Perfect *one*–factorizations in generalized Petersen graphs

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Abstract

A perfectly one-factorable (P1F) regular graph G is a graph admitting a partition of the edge-set into one-factors such that the union of any two of them is a Hamiltonian cycle. We consider the case in which G is a cubic graph. The existence of a P1F cubic graph is guaranteed for each admissible value of the number of vertices. We give conditions for determining P1F graphs within a subfamily of generalized Petersen graphs.

Keywords: factorization 05C70, colouring of graphs and hypergraphs 05C15

1 Introduction

We shall say that a *one*–factorization $\mathcal{F}$ of a regular graph is *perfect* if any two *one*–factors of $\mathcal{F}$ form a Hamiltonian cycle. A regular graph G admitting a perfect *one*–factorization is said to be perfectly *one*–factorable or P1F for short.

In the literature P1F cubic graphs are also called Hamilton graphs in [7] or strongly Hamiltonian graphs in [8], while $\mathcal{F}$ is said to be a Hamilton decomposition, [7, 8].

While the existence spectrum of perfect *one*–factorizations of the complete graph K_n is not yet known, for every even value of n there exists a cubic graph of order n admitting a perfect *one*–factorization. If $u_0, u_1, \ldots, u_{n-1}$ are the vertices of the graph (represented on the regular n–gon with clockwise labelling), then setting $f_1 = \{u_i u_{i+1} : 0 \leqslant i \leqslant n-1, i \equiv 0 \pmod 2\}$,

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$f_2 = \{u_i u_{i+1} : 0 \leqslant i \leqslant n-1, i \equiv 1 \pmod 2\}$ and $f_3 = \{u_0 u_{\frac{n}{2}}, u_i u_{n-i} : 1 \leqslant i \leqslant \frac{n}{2} - 1\}$, we have that $\mathcal{F} = \{f_1, f_2, f_3\}$ is a perfect *one*–factorization of the resulting cubic graph.

Moreover, there also exists a perfectly *one*–factorable bipartite cubic graph of order n for every admissible value of n, that is $\frac{n}{2} \equiv 1 \pmod 2$ (see [8]). In fact the $\frac{n}{2}$–Möbius ladder is bicubic and P1F.

Are there other examples of P1F cubic graphs? The answer is yes: every 2–factor Hamiltonian cubic graph, [4], is P1F since it is *one*–factorable and has the property that any 2–factor is Hamiltonian.
More specifically, it is proved in [6] that each P1F cubic graph can be constructed by repeated application of two modifications, namely the extensions ρ and π, of the graph, called θ–graph, consisting of two vertices and three multiple edges between them.
The procedure described in [6] allows in principle the construction of ALL cubic graphs which are perfectly one-factorable. Still, given any specific cubic graph G, this procedure does not yield a criterion which can easily tell whether G is P1F or not. Generally speaking, one cannot decide whether any one of the inverse operations can be performed on the graph unless a perfect *one*–factorization has already been assigned. For this reason we investigate a large family of cubic graphs, namely the family of generalized Petersen graphs, [10].

For the generalized Petersen graphs $GP(n,k)$, with $n \geqslant 3$ and $1 \leqslant k \leqslant \lfloor \frac{(n-1)}{2} \rfloor$, the following results hold:

(i) $GP(n,1)$ is P1F if and only if $n = 3$;

(ii) $GP(n,2)$ is P1F if and only if $n \equiv 3, 4 \pmod 6$;

(iii) $GP(n,3)$ is P1F if and only if $n = 9$;

(iv) $GP(n,k)$ is not P1F if $n \equiv 0 \pmod 2$ and $k \equiv 1 \pmod 2$;

(v) $GP(3d,k)$ with $d \equiv 1 \pmod 2$, $(k, 3d) = 1$ and $k > 1$ is P1F;

(vi) $GP(3d,d)$ with $d \equiv 1 \pmod 2$ is P1F.

The smallest case which remains undecided even after (i),(ii),(iii),(iv), (v) and (vi) is $GP(10,4)$ (as a matter of fact it is not P1F by direct computation).

