The crossing number of $K_{1,1,3,n}$

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Abstract

In this paper, we show that the crossing number of the complete multipartite graph $K_{1,1,3,n}$ is

$$cr(K_{1,1,3,n}) = 4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + \lfloor\frac{3n}{2}\rfloor.$$

Our proof depends on Kleitman's results for the complete bipartite graphs [D. J. Kleitman, The crossing number of $K_{5,n}$, *J. Combin. Theory*, 9 (1970), 315-323].

1 Introduction

Computing the crossing number of a given graph is, in general, an elusive problem. In fact, computing the crossing number of a graph is NP-complete [3]. Exact values are known only for very restricted classes of graphs. A good, updated survey on crossing numbers is [9].

A longstanding problem of crossing numbers is the Zarankiewicz conjecture, which asserts that the crossing number of the complete bipartite graphs $K_{m,n}$ is given by

$$cr(K_{m,n}) = \lfloor\frac{m}{2}\rfloor\lfloor\frac{m-1}{2}\rfloor\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor. \quad (1)$$

It is only known to be true for $m \leq 6$ [8]; and for $m = 7$ and $n \leq 10$ [10]. Recently, in [2], E. deKlerk et al. give a new lower bound for the crossing number of $K_{m,n}$. In the following, $Z(m,n)$ will denote the right terom of equation (1).

It is natural to ask generalize the Zarankiewicz conjecture and to ask: What are the crossing numbers for the complete multipartite graphs? For

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general upper bound of the crossing number of a complete n-partite graph, see [5]. In [1],it is shown that the crossing numbers of $K_{1,3,n}$ and $K_{2,3,n}$ are as follows;

$$\begin{aligned} cr(K_{1,3,n}) &= Z(4,n) + \lfloor \frac{n}{2} \rfloor, \\ cr(K_{2,3,n}) &= Z(5,n) + n. \end{aligned}$$

In [6], the author has applied a similar technique as in [1] to find the crossing numbers of $K_{1,1,1,1,n}$, $K_{1,2,2,n}$, $K_{1,1,1,2,n}$ and $K_{1,4,n}$. In [7], it is also shown that if Zarankiewicz's conjecture is true for $m = 2k+1$ (for $m = 2k+2$ respectively), then the crossing number of $K_{1,2k,n}$ (of $K_{1,2k+1,n}$) is $Z(2k,n) + (k^2-k)\lfloor \frac{n}{2} \rfloor$ (is $Z(2k+1,n) + k^2\lfloor \frac{n}{2} \rfloor$ respectively). In this paper, we show that the crossing number of $K_{1,1,3,n}$ is $cr(K_{1,1,3,n}) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{3n}{2} \rfloor$.

Here are some notations. Let G be a simple graph with the vertex set $V = V(G)$ and the edge set $E = E(G)$. A *drawing* is a mapping of a graph G into the plane. Vertices go into distinct nodes. An edge and its incident vertices map into a homemorphic image of the closed interval $[0,1]$ with the relevant nodes as endpoints and the interior, an arc, containing no node. A drawing is *good* if it satisfies (i) no two arcs incident with a common node have a common point; (ii) no two arcs have more than one point in common; (iii) no arc has a self-intersection; and (iv) no three arcs have a point in common.

A common point of two arcs is a *crossing*. Let A and B be subsets of E. In a drawing ϕ, the number of crossings of edges in A with edges in B is denoted by $cr_\phi(A,B)$. Especially, $cr_\phi(A,A)$ will be denoted by $cr_\phi(A)$. For a good drawing ϕ, the total number of crossings is $cr_\phi(E)$. The *crossing number* of the graph G, $cr(G)$, is the the minimum of $cr_\phi(E)$ among all good drawings ϕ of G.

Remark. We often make no distinction between a graph-theoretical object (such as a vertex, or a edge) and its drawing. Throughout this work, we have taken special care to ensure that no confusion arises from this practice.

