FURTHER RESULTS ON SUPER EDGE-MAGIC DEFICIENCY OF GRAPHS

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Abstract

Acharya and Hegde have introduced the notion of strongly k-indexable graphs: A (p, q)-graph G is said to be ***strongly k-indexable*** if its vertices can be assigned distinct integers $0, 1, 2, ..., p-1$ so that the values of the edges, obtained as the sums of the numbers assigned to their end vertices can be arranged as an arithmetic progression $k, k+1, k+2, ..., k+(q-1)$. Such an assignment is called a strongly k-indexable labeling of G. Figueroa-Centeno et.al, have introduced the concept of super edge-magic deficiency of graphs: Super edge-magic deficiency of a graph G is the minimum number of isolated vertices added to G so that the resulting graph is super edge-magic. They conjectured that the super edge-magic deficiency of the complete bipartite graph $K_{m,n}$ is $(m-1)(n-1)$ and proved it for the case $m = 2$. In this paper we prove that the conjecture is true for $m = 3$, 4 and 5, using the concept of strongly k-indexable labelings[1].

1 Introduction

For all terminology and notation in graph theory we follow Harary [6] and West [7].

Graph labelings, where the *vertices* and *edges* are assigned *real values* or *subsets of a set* are subject to certain conditions, have often been motivated by their utility to various applied fields and their intrinsic mathematical interest (logico-mathematical). An enormous body of literature has grown around the subject, especially in the last forty years or so, and is still getting embellished due to increasing number of application driven concepts [5].

Acharya and Hegde [1, 2] have introduced the concept of strongly k-indexable graphs.

Given a graph $G = (V, E)$, the set $\mathcal{N}$ of nonnegative integers, a finite subset $\mathcal{A}$ of $\mathcal{N}$ and a commutative binary operation $+ : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$, every vertex function $f : V(G) \to \mathcal{A}$ induces an edge function $f^+ : E(G) \to \mathcal{N}$ such that $f^+(uv) = f(u) + f(v), \forall\ uv \in E(G)$. Such vertex functions are called ***additive vertex functions***. An **additive labeling** of a graph G is an injective additive vertex function f such that the induced edge function f^+ is injective.

[1]Key Words: Strongly k-indexable graphs, Super edge-magic deficiency of graphs

For the given (p, q)-graph $G = (V, E)$.

1. $f(V) = \{f(u) : u \in V(G)\}$.
2. $f^+(E) = \{f^+(e) : e \in E(G)\}$.

Definition 1.1 *An additive labeling $f : V(G) \to \{0, 1, 2, ..., p-1\}$ of a (p, q)-graph G with $f^+(E) = \{k,\ k+d,\ ...,\ k+(q-1)d\}$ is called **strongly $(k,\ d)$-indexable labeling** of G.*

Definition 1.2 *A strongly (k, d)-indexable labeling of a (p, q) graph G with $d = 1$ is called a **strongly k-indexable labeling**. A graph which admits such a labeling for at least one value of k is called strongly k-indexable graph.*

Enomoto et.al.,[3] have introduced the concept of super edge-magic graph.

Definition 1.3 *A graph G is said to be super edge-magic if it admits a bijection $f : V \cup E \to \{1, 2, ..., p+q\}$ with $f(V) = \{1, 2, ..., p\}$ and $f(E) = \{p+1, p+2, ..., p+q\}$ such that $f(u) + f(v) + f(uv) = c(f)$, $uv \in V$ where $c(f)$ is a constant.*

From the above definition one can see that a graph is super edge-magic if and only if it is strongly k-indexable for some k.

R. M. Figueroa-Centenoa et.al.,[4] have introduced the concept of super edge-magic deficiency of graphs.

Definition 1.4 *The super edge-magic deficiency of a graph G is the minimum number of isolated vertices added to G so that the resulting graph is super edge-magic and is denoted by $\mu_s(G)$.*

From the above definitions one can see that $0 \le \mu_s(G) \le \infty$.

Since a graph is super edge-magic if and only if it is strongly k-indexable, super edge-magic deficiency can be equivalently defined as the minimum number of isolated vertices added to a graph G so that the resulting graph is strongly k-indexable for some k. For the sake of convenience we call this parameter as **vertex dependent characteristic** and denote it by $d_c(G)$. Figueroa-Centenoa et.al.,[4] have proved that

Theorem 1.5 : *The vertex dependent characteristic of the complete bipartite graph $K_{m,n}$ is at most $(m-1)(n-1)$.*

They conjectured that

Conjecture 1.6 : *The vertex dependent characteristic of the complete bipartite graph $K_{m,n}$ is equal to $(m-1)(n-1)$.*

Also, they proved that

Theorem 1.7 *The vertex dependent characteristic of the complete bipartite graph $K_{2,n}$ is $(n-1)$.*

2 Results

In this section we prove the above mentioned conjecture for $m = 3$, 4 and 5, using the concept of strongly k-indexable labelings.

Theorem 2.1 : *The vertex dependent characteristic of the complete bipartite graph $K_{3,n}$ is $2(n-1)$.*

Proof: From Theorem 1.5, clearly

$$d_c(K_{3,\,n}) \leq 2(n-1). \tag{1}$$

From Theorem 1.7, $d_c(K_{2,3})= 2$.
Suppose $d_c(K_{3,\,n}) < 2(n-1)$ for some integer $n \geq 3$. Then there exists a strongly k-indexable labeling $f : V(K_{3,\,n} \cup (2n-2-j)K_1) \to \{0,\ 1, ..., 3n-j\}$ for some integer $j \geq 1$ such that

$$f^+(K_{3,\,n}) = f^+(K_{3,\,n} \cup (2n-2-j)K_1) = \{k,\ k+1, ..., k+3n-1\}.$$

Let $A = \{x_i : x_i \in V(K_{3,n}),\ \deg(x_i) = n \text{ and } f(x_i) < f(x_{i+1}),\ i = 1,\ 2\}$.