It remains an open problem to determine all pairs (n,k) for which the generalized Petersen graphs $GP(n,k)$ are P1F. It does not seem that this task can be carried out immediately. As a matter of fact a direct study of some graphs which do not fall in the previous list produce both existence and non–existence results: we are able to exhibit a perfect *one*–factorization of $GP(19,4)$ and to prove that $GP(17,4)$ is not P1F.

2 Some general properties

One–factorizations of cubic graphs are also called 3–edge–colourings, hence *one*–factors will be also called colours.

Lemma 1 *Let G be a cubic graph with a perfect one–factorization $\mathcal{F} = \{f_1, f_2, f_3\}$. Let E be an edge–cut, $|E| = h$, with $h = h_1 + h_2 + h_3$, where h_i denotes the number of edges in E of colour f_i. The following conditions hold:*

1. $h_1 \equiv h_2 \equiv h_3 \equiv h \pmod 2$;
2. $h_1h_2 + h_1h_3 + h_2h_3 > 0$.

Proof. Condition (1) is the parity Lemma proved in [5]. Condition (2) means that at least two integers h_i are different from zero. In fact if $h_j = h_k = 0$, with $j \neq k$, then $f_j \cup f_k$ is a non–Hamiltonian cycle in G. This contradicts the assumption on $\mathcal{F}$.

It is well known that a graph G possessing a Hamiltonian cycle is at least 2–edge–connected. A P1F cubic graph is 3–edge–connected. In fact, if $e_1, e_2 \in E(G)$ disconnect the graph G, then we have a contradiction by condition (2) of Lemma 1.

Observe that a Hamiltonian cycle C of an arbitrary cubic graph G always gives rise to a *one*–factorization of G: namely the cycle C is the union of two distinct *one*–factors, say f_1 and f_2, whereas $E(G) \setminus E(C)$ gives the third *one*–factor. For the rest of the paper we will denote such a *one*–factor by $G - C$ and we will say that the *one*-factorization arises from C.

If G is a cubic graph possessing an odd number of Hamiltonian cycles, from Smith's Theorem, [9] it follows that every edge of G belongs to at least two Hamiltonian cycles. We shall use this property to prove the following general proposition.

Proposition 1 *Let G be a cubic graph possessing exactly three Hamiltonian cycles, then G admits a perfect one–factorization.*

Proof. Let C_1, C_2 and C_3 denote the Hamiltonian cycles of G. Let $\mathcal{F} = \{f_1, f_2, f_3\}$ be the *one*–factorization of G arising from C_1, that is $f_1 \cup f_2 = C_1$ and $f_3 = G - C_1$. To prove that $\mathcal{F}$ is perfect we show that

$C_2 = f_1 \cup f_3$ and $C_3 = f_2 \cup f_3$.
Since G has an odd number of Hamiltonian cycles, from the previous remark it follows that the edges in f_3 are all contained in both cycles C_2 and C_3, that is $f_3 \subset C_2$ and $f_3 \subset C_3$. Then C_2 can be written as the disjoint union of f_3 and a set F of n edges, with $|V(G)| = 2n$. Since C_2 is a Hamiltonian cycle of G and f_3 is a *one*–factor, it follows that also F is a *one*–factor of G. We have that $F \subset C_1$ because $G = C_1 \cup f_3$. Since the unique *one*–factors in C_1 are f_1 and f_2 it follows that either $F = f_1$ or $F = f_2$. Without loss of generality we can assume $F = f_1$ and so $C_2 = f_1 \cup f_3$. The same holds for C_3, that is $C_3 = f_2 \cup f_3$.

It is well–known that a cubic graph G with a unique 3–edge–colouring has exactly three Hamiltonian cycles, then we have that G is P1F by Proposition 1. The class of planar cubic graphs with a unique 3–edge–colouring is fully described in [3]. The author shows that each planar graph has a unique 3–edge–colouring if and only if it can be obtained from K_4 by repeatedly applying star products with copies of K_4. We observe that not all planar P1F cubic graphs are contained in this class: see for instance the graph $GP(10, 2)$, that is the dodecahedral graph.

