# Landmarks in Binary Tree Derived Architectures

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**Abstract**

Let $M = \{v_1, v_2 \dots v_\ell\}$ be an ordered set of vertices in a graph $G$. Then $(d(u, v_1), d(u, v_2) \dots d(u, v_\ell))$ is called the *M-location* of a vertex $u$ of $G$. The set $M$ is called a *locating set* if the vertices of $G$ have distinct $M$-locations. A *minimum locating set* is a set $M$ with minimum cardinality. The cardinality of a minimum locating set of $G$ is called *Location Number* $L(G)$. This concept has wide applications in motion planning and in the field of robotics. In this paper we consider networks with binary tree as an underlying structure and determine minimum locating set of such architectures.We show that the location number of an $n$-level $X$-tree lies between $2^{n-3}$ and $2^{n-3}+2$. We further prove that the location number of an $N \times N$ mesh of trees is greater than or equal to $N/2$ and less than or equal to $N$.

## 1 Introduction

The tree interconnection network lends itself to several suitably structured applications. However, the low connectivity at each node, traffic congestion and single point of failure at the root node reduce reliability and availability. Both the hypertree and $X$-tree are fault tolerant variants of the basic tree network and have been the focus of more recent implementation and research interest.

The tree interconnection network is suitable for tree structured computations (multi-input, single-output) and divide-and-conquer type applications. Tree-based networks have fixed degree nodes and are suitable for massively parallel systems. For the single-rooted complete binary tree shown in Figure 1(a), the node degrees are bounded above by 3. The average inter-node distance in such a tree network increases as $\log N$, where $N$ is the network size. While the tree has smaller diameter than the mesh, it does not exploit physical locality effectively [8, 10]. In particular, many algorithms can make use of direct communication between the leaf nodes if it exists. For applications that require extensive communication between leaf nodes, this property may prove to be a serious disadvantage.

Additionally, the single root node in the case of the single-rooted binary tree could present severe traffic congestion and a single point of failure. Even in

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the case of the double-rooted tree, failure or severe congestion at any root node disconnects both left and right halves of the network.

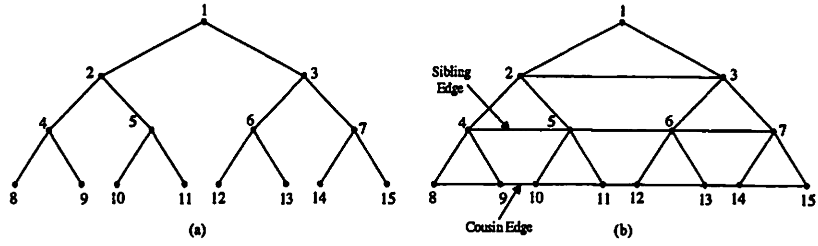


Figure 1: (a) A complete binary tree (b) An $X$-tree

Grids and meshes of trees are among the best known interconnection networks devised. Meshes of trees perform better with algorithms that require data broadcasting in the rows or columns. The mesh of trees is a hybrid interconnection network based on arrays and trees. It owns two advantages of small diameter and large bisection width and is known as the fastest network when considered solely in terms of speed [10]. It can provide logarithmic or log-squared time solutions to many important problems such as packet routing, sorting, pre-fix computation, matrix multiplication, convolution, transitive closure, shortest path, minimum spanning tree, nearest neighbor, convex hull, and so on [10]. The meshes of trees have been proposed as an ingenious hybrid of trees and meshes. The mesh of trees is an area-universal network. Thus it can emulate any network of equal VLSI layout area with a poly-logarithmic slowdown. Due to its small diameter, the mesh of trees can solve a number of problems more quickly than the 2-dimensional mesh.

Let $M = \{v_1, v_2 \ldots v_\ell\}$ be an ordered set of vertices in a graph $G$. Then $f(u) = (d(u,v_1), d(u,v_2) \ldots d(u,v_\ell))$ is called the *$M$-location* of a vertex $u$ of $G$. The set $M$ is called a *locating set* if no two vertices of $G$ have the same $M$-location (that is, $f(x) \neq f(y)$ if $x \neq y$). In other words, if $M$ is a locating set then it is clear that for each pair of vertices $u$ and $v$ of $V \setminus M$, there is a vertex $w \in M$ such that $d(u,w) \neq d(v,w)$ [9]. A *minimum locating set* is a set $M$ with minimum cardinality [6]. The cardinality of a minimum locating set of $G$ is called *Location Number* and is denoted by $L(G)$. The *minimum locating set* problem is to find a minimum *locating set*. The members of a minimum locating set of $G$ are called *landmarks* of $G$. Slater [16, 17] describes applications of these concepts when working with sonar and loran stations. Chartrand et al. [4] calls locating set as a *resolving set*.

