On the existence and embedding of edge-coloured graph decompositions

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Abstract

Wilson's Theorem [8] guarantees that for any given graph G, there exists an integer n such that the complete graph K_n can be decomposed into edge-disjoint isomorphic copies of G. The corresponding result is not true for edge-coloured graph decompositions. Let rK_n^* denote the edge-coloured graph with n vertices, r colours, and precisely one edge of each colour joining each pair of distinct vertices. It is easy to give an example of an edge-coloured graph G^* such that there is no finite integer n for which it is possible to decompose rK_n^* into edge-disjoint colour-identical copies of G^*. We investigate the problem of determining precisely when an edge-coloured graph G^* with r colours admits a G^*-decomposition of rK_n^* for some finite n. We also investigate conditions under which any partial edge-coloured G^*-decomposition of rK_n^* has a finite embedding.

1 Introduction

We use the term graph for multigraph and say explicitly whenever we mean a simple graph. All graphs are assumed finite and do not have loops.

An *edge-coloured graph* G^* is a graph G together with an assignment of colours to its edges. Edge-coloured graphs will always be identified by a superscript asterisk. The vertex set, edge set and the set of colours assigned to the edges of an edge-coloured graph G^* are denoted by $V(G^*)$, $E(G^*)$ and $C(G^*)$ respectively, and $|V(G^*)|$, $|E(G^*)|$ and $|C(G^*)|$ are referred to as the *order*, *size* and *index* of G^*. Two edge-coloured graphs G^* and H^* are *colour-identical* if there is bijection ϕ from $V(G^*)$ to $V(H^*)$ such that ab is an edge of colour α in $E(G^*)$ if and only if $\phi(a)\phi(b)$ is an edge of colour α in $E(H^*)$.

We denote by rK_n^* the edge-coloured graph with n vertices, r colours, and precisely one edge of each colour joining each pair of distinct vertices. A *partial edge-coloured G^*-decomposition* is a set $\mathcal{G}^*$ of edge-coloured subgraphs of rK_n^*, each colour-identical to G^*, such that for each colour α and each pair a and b of distinct vertices of rK_n^*, there is at most one edge-coloured subgraph $G^* \in \mathcal{G}^*$ containing an edge ab of colour α. The *index* of a partial edge-coloured G^*-decomposition is the index of G^*. Clearly, if there exists an edge-coloured G^*-decomposition then parallel edges in G^* have distinct colours. From here on, all edge-coloured graphs will be assumed to have this property.

A partial edge-coloured G^*-decomposition $\mathcal{G}^*$ is an *edge-coloured G^*-decomposition of H^** if H^* is the edge-coloured subgraph of rK_n^* defined by

$$V(H^*) = \bigcup_{G^* \in \mathcal{G}^*} V(G^*) \qquad \text{and} \qquad E(H^*) = \bigcup_{G^* \in \mathcal{G}^*} E(G^*)$$

with the colours of edges being preserved. That is, for each colour α there is an edge of colour α joining vertices a and b in H^* if and only if there is a $G^* \in \mathcal{G}^*$ containing an edge of colour α joining a to b. A *complete edge-coloured G^*-decomposition of order n and index r* is a G^*-decomposition of rK_n^*. Edge-coloured G^*-decompositions will often be referred to simply as G^*-decompositions as the asterisk identifies that the decomposition is edge-coloured. Similarly, we may sometimes refer to the graph G^* rather than the edge-coloured graph G^*. A partial G^*-decomposition $\mathcal{G}^*$ is said to be *embeddable*, or to have an *embedding*, if it is a subset of a complete G^*-decomposition $\mathcal{G}'^*$, and $\mathcal{G}^*$ is then said to be *embedded in* $\mathcal{G}'^*$.

The purpose of this paper is to examine the following two questions:

(1) For which G^* does there exist a complete G^*-decomposition?

(2) For which G^* is it true that every partial G^*-decomposition is embeddable?

We are interested only in determining, for a given G^*, whether or not there exists a complete G^*-decomposition of some finite order, not in more general asymptotic existence results. Similarly, for Question (2) we are interested only in establishing for a given G^* whether or not every partial G^*-decomposition is embeddable, and not in the orders for which the embeddings exist.