We note the following formulas, which can be shown easily,

$$cr_\phi(A \cup B) = cr_\phi(A) + cr_\phi(B) + cr_\phi(A,B), \quad (2)$$

$$cr_\phi(A, B \cup C) = cr_\phi(A,B) + cr_\phi(A,C), \quad (3)$$

where A, B and C are mutually disjoint subsets of E.

Let A be a nonempty subset of V or of E for a graph G, for a graph G. Then $\langle A \rangle$ denotes the subgraph of G induced by A. The set of edges which are incident with a vertex v is denoted by $E(v)$. For a complete k-partite graph $K_{a_1,a_2,\ldots,a_k}$ with the partition $(A_1, A_2, \ldots, A_k)$ of the vertex set V

and the edge set E, where $|A_i| = a_i$, we will write $E_{A_iA_j}$ for the edge sets of $\langle A_i \cup A_j \rangle$.

2 Crossing number of $K_{1,1,3,n}$

Lemma 2.1. *There are 7 non-isomorphic good drawings of* $K_{1,1,3}$.

Proof. From [4], we know that there are 6 non-isomorphic drawings of $K_{2,3}$, namely, the drawings in Figure 1. To obtain a drawing of $K_{1,1,3}$ from these drawings of $K_{2,3}$, we have to draw an edge e connecting the vertices in the partition containing two vertices, that is, the vertices represented by •. Note that the edge e cannot cross any edge in order to be a good drawing. Then, from the drawings in Figure 1, we obtain 7 non-isomorphic drawings of $K_{1,1,3}$, namely, the drawings in Figure 2. □

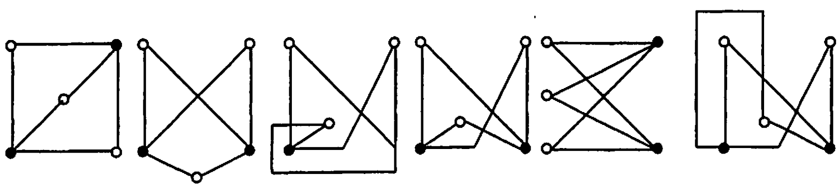

Figure 1.

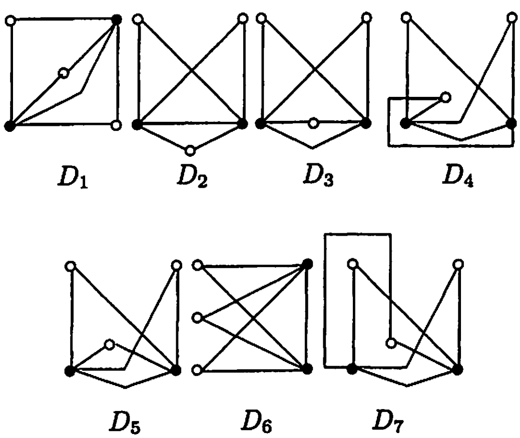

Figure 2.

Theorem 2.1. *The crossing number of the complete 4-partite graph $K_{1,1,3,n}$ is given by*

$$cr(K_{1,1,3,n}) = Z(5,n) + \lfloor \frac{3n}{2} \rfloor.$$

Proof. Let (X, Y, U, Z) be the vertex partition of $K_{1,1,3,n}$ such that $X = \{x\}, Y = \{y\}, U = \{u_1, u_2, u_3\}$ and $Z = \bigcup_{i=1}^{n} \{z_i\}$. To show that $cr(K_{1,1,3,n}) \leq Z(5,n) + \lfloor \frac{3n}{2} \rfloor$, see Figure 3 for $n = 4$ and it can be easily generalized to n. This also follows from the general bound in [5].