For the complete graph $K_p$, the cycle $C_p$ and the complete bipartite graph $K_{m,n}$, Harary et al. [7] have shown that $L(K_p) = p - 1$, $L(C_p) = 2$ and $L(K_{m,n}) = m + n - 2$. This problem has been studied for grids [11], trees, multi-dimensional grids [9], Petersen graphs [2], De Bruijn graphs [12], Torus networks [13], Benes and Butterfly networks [14] and Honeycomb networks [15].

The minimum locating set problem is proved to be *NP*-complete for general graphs by a reduction from 3-dimensional matching [3, 6]. Recently Manuel et al. [14] have proved that this problem is *NP*-complete for bipartite graphs.

In this paper, we study the minimum locating set problem for $X$-trees, mesh of trees and double rooted $X$-trees. We prove that the location number of $n$-level X-tree is greater than or equal to $2^{n-3}$ and less than or equal to $2^{n-3}+2$. Also we show that the location number of $N \times N$ mesh of trees lies between $N/2$ and $N$.

# 2 Topological Properties of Binary Tree Derived Architectures

## 2.1 $X$-Trees

An $X$-tree is a complete binary tree with edges added to connect consecutive nodes on the same level of the tree. Edges are added to the tree so that the vertices on each level are connected, from left to right, in a path. Edges on such paths are called as *horizontal edges.* Horizontal edges are of two types; *sibling edges* and *cousin edges.* A sibling edge denotes a horizontal edge that connects two vertices with the same parent and a cousin edge denotes any of the remaining horizontal edges. Two sibling edges are said to be *adjacent* if there is exactly one cousin edge between them. The term *vertical edge* designates a tree edge. An $n$-level $X$-tree has $2^{n+1}-1$ vertices and $2^{n+2}-n-4$ edges [1]. Vertices at level $n$ are called leaf vertices. See Figure 1(b). An $n$-level $X$-tree or a $2^n$-leaf $X$-tree will be denoted by $X(n)$. The root of $X(n)$ is considered to be at level 0. The vertices of $X(n)$ other than the root and the leaf vertices are called *internal vertices.*

## 2.2 Mesh of trees

An $N \times N$ mesh of trees consists of an $N \times N$ grid of nodes where there is an $N$-leaf complete binary tree on each row and each column. An $N \times N$ mesh of trees is denoted as $MOT(N)$. The number of nodes in $MOT(N)$ is $3N^2 - 2N$. The diameter of $MOT(N)$ is $4 \log N$ [10]. An $N \times N$ mesh of trees is basically a 2-dimensional square mesh whose edges have been removed and replaced by complete binary trees on each column and row. These trees are called *column trees* and *row trees.* The graph in Figure 2 is a $8 \times 8$ mesh of trees. The new nodes introduced are internal nodes of the trees. We follow the notation given in [10]. In the following sections, we provide bounds for the location number of the binary tree derived architectures mentioned earlier.

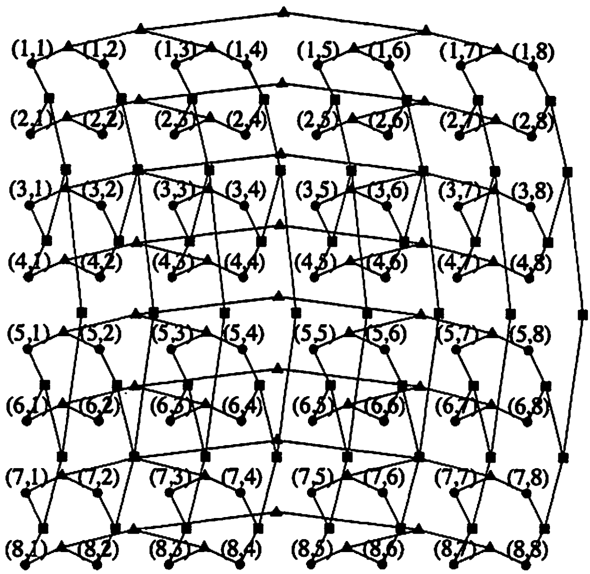


Figure 2: An $8 \times 8$ mesh of trees. The square vertices are internal vertices of column trees and triangles are internal vertices of row trees.