3 P1F generalized Petersen graphs

The generalized Petersen graph $GP(n, k)$, $n \geqslant 3$ and $1 \leqslant k \leqslant \lfloor \frac{(n-1)}{2} \rfloor$, is a graph with vertex–set $\{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, v_{n-1}\}$ and edge–set $\{u_iu_{i+1}, u_iv_i, v_iv_{i+k} : 0 \leqslant i \leqslant n-1\}$, with subscripts reduced modulo n.
Edges u_iu_{i+1} and v_iv_{i+k} are called *outer* and *inner* edges respectively, while the edges u_iv_i are called *spokes*. Two spokes u_iv_i and u_jv_j are consecutive or subsequent, if $j = i + 1$.
It is well known that every $GP(n, k)$ has a 3-edge colouring apart from the original Petersen graph $GP(5, 2)$, [2].

Moreover $GP(n, k)$ is Hamiltonian if and only if it is different from $GP(n, 2)$ and from $GP(n, \frac{n-1}{2})$with $n \equiv 5 \pmod 6$, [1].

Proposition 2 *Let d and k be positive integers with d odd and either $(3d, k) = 1$ $(k > 1)$, or $k = d$. The graph $GP(3d, k)$ is P1F.*

Proof. Let $(3d, k) = 1$, that implies $k \equiv 1 \pmod 3$ or $k \equiv 2 \pmod 3$. We construct a *one*-factorization $\{f_1, f_2, f_3\}$ of $GP(3d, k)$ by colouring the outer edges as follows:

$$u_ju_{j+1} \in f_1 \text{ if } j \equiv 0 \pmod 3;$$
$$u_ju_{j+1} \in f_2 \text{ if } j \equiv 1 \pmod 3;$$
$$u_ju_{j+1} \in f_3 \text{ if } j \equiv 2 \pmod 3.$$

Then the spokes $u_j v_j$ are coloured as stated below:

$$u_j v_j \in f_1 \text{ if } j \equiv 2 \pmod 3;$$
$$u_j v_j \in f_2 \text{ if } j \equiv 0 \pmod 3;$$
$$u_j v_j \in f_3 \text{ if } j \equiv 1 \pmod 3.$$

Finally the colouring of the inner edges $v_j v_{j+k}$ changes according to the fact that $k \equiv 1$ or $k \equiv 2 \pmod 3$. If j is a fixed index in $\{0, \ldots, 3d-1\}$, with $j \equiv 0 \pmod 3$, then the inner edges are coloured as in Figure 1 if $k \equiv 1 \pmod 3$, or as in Figure 2 if $k \equiv 2 \pmod 3$.