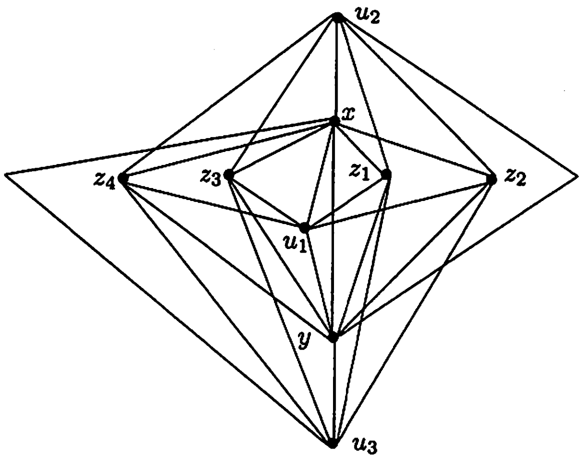

Figure 3.

Therefore it is sufficient to prove that

$$cr(K_{1,1,3,n}) \geq Z(5,n) + \lfloor \frac{3n}{2} \rfloor. \tag{4}$$

We will prove (4) by induction on n. For $n = 1$, $K_{1,1,3,1}$ contains $K_{3,3}$ and it is clear that $K_{3,3}$ is nonplanar, therefore $cr(K_{1,1,3,1}) \geq 1$. Therefore (4) is true for $n = 1$.

For $n = 2$, $\langle E_{YU} \cup E_{YZ} \cup E_{UZ} \rangle$ contains a drawing of $K_{3,3}$. From [4], we know that any drawing of $K_{3,3}$ has a crossing number 1, 3, 5, 7 or 9. Therefore, if the $K_{3,3}$ contained in $\langle E_{YU} \cup E_{YZ} \cup E_{UZ} \rangle$ has at least 3 crossings, we have $cr(K_{1,1,3,2}) \geq 3$. Therefore we may assume the $K_{3,3}$ contained in $\langle E_{YU} \cup E_{YZ} \cup E_{UZ} \rangle$ having only 1 crossing. Again, from [4], there is a unique drawing of $K_{3,3}$ such it has only 1 crossing, namely, the drawing in Figure 4. However, if the $K_{3,3}$ contained in $\langle E_{YU} \cup E_{YZ} \cup E_{UZ} \rangle$ is drawn as in Figure 4, no matter which region x is placed, we have the number of crossings between the edges in $E(x)$ and $E_{YU} \cup E_{YZ} \cup E_{UZ}$ is at least 2, which implies that the total number of crossings is at least 3. This proves (4) for $n = 2$.

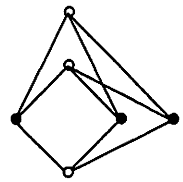

Figure 4.

Now suppose

$$\begin{aligned} cr(K_{1,1,3,n-2}) &\geq Z(5, n-2) + \lfloor \frac{3(n-2)}{2} \rfloor \\ cr(K_{1,1,3,n-1}) &\geq Z(5, n-1) + \lfloor \frac{3(n-1)}{2} \rfloor \\ cr(K_{1,1,3,n}) &< Z(5, n) + \lfloor \frac{3n}{2} \rfloor, \end{aligned} \tag{5}$$

for some $n \geq 3$. Then there exists a good drawing ϕ of $K_{1,1,3,n}$ such that

$$cr_\phi(E) \leq Z(5, n) + \lfloor \frac{3n}{2} \rfloor - 1. \tag{6}$$

Let $W = E_{XY} \cup E_{XU} \cup E_{YU}$. Then, by (2) and (3), we have

$$cr_\phi(E) = cr_\phi(W) + cr_\phi(\bigcup_{i=1}^{n} E(z_i)) + \sum_{i=1}^{n} cr_\phi(W, E(z_i)). \tag{7}$$

Since $\langle \bigcup_{i=1}^{n} E(z_i) \rangle \cong K_{5,n}$, by (1), we have

$$cr_\phi(\bigcup_{i=1}^{n} E(z_i)) \geq Z(5, n). \tag{8}$$

If $cr_\phi(W, E(z_i)) \geq 2$ for all i, by (7) and (8), we have $cr_\phi(E) \geq Z(5, n) + 2n$ which contradicts (6). Therefore, by reordering, we may assume

$$cr_\phi(W, E(z_1)) \leq 1. \tag{9}$$

We will consider two cases:

Case 1. $cr_\phi(W, E(z_i)) = 0$ for some i;

Case 2. $cr_\phi(W, E(z_i)) \geq 1$ for all i

Case 1. By reordering, we may assume that $cr_\phi(W, E(z_1)) = 0$. A drawing of $\langle W \rangle$ divides $\mathbb{R}^2$ into regions and the condition $cr_\phi(W, E(z_1)) = 0$ implies that $X \cup Y \cup U$ is contained in the boundary of one of the regions. Denote $F = W \cup E(z_1)$. Then Figure 5 shows all the possible drawings of $\langle F \rangle$ up to isomorphism because of Figure 2.

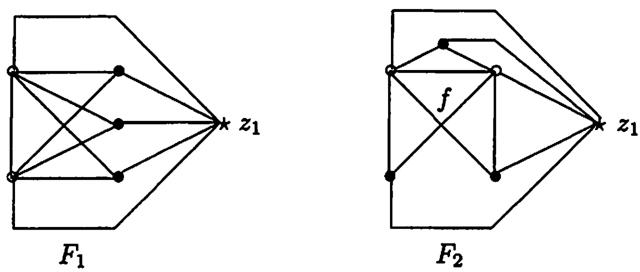

Figure 5.

Suppose the drawing of $\langle F \rangle$ is F_1. Then for $2 \leq j \leq n$, we have

$$cr_\phi(F, E(z_j)) \geq 4. \tag{10}$$

By (2) and (3), we have

$$cr_\phi(E) = cr_\phi(F) + cr_\phi(\bigcup_{i=2}^{n} E(z_i)) + \sum_{i=2}^{n} cr_\phi(F, E(z_i)). \tag{11}$$

Note also that $\bigcup_{i=2}^{n} E(z_i) \cong K_{5,n-1}$, by (1), we have

$$cr_\phi(\bigcup_{i=2}^{n} E(z_i)) \geq Z(5, n-1). \tag{12}$$

Since $cr_\phi(F) = 3$ in F_1, by (10), (11) and (12), we have $cr_\phi(E) \geq 3 + Z(5, n-1) + 4(n-1) \geq Z(5,n) + \lfloor \frac{3n}{2} \rfloor$ which contradicts (6).

Now suppose the drawing of $\langle F \rangle$ is F_2. If z_j for $2 \leq j \leq n$ is located in another region than f, we have

$$cr_\phi(F, E(z_j)) \geq 4. \tag{13}$$

If z_j for $2 \le j \le n$ is located in the region f, we have

$$cr_\phi(W, E(z_j)) \ge 3. \tag{14}$$

Let l be the number of z_j for $2 \le j \le n$ being located in the region f. Combining (6), (7), (8), (14), we have

$$3l \le \lfloor \frac{3n}{2} \rfloor - 1. \tag{15}$$

Since $cr_\phi(F) = 1$, by (11), (12), (13), (14) (15), we have $cr_\phi(E) \ge 1 + Z(5, n-1) + 4(n-1-l) + 3l \ge Z(5,n) + \lfloor \frac{3n}{2} \rfloor$, which contradicts (6).

Case 2. By (9), there must exist a region in the drawing of $\langle W \rangle$ containing at least 4 vertices of $X \cup Y \cup U$ on its boundary. From the drawings in Figure 2, the only possible drawings of W are D_1, D_2, D_3, D_5, D_6.