# 3 Minimum Locating Set of $X$-trees

Consider an $n$-level $X$-tree $X(n)$. The vertices at the bottommost level are called *leaves* and other nodes are called *internal* nodes. Let $V_L$ and $V_I$ denote the sets of leaf and internal vertices of $X(n)$ respectively. We label the root of $X(n)$ as 1. Nodes at any level are labeled from left to right. The children of the node $x$ are labeled $2x$ and $2x+1$. See Figure 3.

**Lemma 1** *Let $S$ be any locating set for $X$-tree $X(n)$. Then $S \nsubseteq V_I$.*

**Proof.** Let $u$, $v \in V_L$ be the children of the same parent. Then $u$ and $v$ are equidistant from every member of $V_I$. Hence $S \nsubseteq V_I$. □

**Corollary 1** *Let $S$ be any locating set for $X$-tree $X(n)$. $V_L \cap S \neq \phi$.* □

We first provide a lower bound for the location number of $X(n)$. To do this we define a set $N(a)$ for each leaf vertex $a$ of $X(n)$ by

$N(a) = \{a-3, a-2, a-1, a, a+1, a+2, a+3, a+4\}$.

We call $N(a)$ as the leaf neighbourhood of $a$. This set contains at most 8 elements and not all elements of $N(a)$ are defined for some leaves. For example, $N(2^n) = \{2^n, 2^n+1, 2^n+2, 2^n+3, 2^n+4\}$.

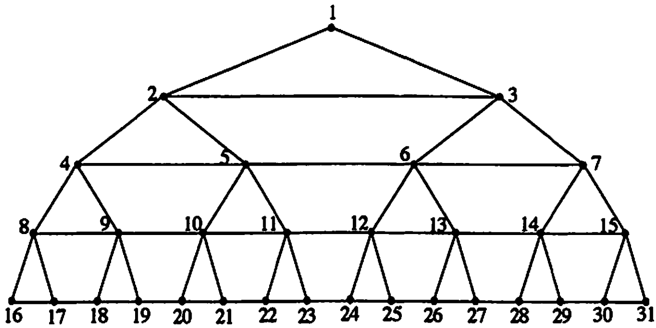


Figure 3: A 4-level $X$-tree with labels

**Lemma 2** *Let $S$ be a locating set of $X(n)$. For $a \in V_L$, $N(a) \cap S \neq \phi$ whenever $(a, a+1)$ is a sibling edge of $X(n)$.*

**Proof.** If possibe let $N(a) \cap S = \phi$. This implies $S \subseteq V(X(n)) \setminus N(a)$. Now let $u$ be any vertex in $V(X(n)) \setminus N(a)$. There exists shortest paths from $a$ to $u$ and $a+1$ to $u$, passing through the parent of $a$ and $a+1$. Hence we have $d(a, u) = d(a+1, u)$. In particular this is true for every $u \in S$; but this contradicts the fact that $S$ is a locating set for $X(n)$. □

Now $|V_L| = 2^n$ and $|N(a)| \leq 8$, for $a \in V_L$. As $V_L$ can be partitioned into $2^{n-3}$ subsets of cardinality 8 each, we have the following result by applying Lemma 2.

**Lemma 3** *(Lowerbound)* $L(X(n)) \geq 2^{n-3}$. □

**Notation 1** *The vertices at level $j$ of $X(n)$ can be partitioned into 4-subsets, each inducing a path. Let $R_j^i$ be the path induced by the vertices $2^j + 4(i-1) + k$, $0 \leq k \leq 3$, $1 \leq i \leq 2^{j-2}$, $2 \leq j \leq n$. Consider $R_{n-1}^i$ and let $m_i$ denote the left descendant of $2^{n-1} + 4(i-1)$. Then $m_i = 2(2^{n-1} + 4(i-1)) = 2^n + 8(i-1)$.*

**Lemma 4** *Any two vertices of $R_j^i$, $1 \leq i \leq 2^{j-2}$, $2 \leq j \leq n-1$, are at unequal distances from $m_i$.*

**Proof.** Let $u_k = 2^j + 4(i-1) + k$, $0 \leq k \leq 3$ be the vertices of $R_j^i$, $1 \leq i \leq 2^{j-2}$, $2 \leq j \leq n-1$. Now $d(u_0, m_i) = n - j$. Hence $d(u_1, m_i) = n - j + 1$, $d(u_2, m_i) = n - j + 2$ and $d(u_3, m_i) = n - j + 3$ proving that the pairs $u_0, u_1$; $u_0, u_2$ and $u_0, u_3$ are at unequal distances from $m_i$. Similar argument applies to the pairs $u_1, u_2$; $u_1, u_3$ and $u_2, u_3$. □

**Lemma 5** *(Upper Bound)* $L(X(n)) \leq 2^{n-3} + 2, n \geq 4$.

**Proof.** We claim that the set $S = \{1, m_i, 2^{n+1} - 1 : 1 \leq i \leq 2^{n-3}\}$ is a locating set for $X(n)$. In otherwords, we prove that for every pair of vertices $u, v$ in $X(n)$ there exists a vertex $w \in S$ such that $d(u, w) \neq d(v, w)$.