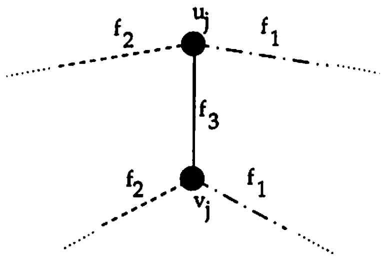

Figure 1: $k \equiv 1 \pmod 3$

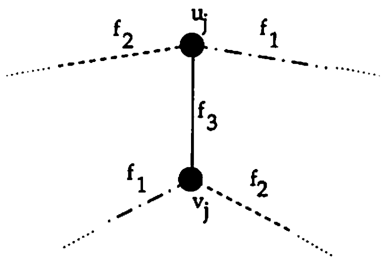

Figure 2: $k \equiv 2 \pmod 3$

Let p_1 denote the path in $GP(3d, k)$ given by $(u_2, v_2, v_{2-k}, v_{2-2k}, u_{2-2k}, u_{3-2k})$ if $k \equiv 1 \pmod 3$, or by $(u_2, v_2, v_{2+k}, v_{2+2k}, u_{2+2k}, u_{3+2k})$ if $k \equiv 2 \pmod 3$. To consider simultaneously both cases $k \equiv 1 \pmod 3$ and $k \equiv 2 \pmod 3$, we write p_1 as $p_1 = (u_2, v_2, v_{2\mp k}, v_{2\mp 2k}, u_{2\mp 2k}, u_{3\mp 2k})$.
One can see that p_1 possesses 6 distinct vertices. In fact $u_2 = u_{2\mp 2k}$ if and only if $2k \equiv 0 \pmod{3d}$ which means $2k \equiv 0 \pmod 3$. That yields a contradiction since $2k \equiv 2, 1 \pmod 3$. For the same reason $v_2 \neq v_{2\mp 2k}$, $v_2 \neq v_{2\mp k}$ and $v_{2\mp k} \neq v_{2\mp 2k}$. Whereas $u_2 = u_{3\mp 2k}$ if and only if $2k \equiv 1 \pmod{3d}$, but $2 \leqslant 2k < 3d - 1$, since $1 \leqslant 2k \leqslant \lfloor \frac{3d-1}{2} \rfloor$, then $u_2 \neq u_{3\mp 2k}$. It is straightforward that $u_{2-2k} \neq u_{3-2k}$.
Observe that in p_1 two edges sharing a vertex do not belong to the same *one*–factor, whereas edges not sharing a vertex belong to the same *one*–factor. In fact we have that p_1 is bicoloured.
For $i = 2, \ldots, d$, let $\phi_i : V \longrightarrow V$ be the map defined on the vertex-set of the graph moving every vertex u_j (v_j) to the vertex u_{j+r} (v_{j+r}), where $r = (i-1)(2-2k)$ if $k \equiv 1 \pmod 3$, or $r = (i-1)(2+2k)$ if $k \equiv 2 \pmod 3$. To consider simultaneously both values of k we set $r = (i-1)(2 \mp 2k)$.
The map ϕ_i is an automorphism of the graph, then p_1 and $\phi_i(p_1) = p_i$ are isomorphic paths and since $(i-1)(2 \mp 2k) \equiv 0 \pmod 3$ we have that every edge e_j of p_1 is coloured as $\phi_i(e_j)$, that is ϕ_i preserves the colours.
Consider the paths p_i and p_j, with $i, j \in \{1, 2, \ldots, d\}$ and $i \neq j$. We prove that p_i and p_j have no common vertices. Without loss of generality we can