$B = \{y_i : y_i \in V(K_{3,\,n}),\ \deg(y_i) = 3 \text{ and } f(y_i) < f(y_{i+1}); 1 \leq i \leq n-1\}$.

$C = \{z_i : z_i \in V((2n-2-j)K_1),\ \deg(z_i) = 0,\ 1 \leq i \leq 2n-2-j\}$.
Let $f(x_1) = a$ then $f(x_2) = a+b$ and $f(x_3) = a+b+c$ where b, c are positive integers.
Consider the following mutually exclusive subsets of $f^+(K_{3,n})$:

$$\begin{aligned}
A_1 &= \{a+f(y_1),\ a+b+f(y_1),\ a+b+c+f(y_1)\}\\
A_2 &= \{a+f(y_2),\ a+b+f(y_2),\ a+b+c+f(y_2)\}\\
A_3 &= \{a+f(y_3),\ a+b+f(y_3),\ a+b+c+f(y_3)\}\\
&\quad \cdots \qquad\qquad \cdots \qquad\qquad \cdots\\
A_n &= \{a+f(y_n),\ a+b+f(y_n),\ a+b+c+f(y_n)\}
\end{aligned} \tag{2}$$

Since f is strongly k-indexable,

$$f^+(K_{3,\,n}) = A_1 \cup A_2 \cup A_3 \cup ... \cup A_n.$$

Therefore $a+f(y_1) = k$ and $a+b+c+f(y_n) = k+3n-1$. There are $(b - 1)$ edge values between each $a+f(y_i)$ and $a+b+f(y_i)$, $1 \leq i \leq n$ in $f^+(K_{3,\,n})$ and $(c - 1)$ edge values between each $a+b+f(y_i)$ and $a+b+c+f(y_i)$, $1 \leq i \leq n$ in $f^+(K_{3,\,n})$. As there are only $3n$ elements in $f^+(K_{3,\,n})$, we must have $(b-1)n+(c-1)n+2 \leq 3n$ which implies

$$(b-1)n + (c-1)n \leq 3n-2 < 3n \Rightarrow b+c < 5.$$

Therefore possible values of b and c are one among the following.
(1) $b = 1$ and $c = 3$.
(2) $b = 3$ and $c = 1$.
(3) $b = 1$ and $c = 2$.
(4) $b = 2$ and $c = 1$.
(5) $b = 2$ and $c = 2$.
(6) $b = 1$ and $c = 1$.

Case 1: $b = 1$ and $c = 3$.

$$\text{From (2)}, weget$$

$$A_1 = \{a + f(y_1),\ a + 1 + f(y_1),\ a + 4 + f(y_1)\}$$
$$A_2 = \{a + f(y_2),\ a + 1 + f(y_2),\ a + 4 + f(y_2)\}$$
$$A_3 = \{a + f(y_3),\ a + 1 + f(y_3),\ a + 4 + f(y_3)\}$$
$$\dots \qquad \dots \qquad \dots$$
$$A_n = \{a + f(y_n),\ a + 1 + f(y_n),\ a + 4 + f(y_n)\}.$$

One can observe that, the increasing order of edge values of $K_{3,\,n}$ are

$$a + f(y_1),\ a + 1 + f(y_1),\ a + f(y_2),\ a + 1 + f(y_2),$$
$$a + 4 + f(y_1),\ a + f(y_3),\ \dots, a + 4 + f(y_3).$$

From this increasing order we get,

$$f(y_2) = 2 + f(y_1) \text{ and } f(y_3) = 5 + f(y_1).$$

But then

$$\begin{aligned} f(x_2) + f(y_3) &= a + 1 + 5 + f(y_1) \\ &= a + 6 + f(y_1) \\ &= f(x_3) + f(y_2) - \text{a contradiction (because } f^+ \text{ is injective).} \end{aligned}$$

Case 2: $b = 3$ and $c = 1$.
By similar arguments as in Case **1**, we get a contradiction.

Case 3: $b = 1$ and $c = 2$.

$$\text{From (2)}, weget$$

$$A_1 = \{a + f(y_1),\ a + 1 + f(y_1),\ a + 3 + f(y_1)\}$$
$$A_2 = \{a + f(y_2),\ a + 1 + f(y_2),\ a + 3 + f(y_2)\}$$
$$A_3 = \{a + f(y_3),\ a + 1 + f(y_3),\ a + 3 + f(y_3)\}$$
$$\dots \qquad \dots \qquad \dots$$
$$A_n = \{a + f(y_n),\ a + 1 + f(y_n),\ a + 3 + f(y_n)\}.$$

One can easily observe that

$$\begin{aligned} f(x_2) + f(y_2) &= a + 1 + f(y_2) \\ &= a + 3 + f(y_1) \\ &= f(x_3) + f(y_1) - \text{a contradiction.} \end{aligned}$$

Case 4: $b = 2$ and $c = 1$.
By similar arguments as in Case 3, we get a contradiction.