Since $1 \in S$, the pairs of vertices of $X(n)$ at different levels are at unequal distances from 1. So we need to consider only pairs of vertices of $X(n)$ at the same level of $X(n)$.

Let $u$ and $v$ be any two vertices at level $j$ of $X(n)$.

**Case 1:** $u, v \in R_j^i$, $1 \leq i \leq 2^{j-2}$, $2 \leq j \leq n-1$.

By Lemma 4, $d(u, m_i) \neq d(v, m_i)$, where $m_i$ is the leftmost descendant of $2^j + 4(i-1)$ lying at level $n$. The possibility that $u = 2^j + 4(i-1)$ is not ruled out.

**Case 2:** $u \in R_j^{i_1}$, $v \in R_j^{i_2}$, $i_1 < i_2$. It is sufficient to consider the case $i_2 = i_1 + 1$. So let $u \in R_j^i$ and $v \in R_j^{i+1}$. Assume that $u = 2^j + 4(i-1) + k_1$, $0 \leq k_1 \leq 3$ and $v = 2^j + 4i + k_2$, $0 \leq k_2 \leq 3$.

Let $u_0 = 2^j + 4(i-1)$. Then $d(u, u_0) = k_1 \leq 3$ and $d(v, u_0) \geq 4$.

Therefore,

$$\begin{aligned} d(u, m_i) &= d(u, u_0) + d(u_0, m_i) \\ &\leq 2^{n-j} + 3 \\ d(v, m_i) &\geq 2^{n-j} + 4. \end{aligned}$$

This argument applies to all levels $j$, $2 \leq j \leq n-1$. Peculiarly in the $n^{th}$ level of the $X(n)$ in each of the 8-subsets beginning with $m_i$ the pairs of vertices $m_i + 4$, $m_i + 5$; and $m_i + 6$, $m_i + 7$ are at equal distances from $m_i$. But these vertices will be at unequal distance from $m_{i+1}$. Finally the pairs $2^{n+1} - 4$, $2^{n+1} - 3$ and $2^{n+1} - 2$, $2^{n+1} - 1$ are at equal distances from 1 and every $m_i$. This forces the inclusion of $2^{n+1} - 1$ in $S$. The cardinality of $S$ is $2^{n-3} + 2$. □

Lemmas 3 and 5 imply the following result.

**Theorem 2** $2^{n-3} \leq L(X(n)) \leq 2^{n-3} + 2$, $n \geq 4$. □

A simple calculation shows that $L(X(n)) = 2^{n-2} + 1$, for $n = 2, 3$.

**Conjecture 1** $L(X(n)) = 2^{n-3} + 1$, $n \geq 4$. □

# 4 Minimum Locating Set of Mesh of Trees

In this section, we shall prove that the location number of an $N \times N$ mesh of trees is greater than or equal to $N/2$ and less than or equal to $N$.

## 4.1 $N \times N$ Mesh of Trees

We first provide a lower bound for the location number of an $N \times N$ mesh of trees.

### 4.1.1 Lower Bound for $L(MOT(N))$

**Lemma 6** *Let $L$ denote a locating set of an $N \times N$ mesh of trees $MOT(N)$. A pair of successive column trees $2j-1$ and $2j$ has at least one vertex of $L$.*

**Proof.** Let us consider the leaves $(1, 2j-1)$ and $(1, 2j)$. Let $w$ be the father of $(1, 2j-1)$ and $(1, 2j)$. Let $u$ be any vertex which is not in the successive column trees $2j-1$ and $2j$. A shortest path between $u$ and $(1, 2j-1)$ passes through $w$. In the same way, every shortest path between $u$ and $(1, 2j)$ passes through $w$. Thus a locating set should contain at least one vertex $v$ in column trees $2j-1$ and $2j$ such that $v$ is at unequal distance from $(1, 2j-1)$ and $(1, 2j)$. □

**Lemma 7** *Let $L$ denote a locating set of $N \times N$ mesh of tree $MOT(N)$. A pair of successive row trees $2i-1$ and $2i$ has at least one vertex of $L$.* □

