Degree sequence conditions for maximally edge-connected and super-edge-connected oriented graphs depending on the clique number

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Abstract

An orientation of a simple graph G is called an oriented graph. If D is an oriented graph, $\delta(D)$ its minimum degree and $\lambda(D)$ its edge-connectivity, then $\lambda(D) \leq \delta(D)$. The oriented graph is called maximally edge-connected if $\lambda(D) = \delta(D)$ and super-edge-connected, if every minimum edge-cut is trivial. If D is an oriented graph with the property that the underlying graph $G(D)$ contains no complete subgraph of order $p+1$, then we say that the clique number $\omega(D)$ of D is less or equal p.

In this paper we present degree sequence conditions for maximally edge-connected and super-edge-connected oriented graphs D with clique number $\omega(D) \leq p$ for an integer $p \geq 2$.

Keywords: *oriented graph, edge-connectivity, super-edge-connectivity, degree sequence, clique number*

1. Introduction and terminology

We consider finite digraphs without loops and multiple edges. A digraph without any directed cycle of length 2 is called an *oriented graph*. For a digraph D the vertex set is denoted by $V(D)$ and the edge set (or arc set) by $E(D)$. If xy is an arc, then we also write $x \to y$ and say x *dominates* y. We define the *order* of D by $n = n(D) = |V(D)|$ and the *size* by $|E(D)|$. For a vertex $v \in V(D)$ of a digraph D let $d^+(v) = d_D^+(v)$ its *out-degree*, $d^-(v) = d_D^-(v)$ its *in-degree* and $d(v) = d_D(v) = \min\{d^+(v), d^-(v)\}$ its *degree*. The *minimum out-degree* and *minimum in-degree* of a digraph D are denoted by $\delta^+ = \delta^+(D)$ and $\delta^- = \delta^-(D)$ and $\delta = \delta(D) = \min\{\delta^+(D), \delta^-(D)\}$ is its *minimum degree*. The *degree sequence*, *out-degree sequence* and *in-degree sequence* of D is defined as the nonincreasing sequence of the degrees, out-degrees and in-degrees of the vertices of D, respectively.

A digraph D is *strongly connected* if for every pair u, v of vertices there exists a directed path from u to v in D. A digraph D is *k-edge-connected* if for any set S of at most $k-1$ edges the subdigraph $D - S$ is strongly connected. The *edge-connectivity* $\lambda = \lambda(D)$ of a digraph D is defined as the largest value of k such that D is k-edge-connected. Because of $\lambda(D) \le \delta(D)$, we call a digraph D *maximally edge-connected* if $\lambda(D) = \delta(D)$. A digraph is *super-edge-connected* or *super-λ*, if every minimum edge-cut is trivial, that means, that every minimum edge-cut consists of edges adjacent to or from a vertex of minimum degree.

For two disjoint vertex sets X and Y of a digraph D let (X, Y) be the set of edges from X to Y. If D is a digraph, then its *underlying graph* $G(D)$ is the graph obtained by replacing each arc of D by an undirected edge joining the same pair of vertices. If D is an oriented graph with the property that the underlying graph $G(D)$ contains no complete subgraph of order $p+1$, then we say that the *clique number* $\omega(D)$ is less or equal p. A *p-partite tournament* is an orientation of a complete p-partite graph. For other graph theory terminology we follow Bondy and Murty [3] or Chartrand and Lesniak [4].

Sufficient conditions for digraphs to be maximally edge-connected or super-λ were given by several authors, for example by Balbuena and Carmona [2], Dankelmann and Volkmann [5], Fàbrega and Fiol [6], Fiol [7], Geller and Harary [8], Hellwig and Volkmann [9, 10, 11], Imase, Soneoka and Okada [12], Jolivet [13], Soneoka [14], Volkmann [16] and Xu [19]. However, closely related conditions for maximally edge-connected and super-edge-connected oriented graphs have received little attention until recently, cf., Ayoub and Frisch [1], Fiol [7] and Volkmann [17, 18]. In this paper we will present some degree sequence conditions for maximally edge-connected and super-λ oriented graphs D with clique number $\omega(D) \le p$ for an integer $p \ge 2$.

2. Preliminary results

The following known results play an important role in our investigations. We start with a well-known result of Turán [15].

