and n-colour partitions enumerated by $A_1(\nu)$. We do this by mapping each column $\begin{smallmatrix} a \\ b \end{smallmatrix}$ of the Frobenius symbol to a single part m_i of the n-colour partition . The mapping ϕ is:

$$\phi : \begin{pmatrix} a \\ b \end{pmatrix} \rightarrow (a+b+1)_{a-b+1},\ a \geq b \tag{2.2.1}$$

and the inverse mapping ϕ^{-1} is given by

$$\phi^{-1} : m_i \rightarrow \begin{pmatrix} (m+i-2)/2 \\ (m-i)/2 \end{pmatrix}. \tag{2.2.2}$$

Since $(a+b+1)$ and $(a-b+1)$ have the same parity, (2.1) implies (1.d). Now for any two adjacent columns $\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}$ in the Frobenius symbol with $\phi \begin{pmatrix} a \\ b \end{pmatrix} = m_i$ and $\phi \begin{pmatrix} c \\ d \end{pmatrix} = n_j$, we have

$$\begin{aligned} ((m_i - n_j)) &= (a+b+1) - (a-b+1) - (c+d+1) - (c-d+1) \\ &= 2b - 2c - 2 \\ &= 0 \text{ (by } (1.b)). \end{aligned} \tag{2.2.3}$$

Equation (2.3) implies (1.e). (1.c) and (2.1) imply (1.f).
To see the reverse implication, we note that

$$\phi^{-1} : m_i \rightarrow \begin{pmatrix} (m+i-2)/2 \\ (m-i)/2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

and

$$\phi^{-1} : n_j \rightarrow \begin{pmatrix} (n+j-2)/2 \\ (n-j)/2 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$$

that is,

$$\begin{aligned} a &= (m+i-2)/2, \\ b &= (m-i)/2, \\ c &= (n+j-2)/2, \\ d &= (n-j)/2, \end{aligned}$$

and so

$$a - b = i - 1, \tag{2.2.4}$$

$$c - d = j - 1, \tag{2.2.5}$$

$$b - c = \frac{1}{2}((m_i - n_j)) + 1. \tag{2.2.6}$$

clearly (2.4) and (2.5) imply (1.a). (1.e) and (2.6) imply (1.b). Finally, in view of the fact that k_k must be the smallest part of its partition (1.f) and (2.2) imply (1.c). This completes the proofs of Theorem 1.
Proofs of Theorems 2-4 are similar to the proof of Theorem 1 and hence are omitted.

Conclusion

It would be of interest to provide combinatorial interpretations for other mock theta functions also by using the method of this paper.

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