Case 5: $b = 1$ and $c = 1$.
If $b = c = 1$ and then

$$\begin{aligned} k &= a + f(y_1) \\ k + 1 &= a + 1 + f(y_1) \\ k + 2 &= a + 2 + f(y_1) \\ &\dots \qquad \dots \\ k + 3n - 1 &= a + 2 + f(y_n). \end{aligned} \tag{3}$$

From (3), we get

$$\begin{aligned} f(y_2) &= 3 + f(y_1) \\ f(y_3) &= 6 + f(y_1) \\ &\dots \qquad \dots \\ f(y_n) &= 3(n-1) + f(y_1). \end{aligned} \tag{4}$$

Hence

$$\begin{aligned} f(y_n) &= 3n - 3 + f(y_1)) \\ &\leq 3n - j \ (\because 3n - j \text{ is the maximum vertex value}) \\ \Rightarrow f(y_1) &\leq 3 - j. \end{aligned}$$

But $f(y_1) \geq 0 \Rightarrow 3 - j \geq 0 \Rightarrow j \in \{1, 2, 3\}$.
Note that

$f(A) = \{a, a+1, a+2\}$.
$f(B) = \{f(y_1), f(y_1) + 3, f(y_1) + 6, ..., f(y_1) + 3(n-1)\}$. (From (4)), *weget*
$f(C) = \{f(z_1), f(z_2), f(z_3), ..., f(z_{2n-2-j})\}$.

Let $F = \{f(y_i) + i : 1 \leq i \leq 3n - 3\} \backslash f(B)$. Clearly $F \subseteq f(K_{3,n} \cup (2n - 2 - j)K_1)$ and F contains $2(n$-$1)$ vertex values.

Sub Case 5.1: $j = 1$.
Then $f(C)$ contains $2n - 3$ vertex values and therefore one element of F must be

in $f(A)$.
Let $f(y_1)+3s-5 \in f(A)$ for some integer s, $2 \le s \le n$. Then

$$a = f(y_1)+3s-5 \quad \Rightarrow a+2 \in f(B) \quad \text{–a contradiction.}$$
$$a+1 = f(y_1)+3s-5 \quad \Rightarrow a \in f(B) \quad \text{–a contradiction.}$$
$$a+2 = f(y_1)+3s-5 \quad \Rightarrow a+1 \in f(B) \quad \text{–a contradiction.}$$

Let $f(y_1)+3r-4 \in f(A)$ for some integer s, $2 \le r \le n$. Then

$$a = f(y_1)+3r-4 \quad \Rightarrow a+1 \in f(B) \quad \text{–a contradiction.}$$
$$a+1 = f(y_1)+3r-4 \quad \Rightarrow a+2 \in f(B) \quad \text{–a contradiction.}$$
$$a+2 = f(y_1)+3r-4 \quad \Rightarrow a \in f(B) \quad \text{–a contradiction.}$$

Therefore $j \neq 1$.

Sub Case 5.2: $j = 2$.
Then $f(C)$ contains $2n-4$ vertex values and therefore two elements of F must be in $f(A)$.
Let $f(y_1)+3t-2$, $f(y_1)+3t-4 \in f(A)$ for some integer t, $1 \le t \le n$. Then, $a+1 = f(y_1)+3t-3 = f(y_1)+3(t-1) \in f(B)$-a contradiction.
Let $f(y_1)+3m-5$, $f(y_1)+3m-4 \in f(A)$ for some integer m, $1 \le m \le n$. Since these two values are consecutive, either $a \in f(B)$ or $a+2 \in f(B)$-a contradiction. Therefore $j \neq 2$.

Sub Case 5.3: $j = 3$.
Then $f(C)$ contains $2n-5$ elements and therefore three elements of F must be in $f(A)$, which is impossible since the elements of $f(A)$ are consecutive. Clearly $j \neq 3$. Thus for $j \ge 1$, $(K_{3,n}) \cup (2n-2-j)K_1$ is not strongly k-indexable.

Case 6: $b = 2$ and $c = 2$.

$$\text{From (2)}, weget$$
$$A_1 = \{a+f(y_1),\ a+2+f(y_1),\ a+4+f(y_1)\}$$
$$A_2 = \{a+f(y_2),\ a+2+f(y_2),\ a+4+f(y_2)\}$$
$$\dots \quad \dots \quad \dots$$
$$A_n = \{a+f(y_n),\ a+2+f(y_n),\ a+4+f(y_n)\}.$$

Then the increasing order of edge values of $K_{3,n}$ are

$$a+f(y_1),\ a+f(y_2),\ a+2+f(y_1),\ a+2+f(y_2),$$
$$a+4+f(y_1),\ a+4+f(y_2),\ a+f(y_3),\ \dots, a+4+f(y_n)$$
$$\Longrightarrow f(y_2) = 1+f(y_1), f(y_3) = 6+f(y_1) \text{ and } f(y_4) = 7+f(y_1).$$

If n is odd, that is $n = 2r + 1$ then there are $4r$ vertex labels which are not used between $f(y_1)$ and $f(y_{2r+1})$. Therefore $2n - 2 - j = 4r - j \geq 4r \Longrightarrow j \leq 0$ -a contradiction to $j \geq 1$.
If n is even then,

$$f(y_n) = 3n - 5 + f(y_1), f(y_{n-1}) = f(y_n) - 1.$$
$$k = a + f(y_1), k + 3n - 1 = a + 4 + f(y_n)$$
$$\Longrightarrow f(y_n) = k + 3n - 5 - a$$
$$\Longrightarrow f(y_n) = k + 3n - 5 - (k - f(y_1))$$
$$\Longrightarrow f(y_n) = 3n - 5 + f(y_1) \leq 3n - j$$
$$\Longrightarrow j \in \{1, 2, 3, 4, 5\}.$$

Threrefore

$$f(A) = \{a,\ a+2,\ a+4\}.$$
$$f(B) = \{f(y_1),\ f(y_1)+1,\ f(y_1)+6,\ f(y_1)+7, ..., f(y_1)+3n-5\}.$$
$$f(C) = \{f(z_1),\ f(z_2),\ f(z_3), ..., f(z_{2n-2-j})\}.$$

Again, let $R = \{f(y_1)+2,\ f(y_1)+3,\ f(y_1)+4,\ f(y_1)+5,\ f(y_1)+8, ...,\}$.
Clearly $R \subseteq f(K_{3,\,n} \cup (2n-2-j)K_1)$ and R contains ($2n$-4) vertex values and $R \cap f(B) = \phi$. Similar to the arguments used for Sub Cases (5.1), (5.2) and (5.3) we can show that $j \neq 1, 2, 3, 4, 5$. Hence from (1), we get $d_c(K_{3,\,n}) = 2(n-1)$. ◇

Theorem 2.2 : *The vertex dependent characteristic of the complete bipartite graph* $K_{4,\,n}$ *is* $3(n-1)$.