Theorem 2.1 (Turán [15]) Let $p \geq 2$ be an integer. If D is an oriented graph with clique number $\omega(D) \leq p$, then

$$|E(D)| \leq \frac{p-1}{2p}|V(D)|^2. \tag{1}$$

Theorem 2.2 (Volkmann [18]) Let $p \geq 2$ be an integer, and let D be an oriented graph with clique number $\omega \leq p$, $\lambda = \lambda(D)$ and $\delta = \delta(D) \geq 1$. If $\lambda < \delta$, then there exist two disjoint sets $X, Y \subset V(D)$ with $X \cup Y = V(D)$ and $|(X,Y)| = \lambda$ such that

$$|X| \geq 2\left\lfloor \frac{p\delta^+(D)}{p-1} \right\rfloor \text{ and } |Y| \geq 2\left\lfloor \frac{p\delta^-(D)}{p-1} \right\rfloor.$$

Theorem 2.3 (Volkmann [18]) Let D be an oriented graph with $\lambda = \lambda(D)$ and $\delta = \delta(D) \geq 2$. If D is not super-λ, then there exist two disjoint sets $X, Y \subset V(D)$ with $X \cup Y = V(D)$ and $|(X,Y)| = \lambda$ such that $|X| \geq 2\delta^+(D)$ and $|Y| \geq 2\delta^-(D)$.

Theorem 2.4 (Volkmann [18]) Let $p \geq 2$ be an integer, and let D be an oriented graph with clique number $\omega \leq p$, $\lambda = \lambda(D)$ and $\delta = \delta(D) \geq 2$. If D is not super-λ, then there exist two disjoint sets $X, Y \subset V(D)$ with $X \cup Y = V(D)$ and $|(X,Y)| = \lambda$ such that

$$|X| \geq 2\left\lfloor \frac{p\delta^+(D)}{p-1} \right\rfloor - 2 \text{ and } |Y| \geq 2\left\lfloor \frac{p\delta^-(D)}{p-1} \right\rfloor - 2.$$

3. Main results

If D is an oriented graph with clique number $\omega(D) \leq p$, then define

$$2\left\lfloor \frac{p\delta^+(D)}{p-1} \right\rfloor = \mu^+(D) = \mu^+ \text{ and } 2\left\lfloor \frac{p\delta^-(D)}{p-1} \right\rfloor = \mu^-(D) = \mu^-.$$

Theorem 3.1 Let $p \geq 2$ be an integer, and let D be an oriented graph of order n, edge-connectivity λ, clique number $\omega \leq p$, with out-degree sequence $d_1^+ \geq d_2^+ \geq \ldots \geq d_n^+ = \delta^+$ and in-degree sequence $d_1^- \geq d_2^- \geq \ldots \geq d_n^- = \delta^-$. Furthermore, let $\nu = 1$ when n is even and $\nu = 0$ when n is odd. If

$$\sum_{i=1}^{\mu^+} d_{n+1-i}^+ \geq \mu^+ \frac{p-1}{p}\frac{n+1+\nu}{4} \text{ and } \sum_{i=1}^{\mu^-} d_{n+1-i}^- \geq \mu^- \frac{p-1}{p}\frac{n+1+\nu}{4},$$

then $\lambda = \delta$.

Proof. Suppose to the contrary that $\lambda < \delta$. Applying Theorem 2.2, we deduce that there exist two disjoint sets $X, Y \subset V(D)$ with $X \cup Y = V(D)$ and $|(X,Y)| = \lambda$ such that $|X| \geq \mu^+$ and $|Y| \geq \mu^-$.

We assume, without loss of generality, that $|X| \leq \frac{n}{2}$. Now let $S \subseteq X$ with $|S| = \mu^+$ such that S contains the μ^+ vertices of smallest out-degree in X. Then it follows from our hypothesis that

$$\sum_{v \in S} d^+(v) \geq \sum_{i=1}^{\mu^+} d^+_{n+1-i} \geq \mu^+ \frac{p-1}{p} \frac{n+1+\nu}{4}.$$

This implies that

$$d^+(u) \geq \frac{p-1}{p} \frac{n+1+\nu}{4}$$

for every $u \in X - S$ and hence

$$\sum_{v \in X} d^+(v) \geq |X| \frac{p-1}{p} \frac{n+1+\nu}{4}. \tag{2}$$

On the other hand, our assumption $\lambda < \delta$ and Turán's bound (1) lead to

$$\sum_{v \in X} d^+(v) = |E(D[X])| + \lambda \leq \frac{p-1}{2p}|X|^2 + \delta^+ - 1. \tag{3}$$