There have been several articles written on edge-coloured graph decompositions and, in particular, some asymptotic existence results have been established. In 2000, Lamken and Wilson [6] proved that for any simple edge-coloured graph G^*, there exist G^*-decompositions of rK_n^* for all sufficiently large integers satisfying simple numerical conditions. This generalises Wilson's well-known proof of the asymptotic existence of G-decompositions for any simple non-edge-coloured graph G [8]. Recently, Li Marzi et al [7] proved a sufficient condition for the existence of edge-coloured G^*-decompositions (where G^* is not necessarily simple) for an infinite family of sufficiently large integers, see Theorem 2.1. This result was established in order to prove certain results on algebras associated with m-cycle systems. Here, we use the result of Li Marzi et al and obtain necessary and sufficient conditions for the existence of a G^*-decomposition (of some finite order).

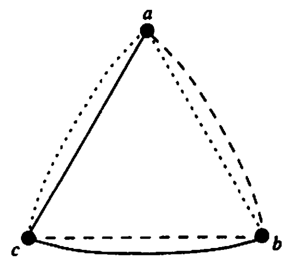
Other results on edge-coloured graph decompositions include Wilson's results on decomposing rK_n^* into edge-coloured copies of K_m [9] and the results of Colbourn and Stinson on decomposing complete edge-coloured graphs into edge-coloured copies of K_4 [5]. As described in [5] and [6], edge-coloured graph decompositions, or the more general *directed* edge-coloured graph decompositions which are studied in [6], are equivalent to other well-known types of designs such as: resolvable and near-resolvable designs, group divisible designs, grid designs, nested designs, self-complementary designs, perpendicular arrays, K-perfect m-cycle systems etc. Also see [1, 2, 3, 4].

2 Existence

For Question (1) we ignore the trivial case of order 1 decompositions. For a subgraph G^* of index 1, the colour can be ignored and a simple graph G (a subgraph of K_n) results. Hence, Wilson's Theorem [8] guarantees the existence of a finite complete G^*-decompositions in the case of index 1.

The same result is not true for G^*-decompositions when G^* is a simple graph of index more than 1. Obviously, if there are more edges of one colour than another in G^* then no finite complete G^*-decomposition exists. However, if G^* is any simple edge-coloured graph in which the number of edges of each colour is the same, then a result of Lamken and Wilson [6] guarantees the existence of a finite complete G^*-decomposition. These above mentioned results of Wilson [8] and Lamken and Wilson [6] are much stronger, actually guaranteeing the existence of complete G-decompositions and G^*-decompositions of all sufficiently large orders satisfying simple numerical conditions.

Now consider arbitrary (non-simple) edge-coloured graphs. It is easy to construct edge-coloured graphs where each colour occurs on the same number of edges, but where no G^*-decomposition of finite order exists. For example, consider the graph T^* shown in the figure below.



T^*

No complete T^*-decomposition exists because (for example) it is impossible to use up a single solid edge between a and b. Denote by $C(a_1b_1)$ the set of colours occurring on the parallel edges joining two vertices a_1 and b_1 of a graph G^* and say that $C(a_1b_1)$ is *complementable* in G^* when there exists a partition

$$C(G^*) = C(a_1b_1) \cup C(a_2b_2) \cup \cdots \cup C(a_tb_t)$$

of $C(G^*)$ for some $a_1, b_1, a_2, b_2, \ldots, a_t, b_t \in V(G^*)$. Then an obvious necessary condition for the existence of a complete G^*-decomposition is that every colour set $C(ab)$ of G^* be complementable. The graph T^* mentioned above does not satisfy this condition.