**Theorem 3** $L(MOT(N)) \geq N/2$.

**Proof.** By Lemma 6, every pair of successive column trees $2j-1$ and $2j$ has at least one vertex of $L(MOT(N))$. There are $N/2$ such pair of column trees. □

In fact the set $D = \{(1,1), (3,3), (5,5) \ldots (N-1, N-1)\}$ intersects every pair successive column trees $2j-1$ and $2j$ as well as every pair of successive row trees $2i-1$ and $2i$. Consider two adjacent vertices $x$ and $y$ of $(1,1)$. Then it is easy to verify that $x$ and $y$ are equidistant from $(k,k)$ for all $k = 1, 2 \ldots N$. Thus, the set $\{(1,1), (2,2), (3,3) \ldots (N,N)\}$ of diagonal vertices is not a locating set. In the same way, the set $\{(1,1), (1,2), (1,3) \ldots (1,N)\}$ of leaves of 1-st row tree is not a locating set. Also, the set $\{(1,1), (2,1), (3,1) \ldots (N,1)\}$ of leaves of 1-st column tree is not a locating set too. □

### 4.1.2 Upper Bound for $L(MOT(N))$

Let $c_j$ denote the root of $j$-th column tree and $r_i$ denote the root of $i$-th row tree. Let $T$ be a complete binary tree with root $r$. A vertex $u$ of $T$ is at level $s$ if the distance between $r$ and $u$ is $s$.

**Lemma 8** *The set $L = \{c_1, c_3 \ldots c_{N-1}, r_1, r_3 \ldots r_{N-1}\}$ of roots of odd column trees and odd row trees is a locating set of an $N \times N$ mesh of tree $MOT(N)$.*

**Proof.** Let $u$ and $v$ denote two arbitrary vertices of $MOT(N)$. There are three cases:

**Case 1($u$ and $v$ are in different trees):** The first possibility is that $u$ and $v$ are in different row trees. Let us assume that $u$ is in $i$-th row tree and $v$ is in $k$-th row tree. Suppose $i$ is odd. Then the root $r_i$ of $i$-th row tree is a member of $L$. Since $u$ is in the $i$-th row tree, $d(u, r_i) < logN$ [10]. In the same way, since $v$ lies outside the $i$-th row tree, $d(v, r_i) > logN$ [10]. Thus $d(u, r_i) \neq d(v, r_i)$.

Suppose $k$ is odd. Then $d(u, r_k) \neq d(v, r_k)$ where $r_k$ is a member of $L$.

Suppose both $i$ and $k$ are even. Then the root $r_{i-1}$ of $(i-1)$-th row tree and root $r_{k-1}$ of $(k-1)$-th row tree are members of $L$. Then if $d(u, r_{i-1}) = d(v, r_{i-1})$, then $d(u, r_{k-1}) \neq d(v, r_{k-1})$. See Figure 4. The proof is the same if $u$ and $v$ are in different column trees. The proof is similar if $u$ is in a row tree and $v$ is in a column tree.

**Case 2 ($u$ and $v$ are in the same tree but at different levels):** Let us assume that $u$ is at $s$-th level in $i$-th row tree and $v$ is at $t$-th level in the

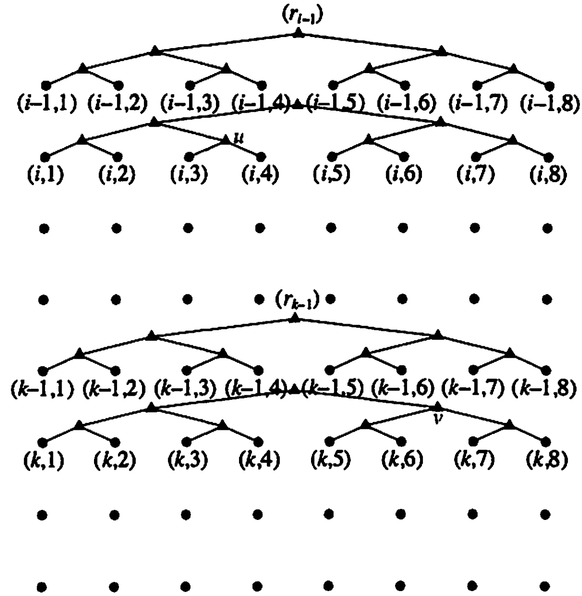


Figure 4: Vertex $u$ lies in $i$-th row tree and $v$ lies in $k$-th row tree. Both are even row trees.

same row tree. If $i$ is odd, the root $r_i$ of $i$-th row tree is a member of $L$. Now $d(u, r_i) = s$ and $d(v, r_i) = t$. Since $s \neq t$, $d(u, r_i) \neq d(v, r_i)$.