Our assumption $|X| \leq \frac{n}{2}$ yields $2|X| \leq n - 1 + \nu$, and thus it follows from (2) that

$$\sum_{v \in X} d^+(v) \geq |X| \frac{p-1}{p} \frac{2|X|+2}{4} = \frac{p-1}{2p}|X|^2 + \frac{(p-1)|X|}{2p}. \tag{4}$$

Combining (3) and (4) and using the bound $|X| \geq \mu^+$, we arrive at the contradiction

$$\begin{aligned}
\delta^+ &\geq \frac{(p-1)|X|}{2p} + 1 \geq \frac{(p-1)\mu^+}{2p} + 1 \\
&\geq \left\lfloor \frac{p\delta^+}{p-1} \right\rfloor \frac{p-1}{p} + 1 \geq \left(\frac{p\delta^+}{p-1} - 1 \right) \frac{p-1}{p} + 1 \\
&= \delta^+ - \frac{p-1}{p} + 1 = \delta^+ + \frac{1}{p} > \delta^+,
\end{aligned}$$

and the proof of Theorem 3.1 is complete. □

The following family of examples will demonstrate that the conditions in Theorem 3.1 are best possible in the sense that

$$\sum_{i=1}^{\mu^+} d^+_{n+1-i} \geq \mu^+ \frac{p-1}{p} \frac{n+1+\nu}{4} - 1$$

and

$$\sum_{i=1}^{\mu^-} d^-_{n+1-i} \geq \mu^- \frac{p-1}{p} \frac{n+1+\nu}{4} - 1$$

do not guarantee that the oriented graph is maximally edge-connected.

Example 3.2 Let $p \geq 3$ be an integer, and let D_1' and D_2' be two $(p-1)$-regular p-partite tournaments with the partite sets $\{u_1, v_1\}$, $\{u_2, v_2\}, \ldots,$ $\{u_p, v_p\}$ and $\{x_1, y_1\}$, $\{x_2, y_2\}, \ldots,$ $\{x_p, y_p\}$. In addition, let D be the p-partite tournament consisting of the disjoint union of D_1 and D_2 such that $\{u_i, v_i, x_i, y_i\}$ are the partite sets of D for $1 \leq i \leq p$ together with the edge set

$$U = \{u_1x_2, u_2x_3, \ldots, u_{p-2}x_{p-1}\}$$

and all further possible edges from D_2 to D_1. The resulting p-partite tournament D is of order $n = n(D) = 4p$ such that $\delta^+(D) = \delta^-(D) = \delta(D) = p-1$, $\mu^+(D) = \mu^-(D) = 2p$ and

$$\begin{aligned} \sum_{i=1}^{\mu^+(D)} d^+_{n+1-i}(D) &= \sum_{i=1}^{\mu^-(D)} d^-_{n+1-i}(D) \\ &= (p-2)p + (p+2)(p-1) = 2p^2 - p - 2 \end{aligned}$$

and

$$\mu^+(D) \frac{p-1}{p} \frac{n+1+\nu}{4} = \mu^-(D) \frac{p-1}{p} \frac{n+1+\nu}{4} = 2p^2 - p - 1.$$

However, since U is a minimum edge-cut, we deduce that $\lambda(D) = p-2 < p-1 = \delta(D)$.

A proof similar to this one of Theorem 3.1 leads to the next result.

Theorem 3.3 Let $p \geq 2$ be an integer, and let D be an oriented graph of order n, edge-connectivity λ, clique number $\omega \leq p$ and degree sequence $d_1 \geq d_2 \geq \ldots \geq d_n = \delta$. Furthermore, let $\nu = 1$ when n is even and $\nu = 0$ when n is odd. If

$$\sum_{i=1}^{2\lfloor \frac{p\delta}{p-1} \rfloor} d_{n+1-i} \geq \left\lfloor \frac{p\delta}{p-1} \right\rfloor \frac{p-1}{p} \frac{n+1+\nu}{2},$$

then $\lambda = \delta$.