We now examine some further necessary conditions for the existence of a complete G^*-decomposition. Consider the graph G^* with vertex set $V(G^*) = \{a, b, c, d, e, f\}$, colour set $C(G^*) = \{\alpha, \beta, \gamma, \delta\}$ and edges defined by

$$\begin{array}{cccc} C(ab) = \{\alpha, \beta\} & C(ac) = \{\gamma, \delta\} & C(ad) = \{\gamma, \delta\} & C(ae) = \{\alpha, \delta\} \\ C(af) = \{\alpha, \beta, \gamma\} & C(bc) = \{\alpha, \delta\} & C(bd) = \{\alpha, \gamma, \delta\} & C(be) = \{\beta\} \\ C(bf) = \{\delta\} & C(cd) = \{\beta\} & C(ce) = \{\beta, \gamma\} & C(de) = \{\alpha, \beta, \gamma\}. \end{array}$$

It is easy to check that every set of parallel edges is complementable in G^*. However, there are only two possible partitions of $C(G^*)$ containing $C(ab)$:

namely $C(ab) \cup C(ac)$ and $C(ab) \cup C(ad)$. Since these are also the only partitions of $C(G^*)$ containing $C(ac)$ or $C(ad)$, it follows that there is no complete G^*-decomposition (since any partial G^*-decomposition contains twice as many occurrences of $\{\gamma, \delta\}$ as $\{\alpha, \beta\}$).

The article [7] by Li Marzi et al gives sufficient conditions for the existence of a complete G^*-decomposition (see Theorem 2.1 below), and it is easy to see that any G^* satisfying these conditions will also satisfy the necessary conditions we have mentioned thus far. We find it convenient to make the following definition. An *existence labeling* or *ex-labeling* of a graph G^* is an assignment of labels from a set X to the edges of G^* such that

- parallel edges are assigned the same label; and
- for each colour $\alpha \in C(G^*)$, the edges of colour α are assigned distinct labels.

Li Marzi et al proved the following theorem.

Theorem 2.1. *Let G^* be edge-coloured graph of size e and index r. If $\frac{e}{r}$ is an integer and there is an ex-labeling of G^* with $\frac{e}{r}$ labels then there exists a complete G^*-decomposition of order q for all sufficiently large prime powers $q \equiv 1 \pmod{\frac{2e}{r}}$.*

Dirichlet's Theorem tells us that there are an infinite number of primes $q \equiv 1 \pmod{\frac{2e}{r}}$, so the theorem guarantees there are G^*-decompositions of order n for infinitely many values of n. It is natural then to ask whether the existence of an ex-labeling of G^* with $\frac{e}{r}$ labels is necessary for the existence of a complete G^*-decomposition. We now show that this is not the case.

Let P be a copy of the Petersen graph on the vertex set $\{0, 1, 2, \ldots, 9\}$ and consider the edge-coloured graph P^* of order 10, size 30 and index 10 constructed as follows. Let $V(P^*) = \{0, 1, 2, \ldots, 9\}$, $C(P^*) =$

$\{c_0, c_1, c_2, \ldots, c_9\}$ and define $E(P^*)$ by joining x to y by an edge of colour c_x and an edge of colour c_y for each edge $xy \in P$. Since P is 3-regular, for each $x \in \{0, 1, 2, \ldots, 9\}$ there are three edges of colour c_x in P^* and these are each incident with the vertex x. It follows that an ex-labeling of P^* with 3 labels induces a proper 3-edge colouring of P. Since no such colouring exists, there is no ex-labeling of P^* with 3 labels.

However, we now use Theorem 2.1 to prove the existence of complete P^*-decompositions. Let P'^* be the union of two vertex disjoint copies of P^*. So P'^* has order 20, size 60 and index 10. The following table gives an ex-labeling of P'^* with the 6 labels a, b, c, d, e and f. Hence by Theorem 2.1, complete P'^*-decompositions exist. In any such decomposition, splitting each copy of P'^* into two copies of P^* yields a complete P^*-decomposition.

(edge, colour)	label in copy 1	label in copy 2	(edge, colour)	label in copy 1	label in copy 2
$(01, c_0)$, $(01, c_1)$	a	c	$(27, c_2)$, $(27, c_7)$	a	f
$(12, c_1)$, $(12, c_2)$	b	d	$(38, c_3)$, $(38, c_8)$	f	b
$(23, c_2)$, $(23, c_3)$	c	e	$(49, c_4)$, $(49, c_9)$	c	f
$(34, c_3)$, $(34, c_4)$	a	d	$(57, c_5)$, $(57, c_7)$	c	b
$(04, c_4)$, $(04, c_0)$	b	e	$(79, c_7)$, $(79, c_9)$	d	e
$(05, c_5)$, $(05, c_0)$	d	f	$(69, c_9)$, $(69, c_6)$	b	a
$(16, c_1)$, $(16, c_6)$	f	e	$(68, c_6)$, $(68, c_8)$	d	c
$(58, c_5)$, $(58, c_8)$	e	a			

It turns out, as the following theorem shows, this idea of examining ex-labelings of disjoint copies of G^* is all we need to give necessary and sufficient conditions for the existence of a complete G^*-decomposition. For any positive integer t and any graph G^*, denote by $t \otimes G^*$ the graph obtained by taking t vertex disjoint copies of G^*.