assume that $j > i$.
Suppose that p_i and p_j have a common vertex, say u_x (or v_x). That means $u_x = u_{y+(i-1)(2\mp 2k)}$ and $u_x = u_{z+(j-1)(2\mp 2k)}$ (or $v_x = v_{y+(i-1)(2\mp 2k)}$ and $v_x = v_{z+(j-1)(2\mp 2k)}$), with $y, z \in \{2, 2\mp 2k, 3\mp 2k\}$ (or $y, z \in \{2, 2\mp k, 2\mp 2k\}$). Therefore $u_{y+(i-1)(2\mp 2k)} = u_{z+(j-1)(2\mp 2k)}$ (or $v_{y+(i-1)(2\mp 2k)} = v_{z+(j-1)(2\mp 2k)}$) if and only if $y + (i-1)(2\mp 2k) \equiv z + (j-1)(2\mp 2k) \pmod{3d}$, that is if and only if $y - z \equiv (j-i)(2\mp 2k) \pmod{3d}$. That implies $y - z \equiv 0 \pmod 3$, since $(2\mp 2k) \equiv 0 \pmod 3$ for every $k \equiv 1, 2 \pmod 3$. Since $y - z \in \{0, 1, 3d-1, (\pm 2k - 1), -(\pm 2k - 1), 2k, -2k)\} \pmod{3d}$ (or $y - z \in \{0, k, -k, 2k, -2k\} \pmod{3d}$), the only admissible value for $y - z$ is 0. We assume $y = z$. It follows that $(j-i)(2\mp 2k) \equiv 0 \pmod{3d}$, that is $(j-i)(2\mp 2k) = 3dq$ with $q \in \mathbb{Z}$. Since $1 \leqslant j - i \leqslant d-1$ we have that $3d \nmid (j-i)$, therefore $3d$ has to divide $(2\mp 2k)$. It follows that $2k - 2 \geqslant 3d$ if $k \equiv 1 \pmod 3$, or $2 + 2k \geqslant 3d$ if $k \equiv 2 \pmod 3$, but $1 \leqslant k < \lfloor \frac{3d-1}{2} \rfloor$, that is $0 \leqslant 2k - 2 \leqslant 3d - 3$ and $4 \leqslant 2 + 2k \leqslant 3d - 1$, then $3d \nmid (2\mp 2k)$, therefore p_i and p_j have no common vertices if $i \neq j$.
Let e_i denote the edge of $GP(3d, k)$ whose endpoints are the last vertex in p_{i-1} and the first vertex in p_i, that is $e_i = u_{1+(i-1)(2\mp 2k)}u_{2+(i-1)(2\mp 2k)}$. Since $1 + (i-1)(2\mp 2k) \equiv 1 \pmod 3$, we have that $e_i \in f_2$.
Note that the first and last edge in each path p_i are both in f_1.
By the properties proved above we can say that $f_1 \cup f_2 = p_1 \cup e_2 \cup p_2 \cup \ldots \cup p_{i-1} \cup e_i \cup p_i \cup \ldots p_d \cup e_d$ is a Hamiltonian cycle.
Let $\phi : V \longrightarrow V$ be a map defined on the vertex-set of $GP(3d, k)$ moving every vertex u_i (v_i) to the vertex u_{i+1} (v_{i+1}). One can see that ϕ is an automorphism of the graph and so does $\phi^2 : V \longrightarrow V$, which moves every vertex u_i (v_i) of the graph to the vertex u_{i+2} (v_{i+2}). One can easily verify that $\phi(f_1 \cup f_2) = f_1 \cup f_3$ and $\phi^2(f_1 \cup f_2) = f_2 \cup f_3$, that is $f_1 \cup f_3$ and $f_2 \cup f_3$ are Hamiltonian cycles. We conclude that the constructed *one*-factorization is perfect.
Let $n = 3d$ and $k = d$, where d is a positive odd integer. If $d = 1$, to construct a perfect *one*-factorization for $GP(3, 1)$, it suffices to colour the spokes with three different colours.
Let $d > 1$, as before we denote by f_1, f_2 and f_3 the *one*-factors, which we construct by colouring the spokes of the graph as follows:

$$u_jv_j \in f_1 \text{ if } 0 \leqslant j \leqslant d-1;$$
$$u_jv_j \in f_2 \text{ if } d \leqslant j \leqslant 2d-1;$$
$$u_jv_j \in f_3 \text{ if } 2d \leqslant j \leqslant 3d-1.$$

Then the outer edges are coloured in this way:

$$u_ju_{j+1} \in f_1 \quad \text{if } d \leqslant j \leqslant 3d-1 \text{ and } j \equiv 1 \pmod 2;$$

$$u_ju_{j+1} \in f_2 \quad \text{if } 0 \leqslant j \leqslant d-1 \text{ and } j \equiv 1 \pmod 2, \text{ or } 2d \leqslant j \leqslant 3d-1 \text{ and } j \equiv 0 \pmod 2;$$

$$u_ju_{j+1} \in f_3 \quad \text{if } 0 \leqslant j \leqslant 2d-1 \text{ and } j \equiv 0 \pmod 2.$$

The inner cycles of the graph are triangles (v_j, v_{j+d}, v_{j+2d}), or equivalently (v_j, v_{j+d}, v_{j-d}). The inner triangles are 3–coloured. More specifically, each edge e of the triangle has the same colour of the spoke whose inner endpoint is the vertex opposite to the edge e in the triangle. Thus, for example, u_jv_j and $v_{j+d}v_{j+2d}$ have the same colour.
One can check that the *one*-factorization above is perfect (for an example see Figure 3). We omit this last part of the proof since it can be obtained readapting the proof given in the previous case $GP(3d, k)$, $(3d, k) = 1$.