Proof: From Theorem 1.5, clearly

$$d_c(K_{4,\,n}) \leq 3(n-1). \tag{5}$$

From Theorems 1.7 and 2.1, we get $d_c(K_{2,\,4}) = 3$ and $d_c(K_{3,\,4}) = 6$. Assume that $d_c(K_{4,\,n}) < 3(n-1)$ for some integer $n \geq 4$. Then there exists a strongly k-indexable labeling $f : V(K_{4,\,n} \cup (3n-3-j)K_1) \to \{0,\ 1, ..., 4n-j\}$ for some integer $j \geq 1$ such that

$$f^{+}(K_{4,\,n}) = f^{+}(K_{4,\,n} \cup (3n-3-j)K_1) = \{k,\ k+1, ..., k+4n-1\}.$$

Let $A = \{x_i : x_i \in V(K_{4,\,n}),\ \deg(x_i) = n \text{ and } f(x_i) < f(x_{i+1}),\ i = 1,\ 2,\ 3\}$.

$B = \{y_i : y_i \in V(K_{4,\,n}),\ \deg(y_i) = 4 \text{ and } f(y_i) < f(y_{i+1}); 1 \leq i \leq n-1\}$.

$C = \{z_i : z_i \in V((3n-3-j)K_1),\ \deg(z_i) = 0,\ 1 \leq i \leq 3n-3-j\}$.

Let $f(x_1) = a$ then $f(x_2) = a+b$, $f(x_3) = a+b+c$ and $f(x_3) = a+b+c+d$ where b, c, d are positive integers.
Similar to previous theorems consider the mutually exclusive subsets of $f^+(K_{4,\,n})$:

$$\begin{aligned}
A_1 &= \{a+f(y_1),\ a+b+f(y_1),\ a+b+c+f(y_1),\ a+b+c+d+f(y_1)\}\\
A_2 &= \{a+f(y_2),\ a+b+f(y_2),\ a+b+c+f(y_2),\ a+b+c+d+f(y_2)\}\\
A_3 &= \{a+f(y_3),\ a+b+f(y_3),\ a+b+c+f(y_3),\ a+b+c+d+f(y_3)\}\\
&\ \ \cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots\\
A_n &= \{a+f(y_n),\ a+b+f(y_n),\ a+b+c+f(y_n),\ a+b+c+d+f(y_n)\}.
\end{aligned} \tag{6}$$

There are (b - 1), (c - 1) and (d - 1) distinct edge values between each $a+f(y_i)$ and $a+b+f(y_i)$, $a+b+f(y_i)$ and $a+b+c+f(y_i)$ and $a+b+c+f(y_i)$ and $a+b+c+d+f(y_i)$, $1 \leq i \leq n$ in $f^+(K_{4,\,n})$ respectively. As there are only $4n$ elements in $f^+(K_{4,\,n})$, we must have $(b-1)n+(c-1)n+(d-1)n+2 \leq 4n$. Therefore we get $b+c+d < 7$.
There are many possible values of b, c and d such that $b+c+d<7$. It is enough to consider the following seven cases since the remaining cases follow by similar arguments.
(1) $b=1$, $c=1$ and $d=2$.
(2) $b=1$, $c=1$ and $d=3$.
(3) $b=1$, $c=1$ and $d=4$.
(4) $b=2$, $c=1$ and $d=2$.
(5) $b=2$, $c=1$ and $d=3$.
(6) $b=1$, $c=1$ and $d=1$.
(7) $b=2$, $c=2$ and $d=2$.

Case 1: $b=1$, $c=1$ and $d=2$.
In this case, note that $f(y_2) = 3+f(y_1)$ and therefore we get

$$f(x_4)+f(y_1) \ = f(x_2)+f(y_2) - \text{a contradiction (because } f^+ \text{ is injective).}$$

Case 2: $b=1$, $c=1$ and $d=3$.
In this case also, note that $f(y_2) = 3+f(y_1)$ and therefore we get

$$f(x_4)+f(y_1) \ = f(x_3)+f(y_2) - \text{a contradiction.}$$

Case 3: $b=1$, $c=1$ and $d=4$.
Similarly, in this case $f(y_3) = 4+f(y_2)$. Therefore,

$$f(x_3)+f(y_3) \ = f(x_4)+f(y_2) - \text{a contradiction.}$$

Case 4: $b=2$, $c=1$ and $d=2$.
Note that $f(y_2) = 1+f(y_1)$

$$f(x_3)+f(y_1) \ = f(x_2)+f(y_2) - \text{a contradiction.}$$

Case 5: $b = 2$, $c = 1$ **and** $d = 3$.
Note that in this case also $f(y_2) = 1 + f(y_1)$

$$f(x_3) + f(y_1) \quad = f(x_2) + f(y_2) - \text{a contradiction.}$$

Case 6: $b = 1$, $c = 1$ **and** $d = 1$. and
Case 7: $b = 2$, $c = 2$ **and** $d = 2$. also arrive at contradiction using analogous arguments of Theorem 2.1 Case-5 and Case-6. Therefore from all these seven cases, clearly $j \ngeq 1$. Hence from (5) $d_c(K_{4,\,n}) = 3(n-1)$. ◇

Theorem 2.3 . *The vertex dependent characteristic of a complete bipartite graph* $K_{5,n}$ *is* $4(n-1)$.