Example 3.2 shows that the condition

$$\sum_{i=1}^{2\lfloor\frac{p\delta}{p-1}\rfloor} d_{n+1-i} \geq \left\lfloor\frac{p\delta}{p-1}\right\rfloor \frac{p-1}{p}\frac{n}{2}$$

does not guarantee that the oriented graph is maximally edge-connected.

Next we will prove an analogue to Theorem 3.3 for oriented graphs to be super-λ. For this analogue we define

$$\mu = \max\left\{2\delta, 2\left\lfloor\frac{p\delta}{p-1}\right\rfloor - 2\right\}.$$

Theorem 3.4 Let $p \geq 2$ be an integer, and let D be an oriented graph of order n, edge-connectivity λ, clique number $\omega \leq p$ and degree sequence $d_1 \geq d_2 \geq \ldots \geq d_n = \delta \geq 2$. Furthermore, let $\nu = 1$ when n is even and $\nu = 0$ when n is odd. If

$$\sum_{i=1}^{\mu} d_{n+1-i} \geq \mu\frac{p-1}{p}\frac{n+2+\nu}{4},$$

then D is super-λ.

Proof. Suppose to the contrary that D is not super-λ. Applying Theorems 2.3 and 2.4, we deduce that there exist two disjoint sets $X, Y \subset V(D)$ with $X \cup Y = V(D)$ and $|(X,Y)| = \lambda$ such that $|X|, |Y| \geq \mu$.

We assume, without loss of generality, that $|X| \leq \frac{n}{2}$. Now let $S \subseteq X$ with $|S| = \mu$ such that S contains the μ vertices of smallest degree in X. Then it follows from our hypothesis that

$$\sum_{v\in S} d(v) \geq \sum_{i=1}^{\mu} d_{n+1-i} \geq \mu\frac{p-1}{p}\frac{n+2+\nu}{4}.$$

As above, this implies that

$$\sum_{v\in X} d(v) \geq |X|\frac{p-1}{p}\frac{n+2+\nu}{4}. \tag{5}$$

On the other hand Turán's bound (1) yields

$$\sum_{v\in X} d(v) \leq \frac{p-1}{2p}|X|^2 + \delta. \tag{6}$$

Our assumption $|X| \leq \frac{n}{2}$ yields $2|X| \leq n - 1 + \nu$, and thus it follows from (5) that

$$\sum_{v \in X} d^+(v) \geq |X| \frac{p-1}{p} \frac{2|X|+3}{4} = \frac{p-1}{2p}|X|^2 + \frac{3(p-1)|X|}{4p}. \qquad (7)$$

If $p \geq 4$, then we arrive togehter with (6), (7) and the inequality $|X| \geq 2\delta$ to the contradiction

$$\delta \geq \frac{3(p-1)|X|}{4p} \geq \frac{3\delta(p-1)}{2p} > \delta.$$

If $p = 2$, then $|X| \geq \mu$ shows that $|X| \geq 4\delta - 2$. But if $|X| = 4\delta - 2$, then the hypothesis $\delta \geq 2$ and (6) lead to the contradiction

$$\begin{aligned} 4\delta^2 - 2\delta &= \delta|X| \leq \sum_{v \in X} d(v) \leq \frac{p-1}{2p}|X|^2 + \delta \\ &= \frac{1}{4}(4\delta - 2)^2 + \delta = 4\delta^2 - 4\delta + 1 + \delta. \end{aligned}$$

Thus $|X| \geq 4\delta - 1$ and therefore (6) and (7) yield the contradiction

$$\delta \geq \frac{3(p-1)|X|}{4p} \geq \frac{3}{8}(4\delta - 1) > \delta.$$

If $p = 3$ and $\delta \geq 4$, then (6), (7) and the bound $|X| \geq 2\lfloor \frac{3\delta}{2} \rfloor - 2$ lead to the contradiction

$$\delta \geq \frac{3(p-1)|X|}{4p} = \frac{|X|}{2} \geq \left\lfloor \frac{3\delta}{2} \right\rfloor - 1 \geq \frac{3\delta}{2} - \frac{3}{2} > \delta.$$