Theorem 2.2. *Let G^* be an edge-coloured graph of size e and index r. There exists a complete G^*-decomposition of order n for some finite n if and only if $\frac{e}{r}$ is an integer and there exists a positive integer t such that there is an ex-labeling of $t \otimes G^*$ with $t\frac{e}{r}$ labels.*

Proof First suppose $t \otimes G^*$ has an ex-labeling with $t\frac{e}{r}$ labels for some positive integer t. Then by Theorem 2.1 there exists a complete $(t \otimes G^*)$-decomposition. Decomposing each copy of $t \otimes G^*$ into t copies of G^* yields a complete G^*-decomposition. Conversely, suppose there exists a G^*-decomposition of order n for some finite n. Let $\pi : \{1, 2, \ldots, \binom{n}{2}\} \mapsto E(K_n)$ be a bijection, let $t = \frac{rn(n-1)}{2e}$ (the number of copies of G^* in a G^*-decomposition of order n), and let $\{G_1^*, G_2^*, \ldots, G_t^*\}$ be a G^*-decomposition of order n. Now take t vertex disjoint copies of G^* and for $i = 1, 2, \ldots, t$, label the edges of the i-th copy with the labels induced by π on the corresponding edges of G_i^*. The result is an ex-labeling of $t \otimes G^*$ with $t\frac{e}{r}$ labels.
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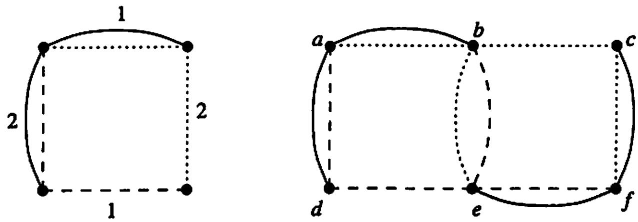
3 Embedding

If G is any simple non-edge coloured graph, then any partial G-decomposition $\mathcal{G}$ is a G-decomposition of some simple graph K. Hence by Wilson's Theorem [8] there is a complete K-decomposition $\mathcal{K}$ of finite order, and so by taking a G-decomposition of each copy of $K \in \mathcal{K}$ we obtain an embedding of $\mathcal{G}$. So we see that it is an immediate consequence of Wilson's Theorem [8] that if G is any simple non-edge coloured graph, then any partial G-decomposition is embeddable. Since we have already seen examples where an edge-coloured graph G^* has no complete G^*-decomposition, a similar result cannot hold for edge-coloured graphs. However, an obvious question is whether the existence of complete G^*-decompositions ensures that every partial G^*-decomposition is embeddable. The following example shows that this is not the case. Consider the graph S^* shown on the left in the figure below.



S^* *A partial S^*–decomposition*

An ex-labeling of S^* is shown in the figure, so complete S^*-decompositions exist by Theorem 2.1. However, the partial S^*-decomposition shown on the right is clearly not embeddable. To see this, observe that the solid edge of $3K_n^*$ joining b and e can never be placed in a copy of S^* (without having two edges of one of the other colours joining b and e).

However, by imposing additional conditions on ex-labelings we are able to prove a theorem, similar to Theorem 2.1, which provides us with sufficient conditions for ensuring every partial G^*-decomposition is embeddable. Thus, we define an *embedding labeling* or an *em-labeling* of G^* to be an ex-labeling, with some set X say, along with the extra condition that for some $x \in X$, x is assigned to simple edges only.