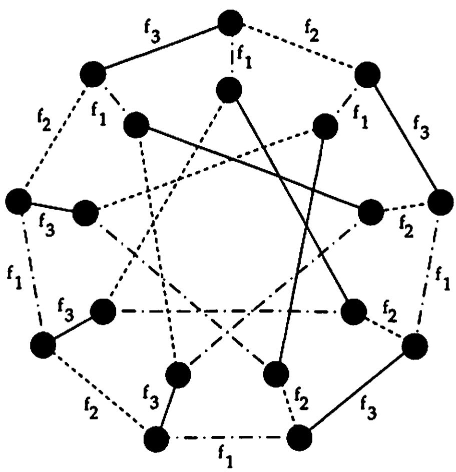

Figure 3: A perfect *one*-factorization of $GP(9, 3)$

Proposition 3 *Let n and k be positive integers, with n even and k odd. There is no perfect one–factorization for the generalized Petersen graph $GP(n, k)$.*

Proof. We show that $GP(n, k)$ with n even and k odd is a bipartite graph.
Let U and W be subsets of the vertex–set V of $GP(n, k)$ defined as follows:

$U = \{u_i : 0 \leqslant i \leqslant n-1, i \equiv 0 \pmod 2\} \cup \{v_j : 0 \leqslant j \leqslant n-1, j \equiv 1 \pmod 2\}$, $W = \{u_i : 0 \leqslant i \leqslant n-1, i \equiv 1 \pmod 2\} \cup \{v_j : 0 \leqslant j \leqslant n-1, j \equiv 0 \pmod 2\}$. It is easy to see that $\{U, W\}$ is a partition of V. Moreover there is no vertex in U and W which is adjacent to a vertex in U and W, respectively. In fact, two adjacent outer vertices, say u_j and u_{j+1}, do not belong to the same subset because $(j+1)-j \not\equiv 0 \pmod 2$; two inner adjacent vertices, say v_j and v_{j+k}, do not belong to the same subset because $k \not\equiv 0 \pmod 2$; an outer vertex u_j and an inner vertex v_j, which are adjacent, do not belong to the same subset because $j - j \equiv 0 \pmod 2$. Then $GP(n,k)$ is a bipartite cubic graph and it does not possess a perfect *one*-factorization since the number of vertices is equivalent to 0 modulo 4 (see [8]).

Proposition 4 *The generalized Petersen graph $GP(n,1)$ is P1F if and only if $n = 3$.*

Proof. If n is even by Proposition 3 we have that $GP(n,1)$ does not admit a perfect *one*-factorization.
We consider $n \equiv 1 \pmod 2$. The graph $GP(3,1)$ is P1F as proved in Proposition 2 so let $n > 3$. Suppose that $GP(n,1)$ admits a perfect *one*-factorization $\mathcal{F} = \{f_1, f_2, f_3\}$. It follows that the spokes u_iv_i cannot have all the same colour, otherwise we find two adjacent edges with the same colour. Hence there exist at least two spokes, say u_iv_i and $u_{i+1}v_{i+1}$, which do not belong to the same *one*-factor. Without loss of generality we can assume that $u_iv_i \in f_1$, whereas $u_{i+1}v_{i+1} \in f_2$. That implies $u_iu_{i+1}, v_iv_{i+1} \in f_3$ and $u_{i-1}u_i, v_{i-1}v_i \in f_2$, then $u_{i-1}v_{i-1} \in f_1$ or $u_{i-1}v_{i-1} \in f_3$. If $u_{i-1}v_{i-1} \in f_1$, we find a 4–cycle $(u_{i-1}, u_i, v_i, v_{i-1})$ in $f_1 \cup f_2$. If $u_{i-1}v_{i-1} \in f_3$, we find a 6–cycle $(u_{i-1}, u_i, u_{i+1}, v_{i+1}, v_i, v_{i-1})$ in $f_2 \cup f_3$. Both these cycles are not Hamiltonian since $n > 3$.