Proof. Consider the complete bipartite graph $K_{5,n}$. From Theorem 1.5, we have

$$d_c(K_{5,n}) \leq 4(n-1) \tag{7}$$

Also, we see that $d_c(K_{2,5}) = 4$, $d_c(K_{3,5}) = 8$ and $d_c(K_{4,5}) = 12$. Assume that $d_c(K_{5,n}) < 4(n-1)$ for some positive integer $n \geq 5$. Then, there exists a strongly k-indexable labeling $f : V(K_{5,n} \cup (4n-4-j)K_1) \to \{0, 1, 2, \ldots, 5n-j\}$ for some positive integer $j \geq 1$ such that $f^+(K_{5,n}) = f^+(K_{5,n} \cup (4n-4-j)K_1) = \{k, k+1, \ldots, k+5n-1\}$.
Let $A = \{x_i : x_i \in V(K_{5,n}), deg(x_i) = n, f(x_i) < f(x_{i+1}), i = 1, 2, 3, 4\}$

$$B = \{y_i : y_i \in V(K_{5,n}), deg(y_i) = 5, f(y_i) < f(y_{i+1}), 1 \leq i \leq n-1\}$$

$$C = \{z_i : z_i \in V((4n-4-j)K_1), deg(z_i) = 0, 1 \leq i \leq 4n-4-j\}.$$

Let $f(x_1) = a$, then $f(x_2) = a+b$, $f(x_3) = a+b+c$, $f(x_4) = a+b+c+d$ and $f(x_5) = a+b+c+d+e$, where b, c, d, e are positive integers. Consider the following mutually exclusive subsets of $f^+(K_{5,n})$:

$$A_1 = \{a+f(y_1), a+b+f(y_1), a+b+c+f(y_1), a+b+c+d+f(y_1), a+b+c+d+e+f(y_1)\}$$

$$A_2 = \{a+f(y_2), a+b+f(y_2), a+b+c+f(y_2), a+b+c+d+f(y_2), a+b+c+d+e+f(y_2)\}$$

$$A_3 = \{a+f(y_3), a+b+f(y_3), a+b+c+f(y_3), a+b+c+d+f(y_3), a+b+c+d+e+f(y_3)\}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$A_n = \{a+f(y_n), a+b+f(y_n), a+b+c+f(y_n), a+b+c+d+f(y_n), a+b+c+d+e+f(y_n)\}. \tag{8}$$

Since f is strongly k-indexable, $f^+(K_{5,n}) = A_1 \cup A_2 \cup \cdots \cup A_n$.

Therefore, $a + f(y_1) = k$ and $a+b+c+d+e+f(y_n) = k+5n-1$. Note that there are $(b-1)$ edge values between $a + f(y_i)$ and $a+b+f(y_i), 1 \leq i \leq n$, $(c-1)$ edge values between $a+b+f(y_i)$ and $a+b+c+f(y_i), 1 \leq i \leq n$, $(d-1)$ edge values between $a+b+c+f(y_i)$ and $a+b+c+d+f(y_i), 1 \leq i \leq n$, $(e-1)$ edge values between $a+b+c+d+f(y_i)$ and $a+b+c+d+e+f(y_i), 1 \leq i \leq n$ in $f^+(K_{5,n})$.

As there are only $5n$ elements in $f^+(K_{5,n})$, we must have $(b-1)n + (c-1)n + (d-1)n + (e-1)n + 2 \leq 5n$, from which we get,

$(b-1)n+(c-1)n+(d-1)n+(e-1)n \leq 5n-2 < 5n$
$\Rightarrow\ b+c+d+e < 9.$

There are many possible values of b, c, d, e such that $b+c+d+e < 9$. It is enough to consider the following twelve cases since the remaining cases follow by similar arguments.

Case 1: b=1, c=1, d=1, e= 5.
From (8), we get

$A_1 = \{a+f(y_1), a+1+f(y_1), a+2+f(y_1), a+3+f(y_1), a+8+f(y_1)\}$

$A_2 = \{a+f(y_2), a+1+f(y_2), a+2+f(y_2), a+3+f(y_2), a+8+f(y_2)\}$

$A_3 = \{a+f(y_3), a+1+f(y_3), a+2+f(y_3), a+3+f(y_3), a+8+f(y_3)\}$

$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$

$A_n = \{a+f(y_n), a+1+f(y_n), a+2+f(y_n), a+3+f(y_n), a+8+f(y_n)\}.$

Then, the increasing order of edge values of $K_{5,n}$ are $a+f(y_1), a+1+f(y_1), a+2+f(y_1), a+3+f(y_1), a+f(y_2), a+1+f(y_2), a+2+f(y_2), a+3+f(y_2), a+8+f(y_1), a+f(y_3), \ldots, a+8+f(y_n)$.

From this increasing order, we get

$a+f(y_2) = a+4+f(y_1)$ and $a+9+f(y_1) = a+f(y_3)$

$\Rightarrow\ f(y_3) = 9+f(y_1)$ and $f(y_2) = 4+f(y_1)$.

But $f(x_4)+f(y_3) = a+3+9+f(y_1)= (a+8)+(4+f(y_1))= f(x_5)+f(y_2)$.
This is a contradiction since f is injective.