Suppose $i$ is even. Then $r_i$ is not a member of $L$. Consider the roots $c_1$ and $c_{N-1}$ which are the roots of 1-st column tree and $(N-1)$-th column tree respectively. See Figure 5. Also, $c_1$ and $c_{N-1}$ are members of $L$. If $d(u, c_1) = d(v, c_1)$, then $d(u, c_{N-1}) \neq d(v, c_{N-1})$ because $u$ and $v$ are at different levels of $i$-th row tree.

The proof is the same if $u$ and $v$ are at different levels of the same column tree.

**Case 3 ($u$ and $v$ are at the same level of the same tree):** Suppose $u$ and $v$ are at the same level of $i$-th row tree. The common ancestor of $u$ and $v$ is denoted as $a$ and the subtree rooted at vertex $a$ is $T(a)$. The leftmost leaf of $T(a)$ is denoted as $\ell$. See Figure 6. It is easy to observe that $\ell$-th column tree is odd. Thus, the root $c_\ell$ of $\ell$-th column tree is a member of $L$. Since $u$ and $v$ are at the same level of $i$-th row tree, $d(u, c_\ell) \neq d(v, c_\ell)$. See Figure 7.

The proof is the same if $u$ and $v$ are in the same column tree. □

In the above theorem, the cardinality of $L$ is $N$. Thus an upper bound of $L(MOT(N)$ is $N$. We have just shown that the location number of an $N \times N$ mesh of trees lies between $N/2$ and $N$. In the following section, we shall prove that the location number $N \times N$ mesh of trees is neither $N/2$ nor $N$. We exhibit

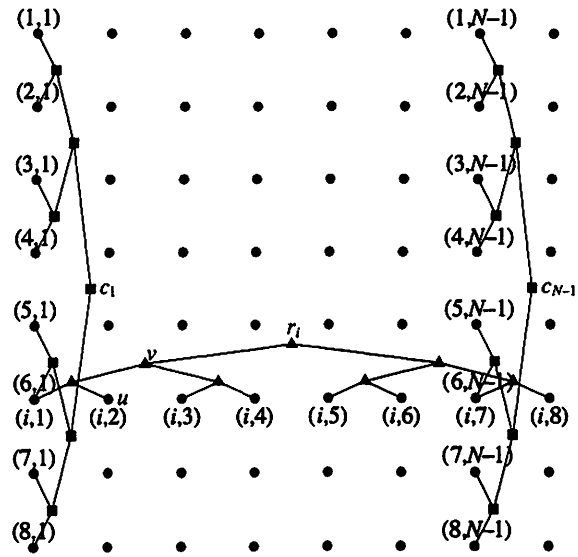


Figure 5: Vertices $u$ and $v$ are in $i$-th row tree but at different levels where $i$ is even.

this by demonstrating that $L(MOT(4)) = 3$ where $N/2 = 2$ and $N = 4$.

## 4.2 $4 \times 4$ Mesh of Trees

If the degree of one vertex is even and that of the other is odd then the vertices are said to be of *different parity.* Otherwise they are said to be of the *same parity.* The distance between two vertices of different parity is odd and that between vertices of the same parity is even.

Two vertices are said to be *diametrically opposite* to each other if the distance between them is equal to the diameter of the graph. Diametrically opposite vertices in $MOT(N)$ will be of the same parity.

**Notation 2** *Given two vertices $u$ and $v$ in $MOT(N)$, let $\lambda(C)$ be the diameter of a shortest cycle $C$ containing $u$ and $v$. Let $d(u,v) = k < \lambda(C)$ and let $P(u,v)$ denote the shortest path on $C$ between them. Let $Q(u,v)$ denote the $(u,v)$- section of $C$ whose length is $> \lambda(C)$. Moreover, $|P(u,v)|$ and $|Q(u,v)|$ denote the length of the path $P(u,v)$ and $Q(u,v)$ respectively.*

In this section we exhibit that the location number of $MOT(4)$ is 3. To show that this number is the minimum, we need to show that $L(MOT(4)) > 2$. A brute force method of showing no 2-subset of $V$ is a locating set is not feasible as there are $40C_2 = 780$ pairs. We provide a rigorous and an elegant proof to show that $L(MOT(4) > 2$.