Suppose that $\langle W \rangle$ is drawn as D_1, D_2 or D_5. It can be checked that if $cr_\phi(W, E(z_i)) \ge 1$, then either $cr_\phi(E_{XU}, E(z_i)) \ge 1$ or $cr_\phi(E_{YU}, E(z_i)) \ge 1$. Hence, by our assumption that $cr_\phi(W, E(z_i)) \ge 1$ for all i, we have

$$\sum_{i=1}^{n} cr_\phi(E_{XU}, E(z_i)) + \sum_{i=1}^{n} cr_\phi(E_{YU}, E(z_i)) \ge n. \tag{16}$$

Note that $E - E_{XU} \cong K_{1,4,n}$. From [6] (see also [7]), we know that $cr(K_{1,4,n}) = Z(5,n) + 2\lfloor \frac{n}{2} \rfloor$. This implies that

$$cr_\phi(E - E_{XU}) \ge Z(5,n) + 2\lfloor \frac{n}{2} \rfloor. \tag{17}$$

By (2), (3) and the fact that $cr_\phi(E_{XU}) = 0$, we have

$$cr_\phi(E) = cr_\phi(E - E_{XU}) + cr_\phi(E_{XU}, E - E_{XU}). \tag{18}$$

By (6), (17), (18) and the fact $cr_\phi(E_{XU}, E - E_{XU}) \ge \sum_{i=1}^{n} cr_\phi(E_{XU}, E(z_i))$, it follows

$$\sum_{i=1}^{n} cr_\phi(E_{XU}, E(z_i)) \le \lceil \frac{n}{2} \rceil - 1. \tag{19}$$

Note that also $E - E_{YU} \cong K_{1,4,n}$, and by exactly the same argument we obtain (19), we get

$$\sum_{i=1}^{n} cr_\phi(E_{YU}, E(z_i)) \le \lceil \frac{n}{2} \rceil - 1. \tag{20}$$

But (19) and (20) together contradict (16).

Now it remains the case $\langle W \rangle$ is drawn as in D_3 or D_6. First suppose that $\langle W \rangle$ is drawn as in D_3. By (9) and our assumption that $cr_\phi(W, E(z_i)) \geq 1$ for all i, we have $cr_\phi(W, E(z_1)) = 1$. Then z_1 must lie in the region containing 4 vertices in $X \cup Y \cup U$, that is, the outer region of D_3. Let $F = W \cup E(z_1)$. Note that if z_1 is drawn in the outer region of D_3, in order to satisfy $cr_\phi(W, E(z_1)) = 1$, $\langle F \rangle$ can only be drawn as in Figure 6.

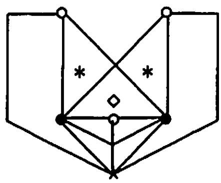

Figure 6.

For z_j $(2 \leq j \leq n)$ lying in the region marked with ◇, we have

$$cr_\phi(W, E(z_j)) \geq 2. \tag{21}$$

We claim that the equality in (21) is impossible. Suppose not, without loss of generality, we may assume that z_2 lies in the region marked with ◇ and $cr_\phi(W, E(z_2)) = 2$. Then we must have

$$cr_\phi(E(z_1), E(z_2)) = 0. \tag{22}$$

For $3 \leq k \leq n$, $\langle E(z_1) \cup E(z_2) \cup E(z_k) \rangle$ is isomorphic to $K_{5,3}$. Then by (2), (3), (22), and the fact that $cr(K_{5,3}) = 4$ (see [8]), we have

$$cr_\phi(E(z_1) \cup E(z_2), E(z_k)) \geq 4 \text{ for } 3 \leq k \leq n. \tag{23}$$

Let $E' = E - (E(z_1) \cup E(z_2))$. Then $\langle E' \rangle = K_{1,1,3,n-2}$ and

$$\begin{aligned} cr_\phi(E) &= cr_\phi(E') + cr_\phi(E(z_1) \cup E(z_2)) + cr_\phi(W, E(z_1)) \\ &\quad + cr_\phi(W, E(z_2)) + \sum_{i=3}^{n} cr_\phi(E(z_1) \cup E(z_2), E(z_i)). \end{aligned} \tag{24}$$

By (2), (3), (5), (23), (24), and the fact that $cr_\phi(W, E(z_1)) = 1$ and $cr_\phi(W, E(z_2)) = 2$, we have $cr_\phi(E) \geq Z(5, n-2) + \lfloor \frac{3(n-2)}{2} \rfloor + 1 + 2 + 4(n-2) \geq Z(5, n) + \lfloor \frac{3n}{2} \rfloor$ which contradicts to (6). This proves our claim.