Case 2: b=1, c=1, d=1, e= 4.
From (8), we get

$A_1 = \{a+f(y_1), a+1+f(y_1), a+2+f(y_1), a+3+f(y_1), a+7+f(y_1)\}$

$A_2 = \{a+f(y_2), a+1+f(y_2), a+2+f(y_2), a+3+f(y_2), a+7+f(y_2)\}$

$A_3 = \{a+f(y_3), a+1+f(y_3), a+2+f(y_3), a+3+f(y_3), a+7+f(y_3)\}$

$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$

$A_n = \{a+f(y_n), a+1+f(y_n), a+2+f(y_n), a+3+f(y_n), a+7+f(y_n)\}$

Then, one can easily observe that

$a+4+f(y_1) = a+f(y_2)$ and $a+8+f(y_1) = a+f(y_3)$

$\Rightarrow\ f(y_2) = 4+f(y_1)$ and $f(y_3) = 8+f(y_1)$.

But $f(x_4)+f(y_2) = (a+3)+(4+f(y_1) = (a+7)+f(y_1) = f(x_5)+f(y_1))$.

This is again a contradiction.

Case 3: b=1, c=1, d=1, e= 3.
From (8), we get

$A_1 = \{a + f(y_1), a + 1 + f(y_1), a + 2 + f(y_1), a + 3 + f(y_1), a + 6 + f(y_1)\}$

$A_2 = \{a + f(y_2), a + 1 + f(y_2), a + 2 + f(y_2), a + 3 + f(y_2), a + 6 + f(y_2)\}$

$A_3 = \{a + f(y_3), a + 1 + f(y_3), a + 2 + f(y_3), a + 3 + f(y_3), a + 6 + f(y_3)\}$

$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$

$A_n = \{a + f(y_n), a + 1 + f(y_n), a + 2 + f(y_n), a + 3 + f(y_n), a + 6 + f(y_n)\}$.

Then, one can easily observe that

$a + 4 + f(y_1) = a + f(y_2)$ and $a + 7 + f(y_1) = a + f(y_3)$

$\Rightarrow f(y_2) = 4 + f(y_1)$ and $f(y_3) = 7 + f(y_1)$.

But $f(x_3) + f(y_2) = a + 2 + 4 + f(y_1) = (a + 6) + f(y_1) = f(x_5) + f(y_1)$.

This is again a contradiction.

Case 4: b=1, c=1, d=1, e= 2.

From (8), we get

$A_1 = \{a + f(y_1), a + 1 + f(y_1), a + 2 + f(y_1), a + 3 + f(y_1), a + 5 + f(y_1)\}$

$A_2 = \{a + f(y_2), a + 1 + f(y_2), a + 2 + f(y_2), a + 3 + f(y_2), a + 5 + f(y_2)\}$

$A_3 = \{a + f(y_3), a + 1 + f(y_3), a + 2 + f(y_3), a + 3 + f(y_3), a + 5 + f(y_3)\}$

$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$

$A_n = \{a + f(y_n), a + 1 + f(y_n), a + 2 + f(y_n), a + 3 + f(y_n), a + 5 + f(y_n)\}$.

Then, one can easily observe that

$a + 4 + f(y_1) = a + f(y_2)$ and $a + 6 + f(y_1) = a + f(y_3)$

$\Rightarrow f(y_2) = 4 + f(y_1)$ and $f(y_3) = 6 + f(y_1)$.

But $f(x_2) + f(y_2) = a + 1 + 4 + f(y_1) = (a + 5) + f(y_1) = f(x_5) + f(y_1)$.

This is again a contradiction.

Case 5: b=3, c=3, d=1, e= 1.

From (8), we get

$A_1 = \{a + f(y_1), a + 3 + f(y_1), a + 6 + f(y_1), a + 7 + f(y_1), a + 8 + f(y_1)\}$

$A_2 = \{a + f(y_2), a + 3 + f(y_2), a + 6 + f(y_2), a + 7 + f(y_2), a + 8 + f(y_2)\}$

$A_3 = \{a + f(y_3), a + 3 + f(y_3), a + 6 + f(y_3), a + 7 + f(y_3), a + 8 + f(y_3)\}$

$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$

$A_n = \{a + f(y_n), a + 3 + f(y_n), a + 6 + f(y_n), a + 7 + f(y_n), a + 8 + f(y_n)\}$.

Then, one can easily observe that

$a + 4 + f(y_1) = a + f(y_2)$ and $a + 9 + f(y_1) = a + f(y_3)$

$\Rightarrow f(y_2) = 4 + f(y_1)$ and $f(y_3) = 9 + f(y_1)$.

But $f(x_2) + f(y_2) = a + 3 + 4 + f(y_1) = (a + 7) + f(y_1) = f(x_4) + f(y_1)$.

This is again a contradiction.

Case 6: b=2, c=2, d=2, e= 1.
From (8), we get

$$A_1 = \{a + f(y_1), a + 2 + f(y_1), a + 4 + f(y_1), a + 6 + f(y_1), a + 7 + f(y_1)\}$$
$$A_2 = \{a + f(y_2), a + 2 + f(y_2), a + 4 + f(y_2), a + 6 + f(y_2), a + 7 + f(y_2)\}$$
$$A_3 = \{a + f(y_3), a + 2 + f(y_3), a + 4 + f(y_3), a + 6 + f(y_3), a + 7 + f(y_3)\}$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$
$$A_n = \{a + f(y_n), a + 2 + f(y_n), a + 4 + f(y_n), a + 6 + f(y_n), a + 7 + f(y_n)\}.$$

Then, one can easily observe that
$a + f(y_2) = a + 3 + f(y_1)$ and $a + f(y_3) = a + 8 + f(y_1)$

$\Rightarrow f(y_2) = 3 + f(y_1)$ and $f(y_3) = 8 + f(y_1)$.