If $p = \delta = 3$, then $|X| \geq 2\delta = 6$. But in the case $|X| = 6$, inequality (6) yields the contradiction $18 \leq 15$ and thus $|X| \geq 7$. Applying now (6) and (7), we obtain the contradiction

$$3 = \delta \geq \frac{3(p-1)|X|}{4p} \geq \frac{7}{2}.$$

In the remaining case that $p = 3$ and $\delta = 2$, it follows from (6) and (7) that $2 = \delta \geq \frac{|X|}{2}$ and thus $|X| \leq 4$. Because of $|X| \geq 2\delta$, we conclude that $|X| = 4$, Since $p = 3$, we finally arrive the contradiction

$$8 = 4\delta \leq \sum_{v \in X} d(v) \leq 5 + \delta = 7.$$

Since we have discussed all posible cases, the proof of Theorem 3.4 is complete. □

The next family of examples will demonstrate that the condition in Theorem 3.4 is best possible in the sense that

$$\sum_{i=1}^{\mu} d_{n+1-i} \geq \mu \frac{p-1}{p} \frac{n+1+\nu}{4}$$

does not guarantee that the oriented graph is super-edge-connected.

Example 3.5 Let $p \geq 3$ be an integer, and let D_1' be a $(p-1)$-regular p-partite tournament with the partite sets $\{u_1, v_1\}$, $\{u_2, v_2\}, \ldots, \{u_p, v_p\}$ such that $\{u_2, u_3, \ldots, u_p\} \to u_1$. In addition, let D_2' be a $(p-1)$-regular p-partite tournament with the partite sets $\{x_1, y_1\}$, $\{x_2, y_2\}, \ldots, \{x_p, y_p\}$ such that $x_p \to \{x_1, x_2, \ldots, x_{p-1}\}$. If $D_1 = D_1' - u_1$ and $D_2 = D_2' - x_p$, then let D be the p-partite tournament consisting of the disjoint union of D_1 and D_2 such that $\{v_1, x_1, y_1\}$, $\{u_p, v_p, y_p\}$ and $\{u_i, v_i, x_i, y_i\}$ for $2 \leq i \leq p-1$ are the partite sets of D together with the edge set

$$U = \{u_2x_1, u_3x_2, u_4x_3, \ldots, u_px_{p-1}\}$$

and all further possible edges from D_2 to D_1. The resulting p-partite tournament D is of order $n = n(D) = 4p - 2$ such that $\delta^+(D) = \delta^-(D) = \delta(D) = p - 1$, $\mu(D) = 2p - 2$ and

$$\sum_{i=1}^{\mu(D)} d_{n+1-i}(D) = (2p-2)(p-1) = 2p^2 - 4p + 2$$

and

$$\begin{aligned} \mu(D) \frac{p-1}{p} \frac{n+1+\nu}{4} &= (2p-2)\frac{p-1}{p} \frac{4p}{4} \\ &= 2p^2 - 4p + 2. \end{aligned}$$

However, since U is a minimum edge-cut, we deduce that D is not super-λ.

Using the definitions

$$\mu_1^+ = \max\left\{2\delta^+, 2\left\lfloor \frac{p\delta^+}{p-1} \right\rfloor - 2\right\}$$

and

$$\mu_1^- = \max\left\{2\delta^-, 2\left\lfloor \frac{p\delta^-}{p-1} \right\rfloor - 2\right\},$$

and applying again Theorems 2.1, 2.3 and 2.4, we obtain the following analogue to Theorem 2.4 with the same arguments.

Theorem 3.6 Let $p \geq 2$ be an integer, and let D be an oriented graph of order n, edge-connectivity λ, clique number $\omega \leq p$, with out-degree sequence $d_1^+ \geq d_2^+ \geq \ldots \geq d_n^+ = \delta^+$ and in-degree sequence $d_1^- \geq d_2^- \geq \ldots \geq d_n^- = \delta^-$. Furthermore, let $\nu = 1$ when n is even and $\nu = 0$ when n is odd. If

$$\sum_{i=1}^{\mu_1^+} d_{n+1-i}^+ \geq \mu_1^+ \frac{p-1}{p} \frac{n+2+\nu}{4} \quad \text{and} \quad \sum_{i=1}^{\mu_1^-} d_{n+1-i}^- \geq \mu_1^- \frac{p-1}{p} \frac{n+2+\nu}{4},$$

then D is super-λ.

Example 3.5 shows that Theorem 3.6 is best possible in the same sense as Theorem 3.4.

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