Theorem 3.1. *Let G^* be edge-coloured graph of size e and index r. If $\frac{e}{r}$ is an integer and there is an em-labeling of G^* with $\frac{e}{r}$ labels then every partial G^*-decomposition of order n, where n is finite, embeds in a complete G^*-decomposition of order n' for some integer n'.*

Proof Let $\mathcal{G}^*$ be a partial G^*-decomposition of finite order and let $|\mathcal{G}^*| = t$. Now, $\mathcal{G}^*$ is an edge-coloured G^*-decomposition of some edge-coloured graph H^* with size te and index r say. Find an ex-labeling of H^* with $\{1,2,\ldots,M\}$ where M is some sufficiently large integer ($M \geq t\frac{e}{r}$).

If $M = t\frac{e}{r}$ then we have an ex-labeling of H^* with $\{1,\ldots,t\frac{e}{r}\}$. Thus by Theorem 2.1, an H^*-decomposition of finite order exists. Decomposing each copy of H^* into copies of G^* yields an embedding of $\mathcal{G}^*$.

If $M > t\frac{e}{r}$ let H'^* be the graph resulting from the union of H^* with t' vertex disjoint copies of G^*, $\{G_1^*,\ldots,G_{t'}^*\}$, where $t' = M - t\frac{e}{r}$. Now H'^* is a graph of size $(t+t')e$ and index r. If there is an ex-labeling of H'^* with $(t+t')\frac{e}{r}$ labels we can use Theorem 2.1 to prove the existence of an H'^*-decomposition. Decomposing each copy of H'^* into copies of G^* yields an embedding of $\mathcal{G}^*$.

Thus, we need to find an ex-labeling of H'^* with $(t+t')\frac{e}{r}$ labels. Firstly, ex-label the subgraph H^* with integers $1,\ldots,M$. Then em-label each of the subgraphs G_i^* with integers $M+(i-1)\frac{e}{r}+1,\ldots,M+i\frac{e}{r}$ so that $M+i\frac{e}{r}$ occurs only on simple edges. Now consider colour $c \in C(G^*)$. In the subgraph H^*, c colours $t\frac{e}{r}$ edges so there are $t' = M - t\frac{e}{r}$ labels in the set $\{1,\ldots,M\}$ which are not used on edges of colour c. Let these labels be $x_1,\ldots,x_{t'}$. In each of the subgraphs G_i^* there is a simple edge of colour c with label $M+i\frac{e}{r}$. Replace each label $M+i\frac{e}{r}$ with x_i for $i=1,\ldots,t'$. Repeat this for all of the colours in $C(G^*)$. It is easy to check that we have an ex-labeling of H'^* with $\{1,2,\ldots,M+t'\frac{e}{r}\}\backslash\{M+\frac{e}{r},M+2\frac{e}{r},\ldots,M+t'\frac{e}{r}\}$; so $M+t'\frac{e}{r}-t' = (t+t')\frac{e}{r}$ labels are used. ∘

Corollary 3.1. *If G^* is a simple edge-coloured graph of size e and index r such that $\frac{e}{r}$ is an integer and each of the colours $C(G^*)$ is assigned to exactly $\frac{e}{r}$ edges of G, then every partial G^*-decomposition of finite order is embeddable.*

Proof For each colour $\alpha \in C(G^*)$, arbitrarily assign the integers $\{1,\ldots,\frac{e}{r}\}$ to the edges of colour α. Clearly this is an em-labeling of G^* with $\frac{e}{r}$ labels and so the result follows by Theorem 3.1. ∘

Theorem 3.1 says that the existence of an em-labeling of an edge-coloured graph G^* is a sufficient condition for embeddability. However, it is not a necessary condition. For example, it is easy to see that any partial $2K_2^*$-decomposition is embeddable, but clearly $2K_2^*$ has no em-labeling as it has no simple edges. We now prove two further theorems on embedding partial G^*-decompositions. Unfortunately, neither gives a simple characterisation of embeddability of partial G^*-decompositions just in terms of the properties of G^*. Finding such properties seems an interesting problem. Theorem 3.2 involves looking at particular partial G^*-decompositions and Theorem 3.3 involves looking at all possible ex-labelings of multiple copies of G^*.