But $f(x_3) + f(y_2) = a + 4 + 3 + f(y_2) = (a + 7) + f(y_1) = f(x_5) + f(y_1)$.

This is again a contradiction.

Case 7: b=2, c=2, d=1, e= 1.
From (8), we get

$$A_1 = \{a + f(y_1), a + 2 + f(y_1), a + 4 + f(y_1), a + 5 + f(y_1), a + 6 + f(y_1)\}$$
$$A_2 = \{a + f(y_2), a + 2 + f(y_2), a + 4 + f(y_2), a + 5 + f(y_2), a + 6 + f(y_2)\}$$
$$A_3 = \{a + f(y_3), a + 2 + f(y_3), a + 4 + f(y_3), a + 5 + f(y_3), a + 6 + f(y_3)\}$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$
$$A_n = \{a + f(y_n), a + 2 + f(y_n), a + 4 + f(y_n), a + 5 + f(y_n), a + 6 + f(y_n)\}.$$

Then, one can easily observe that
$a + f(y_2) = a + 3 + f(y_1)$ and $a + f(y_3) = a + 7 + f(y_1)$
$\Rightarrow f(y_2) = 3 + f(y_1)$ and $f(y_3) = 7 + f(y_1)$.

But $f(x_2) + f(y_2) = (a + 2) + (3 + f(y_1)) = (a + 5) + f(y_1) = f(x_4) + f(y_1)$.
This is again a contradiction.

Case 8: b=1, c=2, d=2, e= 3.
From (8), we get

$$A_1 = \{a + f(y_1), a + 1 + f(y_1), a + 3 + f(y_1), a + 5 + f(y_1), a + 8 + f(y_1)\}$$
$$A_2 = \{a + f(y_2), a + 1 + f(y_2), a + 3 + f(y_2), a + 5 + f(y_2), a + 8 + f(y_2)\}$$
$$A_3 = \{a + f(y_3), a + 1 + f(y_3), a + 3 + f(y_3), a + 5 + f(y_3), a + 8 + f(y_3)\}$$
$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$
$$A_n = \{a + f(y_n), a + 1 + f(y_n), a + 3 + f(y_n), a + 5 + f(y_n), a + 8 + f(y_n)\}$$

Then, one can easily observe that

$$a + f(y_2) = a + 4 + f(y_1) \text{ and } a + f(y_3) = a + 9 + f(y_1)$$

$$\Rightarrow \quad f(y_2) = 4 + f(y_1) \text{ and } f(y_3) = 9 + f(y_1).$$

But $f(x_2) + f(y_2) = a + 1 + 4 + f(y_1) = (a+5) + f(y_1) = f(x_4) + f(y_1)$. This is again a contradiction.

Case 9: b**=1,** c**=1,** d**=2,** e**= 4.**
From (8), we get

$$A_1 = \{a + f(y_1), a + 1 + f(y_1), a + 2 + f(y_1), a + 4 + f(y_1), a + 8 + f(y_1)\}$$

$$A_2 = \{a + f(y_2), a + 1 + f(y_2), a + 2 + f(y_2), a + 4 + f(y_2), a + 8 + f(y_2)\}$$

$\Longrightarrow a + f(y_1) < a + 1 + f(y_1) < a + 2 + f(y_1)$ are three consecutive numbers

$$\Longrightarrow a + f(y_2) = a + 3 + f(y_1)$$

$$\Longrightarrow a + 1 + f(y_2) = a + 4 + f(y_1)$$

$$\Longrightarrow A_1 \cap A_2 \neq \phi.$$

This is again a contradiction.

Case 10: b**=1,** c**=1,** d**=2,** e**= 3.**
From (8), we get

$$A_1 = \{a + f(y_1), a + 1 + f(y_1), a + 2 + f(y_1), a + 4 + f(y_1), a + 7 + f(y_1)\}$$

$$A_2 = \{a + f(y_2), a + 1 + f(y_2), a + 2 + f(y_2), a + 4 + f(y_2), a + 7 + f(y_2)\}$$

$$A_3 = \{a + f(y_3), a + 1 + f(y_3), a + 2 + f(y_3), a + 4 + f(y_3), a + 7 + f(y_3)\}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$A_n = \{a + f(y_n), a + 1 + f(y_n), a + 2 + f(y_n), a + 4 + f(y_n), a + 7 + f(y_n)\}.$$

Then, one can easily observe that

$$a + f(y_2) = a + 4 + f(y_1) \text{ and } a + f(y_3) = a + 7 + f(y_1)$$

$$\Rightarrow \; f(y_2) = 5 + f(y_1) \text{ and } f(y_3) = 8 + f(y_1).$$

But $f(x_3) + f(y_2) = a + 2 + 5 + f(y_2) = (a+7) + f(y_1) = f(x_5) + f(y_1)$.

This is again a contradiction.

Case 11: b**=2,** c**=2,** d**=2,** e**= 2.**
From (8), we get

$$A_1 = \{a + f(y_1), a + 2 + f(y_1), a + 4 + f(y_1), a + 6 + f(y_1), a + 8 + f(y_1)\}$$
$$A_2 = \{a + f(y_2), a + 2 + f(y_2), a + 4 + f(y_2), a + 6 + f(y_2), a + 8 + f(y_2)\}$$
$$A_3 = \{a + f(y_3), a + 2 + f(y_3), a + 4 + f(y_3), a + 6 + f(y_3), a + 8 + f(y_3)\}$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$A_n = \{a + f(y_n), a + 2 + f(y_n), a + 4 + f(y_n), a + 6 + f(y_n), a + 8 + f(y_n)\}.$$

Then, one can easily observe that

$a+2+f(y_1)=a+f(y_2)$ and $a+8+f(y_1)=a+f(y_3)$
$\Rightarrow f(y_2)=3+f(y_1)$ and $f(y_3)=9+f(y_1)$.
But $f(x_1)+f(y_3)=a+9+f(y_1)=(a+6)+(3+f(y_1))=f(x_4)+f(y_2)$.
This is again a contradiction.