Proposition 5 *The generalized Petersen graph $GP(n,2)$ is P1F if and only if $n \equiv 3, 4 \pmod 6$.*

Proof. Let n be an arbitrary positive integer, $n \geqslant 5$, and let $\mathcal{F} = \{f_1, f_2, f_3\}$ be a perfect *one*-factorization of $GP(n,2)$. We prove that necessarily $n \equiv 3, 4 \pmod 6$. Let $S = \{u_{j_i}v_{j_i} : 1 \leqslant i \leqslant m\}$ denote a set of m, $m \leqslant n-2$, spokes of the same colour, namely f_1, such that $u_{j_0}v_{j_0}$ and $u_{j_{m+1}}v_{j_{m+1}}$ have not colour f_1. Without loss of generality we can set $u_{j_0}v_{j_0} = u_0v_0$ and $u_{j_{m+1}}v_{j_{m+1}} = u_{m+1}v_{m+1}$. Observe that if m is even, then u_0v_0 and $u_{m+1}v_{m+1}$ have the same colour, whereas if m is odd they have different colours.
We show that $m \in \{1, 2, 6\}$. Assume $m > 6$, if $m \equiv 3 \pmod 4$ we have that $f_2 \cup f_3$ contains the $(\frac{3m+7}{2})$–cycle $(u_0, u_1, \ldots, u_{m+1}, v_{m+1}, v_{m-1}, v_{m-3},$

$\ldots, v_0)$ which is not Hamiltonian; if $m \equiv 1 \pmod 4$ it is not possible to complete the colouring; if m is even we find two sets of edges, namely $E_1 = \{u_0u_1, v_0v_2, v_1v_{n-1}, u_4u_5, v_4v_6, v_5v_3\}$ or $E_2 = \{u_3u_4, v_3v_5, v_4v_2, u_7u_8, v_7v_9, v_8v_6\}$, according to the fact that $m \equiv 0 \pmod 4$ or $m \equiv 2 \pmod 4$ respectively, disconnecting the graph, but not satisfying Lemma 1. In both cases we have a contradiction, then $m \leqslant 6$.
We consider $1 \leqslant m \leqslant 6$. Assume $m = 3$, we have that $f_2 \cup f_3$ contains the non–Hamiltonian cycle $(u_0, u_1, u_2, u_3, u_4, v_4, v_2, v_0, u_0)$. That yields a contradiction, then $m \neq 3$.
Assume $m = 4$, we find that $f_1 \cup f_2$ (or $f_1 \cup f_3$) contains the non–Hamiltonian cycle $(u_1, u_2, v_2, v_4, u_4, u_3, v_3, v_1, u_1)$. That yields a contradiction, then $m \neq 4$.
Assume $m = 5$, we have that $u_0v_0, u_6v_6 \notin f_1$ and as remarked before they have different colours. We set $u_0v_0 \in f_2$ and $u_6v_6 \in f_3$. Since u_2v_2 and u_4v_4 have colour f_1, it follows that $v_0v_2 \in f_3$ whereas $v_4v_6 \in f_2$. That implies $v_2v_4 \in f_1$, which gives a contradiction.
We have proved that necessarily $m \in \{1, 2, 6\}$.
We show that if $\mathcal{F}$ has a *one*–factor possessing a set S of 6 subsequent spokes, then there is no set of 6 subsequent spokes with the same colour other than S and $n \equiv 4 \pmod 6$. In the case no element of $\mathcal{F}$ possesses a set of 6 consecutive spokes, then $n \equiv 3 \pmod 6$.
Suppose that there exist two distinct sets of 6 subsequent spokes, say $S_1 = \{u_iv_i : 1 \leqslant i \leqslant 6\}$ and $S_2 = \{u_{t+i}v_{t+i} : 1 \leqslant i \leqslant 6\}$, with the property that all edges in the same set have the same colour. Without loss of generality we can assume that $S_1 \subset f_1$. As already remarked u_0v_0 and $u_{m+1}v_{m+1}$ have the same colour. We assume that $u_0v_0, u_{m+1}v_{m+1} \in f_3$.
We consider the spokes between S_1 and S_2, that is the edges u_iv_i with $i = m+1, \ldots, t$.
Since the number of subsequent edges with the same colour can be only $1, 2$ or 6, it is easy to verify that the sequence of consecutive spokes between S_1 and S_2 which have the same colour is necessarily of type $6, 1, 2, 1, 2, \ldots, 1, 2, 1, 6$, that is we have 6 spokes of colour f_1, one spoke of colour f_3, two spokes of colour f_2, one of colour f_3, two of colour f_1 and so on. That means $S_2 \subset f_1$ or $S_2 \subset f_2$. Nevertheless in both cases we have a contradiction, since the set $E = \{u_3u_4, v_3v_5, v_4v_2, u_{t+3}u_{t+4}, v_{t+3}v_{t+5}, v_{t+4}v_{t+2}\}$ disconnects the graph, but does not satisfy Lemma 1. Then $S_1 = S_2$.
Observe that if $S_1 = S_2$, then $\frac{n}{2} - 2 \equiv 0 \pmod 6$, that is $n \equiv 4 \pmod 6$ since $\mathcal{F}$ is a *one*–factorization.
We consider the case in which the sets of subsequent spokes with the same colour have cardinality at most 2. Suppose that there exists a set S containing exactly two consecutive spokes with the same colour. Without loss of generality we can assume $S = \{u_1v_1, u_2v_2\} \subset f_1$. It follows that the