Theorem 3.2. *Let G^* be an edge-coloured graph of size e and index r. A partial G^*-decomposition $\mathcal{G}^*$ is embeddable if and only if there exists a positive integer t such that $\mathcal{G}^* \cup (t \otimes G^*)$ can be ex-labeled with a set of $(|\mathcal{G}^*| + t)\frac{e}{r}$ labels.*

Proof First suppose $\mathcal{G}^* \cup (t \otimes G^*)$ has an ex-labeling with $(|\mathcal{G}^*| + t)\frac{e}{r}$ labels for some positive integer t. Then by Theorem 2.1 there exists a complete $\mathcal{G}^* \cup (t \otimes G^*)$-decomposition. Decomposing each copy of $\mathcal{G}^* \cup (t \otimes G^*)$ into $|\mathcal{G}^*| + t$ copies of G^* yields a complete G^*-decomposition with $\mathcal{G}^*$ embedded in it. Conversely, suppose $\mathcal{G}^*$ is embedded in a complete G^*-decomposition of order n. Let $\pi : \{1, 2, \ldots, \binom{n}{2}\} \mapsto E(K_n)$ be a bijection, let $t = \frac{rn(n-1)}{2e} - |\mathcal{G}^*|$ and let $\mathcal{G}^* \cup \{G_1^*, G_2^*, \ldots, G_t^*\}$ be the complete G^*-decomposition of order n. This induces an ex-labeling of $\mathcal{G}^*$. Furthermore, take t vertex disjoint copies of G^* and for $i = 1, 2, \ldots, t$, label the edges of the i-th copy with the labels induced by π on the corresponding edges of G_i^*. The result is an ex-labeling of $\mathcal{G}^* \cup (t \otimes G^*)$ with $(|\mathcal{G}^*| + t)\frac{e}{r}$ labels. ∘

Theorem 3.3. *Let G^* be an edge-coloured graph of size e and index r with $\frac{e}{r}$ and integer. Every partial G^*-decomposition is embeddable if and only if for each $t \in \mathbb{N}$ and each ex-labeling of $t \otimes G^*$ there is some integer t' such that*

- *there is an ex-labeling $t' \otimes G^*$, and*
- *taking the ex-labelings of $t \otimes G^*$ and $t' \otimes G^*$ together we have an ex-labeling of $(t + t') \otimes G^*$ with $(t + t')\frac{e}{r}$ labels.*

Proof ($\Leftarrow$) Let $\mathcal{G}^* = \{G_1^*, \ldots, G_t^*\}$ be a partial G^*-decomposition. Then $\mathcal{G}^*$ is a G^*-decomposition of some edge-coloured graph H^*. Find some ex-labeling of H^* with a sufficiently large set X of labels. Take t vertex disjoint copies of G^* and for $i = 1, 2, \ldots, t$ label the edges of the i-th copy with the labels on the corresponding edges of G_i^*. This is an ex-labeling of $t \otimes G^*$ with X.

Now assume we can find some integer t' and an ex-labeling of $t' \otimes G^*$ so that taking the ex-labelings of $t \otimes G^*$ and $t' \otimes G^*$ together we have an ex-labeling of $(t+t') \otimes G^*$ with $(t+t')\frac{e}{r}$ labels. Replace the ex-labeling of $t \otimes G$ with the ex-labeling of H^*. Then we have an ex-labeling of $H^* \cup (t' \otimes G^*)$ (the vertex disjoint union of H^* with $t' \otimes G^*$) with $(t + t')\frac{e}{r}$ labels. By Theorem 2.1 an $H^* \cup (t' \otimes G^*)$-decomposition exists. By decomposing each copy of $H^* \cup (t' \otimes G^*)$ into copies of G^* we have a G^*-decomposition in which $\mathcal{G}^*$ is embedded.

($\Rightarrow$) We assume every partial G^*-decomposition is embeddable. Suppose we have an ex-labeling of $t \otimes G^*$ with a set X of labels. The graph $t \otimes G^*$ is a partial G^*-decomposition. We will say the set of edges $\{e_1, \ldots, e_s\}$ *underlies* G^* if no two edges in $\{e_1, \ldots, e_s\}$ are parallel and for all $e \in E(G^*)$, e is parallel to or equal to e_i for some $i \in \{1, 2, \ldots, s\}$. Suppose a set of s edges underlies G^*. Choose a set Y of labels so that $X \cap Y = \emptyset$ and $|Y| = st$. Then ex-label $st \otimes G^*$ with Y so that two edges have the same label if and only if they are parallel. By considering the disjoint union of $t \otimes G^*$ and $st \otimes G^*$ we have an ex-labeling of $(s+1)t \otimes G^*$ with $X \cup Y$.