Case 12: b=1, c=1, d=1, e= 1.
Then

$$\begin{aligned}
k &= a+f(y_1)\\
k+1 &= a+1+f(y_1)\\
k+2 &= a+2+f(y_1)\\
k+3 &= a+3+f(y_1)\\
k+4 &= a+4+f(y_1)\\
k+5 &= a+f(y_2)\\
k+6 &= a+1+f(y_2)\\
\ldots & \quad \ldots\\
k+5n-1 &= a+4+f(y_n)
\end{aligned} \tag{9}$$

From (9), we get

$$\begin{aligned}
f(y_2) &= 5+f(y_1)\\
f(y_3) &= 10+f(y_1)\\
f(y_4) &= 15+f(y_1)\\
\ldots & \quad \ldots\\
f(y_n) &= 5(n-1)+f(y_1)
\end{aligned} \tag{10}$$

From (10), we get

$$\begin{aligned}
f(y_n) &= k+5n-1-a-4\\
&= k+5n-5-(k-f(y_1))\\
&= 5n-5+f(y_1)\\
&\leq 5n-j \text{ (since } 5n-j \text{ is the maximum vertex value)}\\
&\Rightarrow f(y_1)\leq 5-j.
\end{aligned}$$

But $f(y_1)\geq 0 \Rightarrow 5-j\geq 0$
$\Rightarrow j\in\{1,2,3,4,5\}$. Note that $f(A)=\{a,a+1,a+2,a+3,a+4\}$,

$f(B)=\{f(y_1),5+f(y_1),10+f(y_1),\ldots,5(n-1)+f(y_1)\}$,
$f(C)=\{f(z_1),f(z_2),\ldots,f(z_{(4n-4-j)})\}$.
Let $F=\{f(y_1)+1,f(y_1)+2,f(y_1)+3,f(y_1)+4,f(y_1)+6,f(y_1)+7,f(y_1)+8,f(y_1)+9,\ldots,f(y_1)+5n-6\}$.

Clearly $F \subseteq f(K_{5,n} \cup (4n-4-j)K_1)$ and F contains $4(n-1)$ vertex values. Also $F \cap f(B) = \emptyset$.
We have three sub cases.

Sub Case 12.1: j**=1.**
Then, $f(C)$ contains $4n-5$ vertex values and hence one element of F must be in $f(A)$. Let $f(y_1)+5m-7 \in f(A)$ for some positive integer m, $2 \leq m \leq n$. Then

$a = f(y_1)+5m-7 \Rightarrow a+2 \in f(B)$, a contradiction.

$a+1 = f(y_1)+5m-7 \Rightarrow a+3 \in f(B)$- a contradiction

$a+2 = f(y_1)+5m-7 \Rightarrow a+4 \in f(B)$- a contradiction

$a+3 = f(y_1)+5m-7 \Rightarrow a+4 \in f(B)$- a contradiction

$a+4 = f(y_1)+5m-7 \Rightarrow a+1 \in f(B)$- a contradiction

Let $f(y_1)+5r-6 \in f(A)$ for some integer $r, 2 \leq r \leq n$. Then,

$a = f(y_1)+5r-6 \Rightarrow a+1 \in f(B)$- a contradiction

$a+1 = f(y_1)+5r-6 \Rightarrow a+2 \in f(B)$- a contradiction

$a+2 = f(y_1)+5r-6 \Rightarrow a+3 \in f(B)$- a contradiction

$a+3 = f(y_1)+5r-6 \Rightarrow a+4 \in f(B)$- a contradiction

$a+4 = f(y_1)+5r-6 \Rightarrow a+3 \in f(B)$- a contradiction

Therefore $j \neq 1$.

Sub Case 12.2: j**=2.**
Then, $f(C)$ contains $4n-6$ vertex values and therefore two elements of F must be in $f(A)$. Let $f(y_1)+5t-4, f(y_1)+5t-6 \in f(A)$ for some positive integer $t, 1 \leq t \leq n$. Then, $a+1 = f(y_1)+5t-5 = f(y_1)+5(t-1) \in f(B)$- a contradiction. Let $f(y_1)+5w-7, f(y_1)+5w-6 \in f(A)$ for some positive integer $w, 1 \leq w \leq n$. Since these two values are consecutive, either $a \in f(B)$ or $a+2 \in f(B)$- a contradiction. Therefore, $j \neq 2$.

Sub Case 12.3: j**=3.**
Then, $f(C)$ contains $4n-7$ vertex values and therefore three elements of F must be in $f(A)$. This is impossible since the elements of $f(A)$ are consecutive. Clearly $j \neq 3$.

Proceeding on similar lines to sub case 12.3 above, we get contradictions when $j = 4, 5$. Thus for $j \geq 1$, $K_{5,n} \cup (4n-4-j)K_1$ is not strongly k-indexable. Hence From (7), we get $d_c(K_{5,n}) = 4(n-1)$. This completes the proof. ◇

Remark 1. In strongly k-indexable labelings it is enough to consider only vertex labelings (as vertex labelings induces edge labelings) whereas in super edge-magic labelings one has to deal with two functions. From the proof of theorem 1.7 mentioned in Figueroa-Conteno et.al., one can see that it is easier to prove the results on super edge-magic deficiency of graphs using the concept of strongly k-indexable labelings rather than super edge-magic labelings.

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