spokes of the graph are coloured as follows:

$$\begin{aligned} u_iv_i \in f_1 \text{ if } i \equiv 1,2 \pmod 6; \\ u_iv_i \in f_2 \text{ if } i \equiv 4,5 \pmod 6; \\ u_iv_i \in f_3 \text{ if } i \equiv 0,3 \pmod 6. \end{aligned}$$

whence $n \equiv 0 \pmod 6$. One can check that $f_1 \cup f_2$ contains two cycles of length n. That yields a contradiction then necessarily a set of subsequent spokes has cardinality 1 and $n \equiv 3,4 \pmod 6$. The case $n \equiv 0 \pmod 6$ is excluded since in this case $f_1 \cup f_2$ contains two disjoint n–cycles.
Viceversa if $n \equiv 3 \pmod 6$ the existence of a perfect *one*–factorization follows from Proposition 2. In the case $n \equiv 4 \pmod 6$ and $n > 4$, we obtain a perfect *one*–factorization $\mathcal{F} = \{f_1, f_2, f_3\}$ by colouring the spokes as follows:

$$\text{for } 0 \leqslant j \leqslant 5 \text{ we set } u_jv_j \in f_1$$

$$\text{for } 6 \leqslant j \leqslant \tfrac{n}{2} + 2 \text{ we set}$$

$$\begin{aligned} u_jv_j, u_{n+5-j}v_{n+5-j} \in f_1 \text{ if } j \equiv 4,5 \pmod 6; \\ u_jv_j, u_{n+5-j}v_{n+5-j} \in f_2 \text{ if } j \equiv 0,3 \pmod 6; \\ u_jv_j, u_{n+5-j}v_{n+5-j} \in f_3 \text{ if } j \equiv 1,2 \pmod 6. \end{aligned}$$

We now turn our attention to the graphs $GP(n,3)$. Despite an increased number of cases which must be considered, the following result can be proved with methods similar to those of the previous Proposition. The conclusion is rather different this time in the sense that only one graph in the class $GP(n,3)$ is P1F.

Proposition 6 *The generalized Petersen graph $GP(n,3)$ is P1F if and only if $n = 9$.*

We remark that the perfect *one*–factorization of $GP(3d,d)$ in Proposition 2 is obtained from the perfect *one*–factorization $\mathcal{F}$ of $GP(3,1)$ by creating d consecutive copies of u_jv_j, for every spoke u_jv_j of $\mathcal{F}$, thus obtaining a monochromatic set of d subsequent spokes.

One can apply the same procedure to a perfect *one*–factorization of $GP(n,k)$, with arbitrary parameters n, k. However the *one*–factorization obtained in this case might not be perfect.

Finally, we observe that, making use of the well–known isomorphisms of $GP(n,k)$ (see [10]), together with Proposition 5 and Proposition 6, it is sometimes possible to determine whether a generalized Petersen graph, with $k \geqslant 4$, is P1F.

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