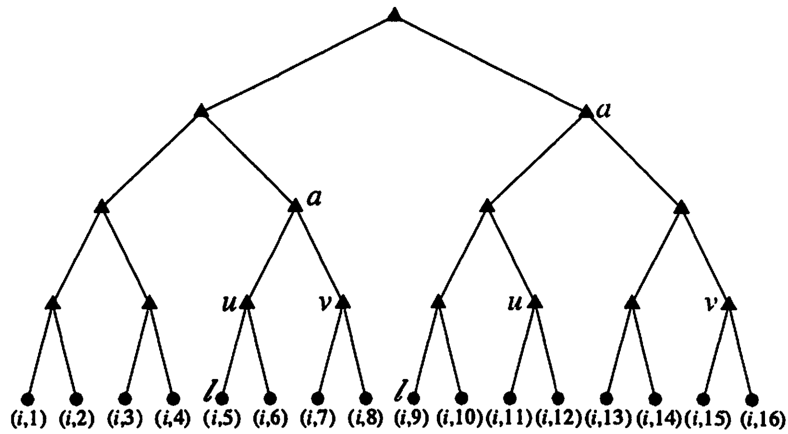


Figure 6: The common ancestor of $u$ and $v$ is denoted as $a$ and the subtree rooted at $a$ is $T(a)$. The leftmost leaf of $T(a)$ is denoted as $\ell$. There are two different examples in the figure

**Proposition 1** *Given two vertices $u$ and $v$ in $MOT(N)$, let $C$ be the shortest cycle containing $u$ and $v$. Let $P(u,v)$ denote the shortest path on $C$ between $u$ and $v$ and $Q(u,v)$ denote the longest path on $C$ between $u$ and $v$. Then $|Q(u,v)| \neq |P(u,v)| + 1$.*

**Proof.** If it were so, then $|P(u,v)| + |Q(u,v)| = 2|P(u,v)| + 1$, which is not possible since $|P(u,v)|+|Q(u,v)|$ is even. Consequently $|Q(u,v)| \geq |P(u,v)|+2$. □

**Observation 1** *If $u$ and $v$ are of different parity and $d(u,v) = k$, then $k \leq \lambda(C) - 1$ and hence $k' \geq \lambda(C) + 1$.*

**Observation 2** *If $u$ and $v$ are of the same parity and $d(u,v) = k$, then $k \leq \lambda(C) - 2$ and hence $k' \geq \lambda(C) + 2$.*

**Lemma 9** $L(MOT(4)) > 2$.

**Proof.** Suppose that $\{u,v\}$ is a locating set for $G$. Then there is a unique shortest path $P$ between $u$ and $v$ [9]. Since $MOT(N)$ is 2-connected, $u$ and $v$ will lie on a common cycle $C$. Choose $C$ so that the length of $C$ is minimum. The uniqueness of $P$ implies that $u$ and $v$ are not diametrically opposite in $C$ and hence $d(u,v) = k < \lambda(C)$.

**Case 1:** The vertices $u$ and $v$ are of different parity. Without loss of generality let $d(u) = 2$ and $d(v) = 3$. Let $v_1, v_2$ be vertices adjacent to $v$, but not lying on $P$. Let $v_1 \in V(C)$. Then

$$d(v, v_1) = d(v, v_2) = 1 \qquad (1)$$

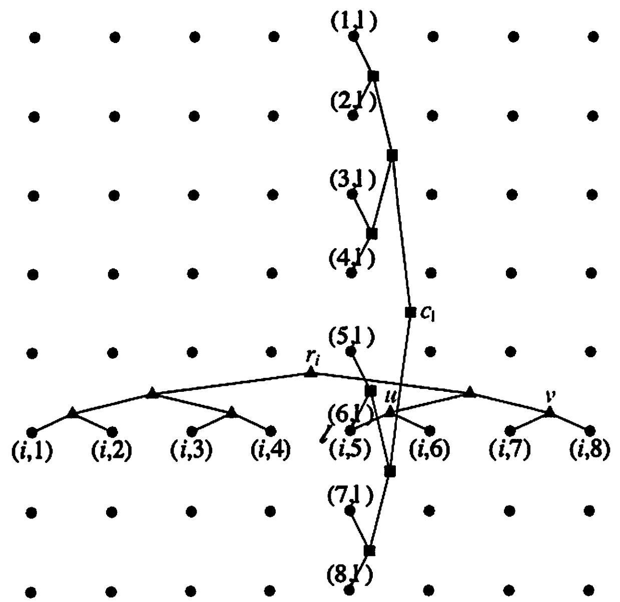


Figure 7: Vertices $u$ and $v$ are at the same level of $i$-th row tree.