By our claim and (21), we know that if z_j for $2 \leq j \leq n$ lies in the region marked with $\diamond$, we have

$$cr_\phi(F, E(z_j)) \geq cr_\phi(W, E(z_j)) \geq 3. \tag{25}$$

If z_j for $2 \leq j \leq n$ lies in the regions marked with $*$, we have

$$cr_\phi(F, E(z_j)) \geq cr_\phi(W, E(z_j)) \geq 3. \tag{26}$$

If z_j for $2 \leq j \leq n$ lies in the regions which are not marked with $*$ or $\diamond$, we have

$$cr_\phi(F, E(z_j)) \geq 4. \tag{27}$$

Let l be the number of z_j for $2 \leq j \leq n$ lying in the regions marked with $*$ or $\diamond$. By (25), (26) and our assumption that $cr_\phi(W, E(z_j)) \geq 1$ for all j, we have

$$\sum_{j=1}^{n} cr_\phi(W, E(z_j)) \geq 3l + (n - l) = n + 2l. \tag{28}$$

By (6), (7), (8), (28), we get

$$2l \leq \lfloor \frac{n}{2} \rfloor - 1. \tag{29}$$

Then by (11), (12), (25), (26), (27), (29) and $cr_\phi(F) = 2$, we have

$$\begin{aligned} cr_\phi(E) &\geq 2 + Z(5, n-1) + 3l + 4(n-1-l) \\ &= Z(5, n-1) + 4n - 2 - l \\ &\geq Z(5, n-1) + 4n - 2 - (\lfloor \frac{n}{2} \rfloor - 1)/2 \\ &\geq Z(5, n) + \lfloor \frac{3n}{2} \rfloor, \end{aligned}$$

which contradicts (6).

Now consider $\langle W \rangle$ is drawn as in D_6. By (9) and our assumption that $cr_\phi(W, E(z_i)) \geq 1$ for all i, we have $cr_\phi(W, E(z_1)) = 1$. Then z_1 must lie in the region containing 4 vertices in $X \cup Y \cup U$, that is, the outer region of D_6. Let $F = W \cup E(z_1)$. Note that if z_1 is drawn in the outer region of D_6, in order to satisfy $cr_\phi(W, E(z_1)) = 1$, $\langle F \rangle$ can only be drawn as in Figure 7.

If z_j for $2 \leq j \leq n$ lies in the regions which are not marked with $*$, we have

$$cr_\phi(F, E(z_j)) \geq 4. \tag{30}$$

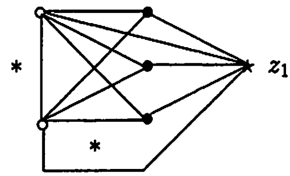

Figure 7.

If z_j for $2 \leq j \leq n$ lies in the regions marked with $*$, we have

$$cr_\phi(F, E(z_j)) \geq 3. \tag{31}$$

However, under the condition $cr_\phi(W, E(z_j)) \geq 1$ for $2 \leq j \leq n$, the equality in (31) is impossible. (Otherwise, $cr_\phi(W, E(z_j)) = 0$.) Hence, by (30) and (31), we get

$$cr_\phi(F, E(z_j)) \geq 4 \text{ for } 2 \leq j \leq n. \tag{32}$$

Since $cr_\phi(F) = 4$, by (11), (12) and (32), we have $cr_\phi(E) \geq 4 + Z(5, n-1) + 4(n-1) \geq Z(5,n) + \lfloor \frac{3n}{2} \rfloor$, which contradicts (6).

□

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