Let the copies of G^* in $(s+1)t \otimes G^*$ be G_1^*, $G_2^* \ldots, G_{(s+1)t}^*$ and let the vertex set of each G_i^* be $\{v_{i,1}, \ldots, v_{i,n}\}$ so that for all $i, j \in \{1, 2, \ldots, st\}$ the map $v_{i,k} \to v_{j,k}$ for $k \in \{1, \ldots, n\}$ is an isomorphism of G_i^* and G_j^*. Let $G_1^*, \ldots, G_t^*$ be labeled with X and $G_{t+1}^*, \ldots, G_{(s+1)t}^*$ be labeled with Y.

Now note that if we swap the labels on all the edges between $v_{i,k}$ and $v_{i,l}$ with the labels on all the edges between $v_{j,k}$ and $v_{j,l}$ then we still have an ex-labeling of $(s+1)t \otimes G^*$ with $X \cup Y$. Partition the copies of G^* into t classes $\mathcal{G}_1, \ldots, \mathcal{G}_t$ where $\mathcal{G}_i = \{G_i^*, G_{i+t}^*, \ldots G_{i+st}^*\}$ for $i = 1, 2, \ldots, t$. Let $\{e_{i,1}, \ldots, e_{i,s}\}$ underlie G_i^*. For each $i \in \{1, \ldots, t\}$ and each $j \in \{1, \ldots, s\}$ switch the labels between edges parallel to $e_{i,j}$ and the corresponding edges in G_{i+jt}^*. This produces an ex-labeling of $(s+1)t \otimes G^*$ with $X \cup Y$ such that:

(a) if x and y label edges in the same copy of G^* and $x \in X$, then $y \notin X$,

(b) if $y \in Y$, the edges y labels are all parallel to each other.

Using this new ex-labeling of $(s+1)t \otimes G^*$ identify vertices $v_{i,k}$ and $v_{j,l}$ if and only if there exist k' and l' satisfying either

- $k' \geq k$ and $l' \geq l$; or
- $k' \leq k$ and $l' \leq l$;

and such that the edges between $v_{i,k}$ and $v_{i,k'}$ and the edges between $v_{j,l}$ and $v_{j,l'}$ are assigned the same label $x \in X$. Properties (a) and (b) ensure that parallel edges have distinct colours and are assigned the same label. So we have a partial G^*-decomposition $\mathcal{G}^*$ labeled with $X \cup Y$ so that edges have the same label if and only if they are parallel. Note that by decomposing $\mathcal{G}^*$ into copies of G^* we obtain the new ex-labeling of $(s+1)t \otimes G^*$ with $X \cup Y$.

Now, because we are assuming every partial G^*-decomposition embeds in a complete G^*-decomposition, we can embed $\mathcal{G}^*$ in a complete G^*-decomposition $\mathcal{G}'^*$ of order n for some integer n. Choose a set Z of labels so that $(X \cup Y) \cap Z = \emptyset$ and $|X \cup Y \cup Z| = \binom{n}{2}$. Ex-label rK_n^* with $X \cup Y \cup Z$ so that $\mathcal{G}^*$ has the same labeling as above. Now decompose rK_n^* into copies of G^* keeping the labels on the edges. This gives an ex-labeling of $\frac{rn(n-1)}{2e} \otimes G^*$ with $X \cup Y \cup Z$. Within this labeling we have our new ex-labeling of $(s+1)t \otimes G^*$ with $X \cup Y$. Switch the labels back to the original ex-labeling of $(s+1)t \otimes G^*$. This contains our original ex-labeling of $t \otimes G^*$ with X. If we let $t' = \frac{rn(n-1)}{2e} - t$ then it is clear that we have found an integer t' and an ex-labeling of $t' \otimes G^*$ so that taking the ex-labelings of $t \otimes G^*$ and $t' \otimes G^*$ together we have an ex-labeling of $(t+t') \otimes G^*$ with $(t+t')\frac{e}{r}$ labels. ○

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