Also $d_Q(u, v) = k' \geq \lambda(C) + 1$ by Observation 1. Hence $d_Q(u, v_1) \geq \lambda(C) > k$. This means that $d_Q(u, v_1) \geq k + 1$. Since $d_P(u, v) = k$ and the path $P$ followed by the edge $vv_1$ is of length $k + 1$, we have $d(u, v_1) = k + 1$. Now, if $d(u, v_2) < k - 1$, then there is a $(u, v)$-path of length $< k$, a contradiction. If $d(u, v_2) = k - 1$, then there are two distinct $(u, v)$- paths, again a contradiction. If $d(u, v_2) = k$, a cycle of odd length will be formed. Hence $d(u, v_2) \geq k + 1$. Since $P$ followed by the edge $vv_2$ is of length $k+1$, we must have $d(u, v_2) = k+1$. Thus

$$d(u, v_1) = d(u, v_2) = k + 1 \tag{2}$$

**Case 2:** The vertices $u$ and $v$ are of same parity. Then $d(u, v) = k \leq \lambda(C) - 2$ and $k' = d_Q(u, v) \geq \lambda(C) + 2$.

**Case 2*a*:** Let $d(u) = d(v) = 3$. Then proceeding as in Case 1, the vertices $v_1$ and $v_2$ adjacent to $v$ are equidistant from both $u$ and $v$.

**Case 2*b*:** Let $d(u) = d(v) = 2$. Let $w$ be adjacent to $v$ on $C$, not lying on $P$. Let $w_1$ and $w_2$ be adjacent to $w$ with $w_1 \in V(C)$ and $w_2 \notin V(C)$. Now $u$ and $w$ are of different parity. We observe that $d(u, w) \not< k - 1$, for otherwise $d(u, v)$ would be less than $k$, a contradiction. Also $d(u, w) \neq k-1$, for otherwise there would be two shortest paths between $u$ and $v$, a contradiction. Moreover $d(u, w) \neq k$. If it were the case, an odd cycle would be formed which is not possible. Hence $d(u, w) = k + 1$. Now $k \leq \lambda(C) - 2$. This means that $d(u, w) = k + 1 \leq \lambda(C) - 1 < \lambda(C)$. As $u$ and $w$ are of different parity, by Case

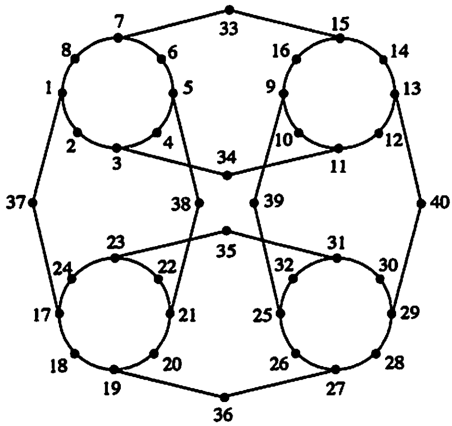


Figure 8: A 4 × 4 Mesh of trees

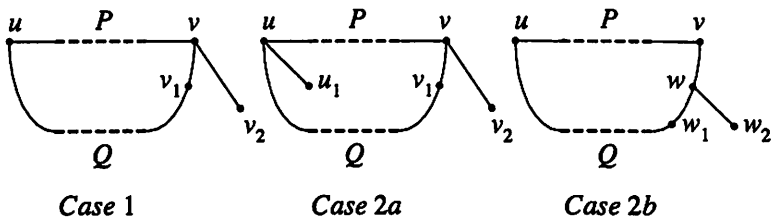


Figure 9: Possible Cases

1, we have that $w_1$ and $w_2$ are equidistant from $u$ and $w$. That is

$$d(u, w_1) = d(u, w_2) \tag{3}$$

and $d(w, w_1) = d(w, w_2) = 1$. Since $G$ has no cycles of length 3, we have

$$d(v, w_1) = d(v, w_2) = 2 \tag{4}$$

The possible cases are illustrated in Figure 9.

The equations (1), (2), (3), (4) contradict our assumption. □

A simple calculation shows that {8, 14, 35} shown in Figure 8 is a locating set of $G$. Thus we have the following theorem.

**Theorem 4** $L(MOT(4)) = 3$. □

## 5 Conclusion

In this paper, we have provided bounds for the location number of $X$-trees, double rooted $X$-trees and mesh of trees. We have proved that the lower bound of $N \times N$ mesh of trees is $N/2$ and its upper bound is $N$. Finding the exact value of location number of mesh of trees is still a challenging mathematical problem. It would be interesting to consider architectures derived from ternary trees and $k$-ary trees. □

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