Circular choosability of planar graphs with large girth *

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Abstract

In this paper, we study the circular choosability recently introduced by Mohar[5] and Zhu [11]. In this paper, we show that the circular choosability of planar graphs with girth at least $\frac{10n+8}{3}$ is at most $2+\frac{2}{n}$, which improves the earlier results.

keywords: circular chromatic number, circular choosability, planar graph

1 Introduction

Our terminology and notation will be standard. The reader is referred to [2] for the undefined terms. Throughout this paper a graph $G(V,E)$ has a finite vertex set V and a finite edge set E. The length of the shortest cycle of a graph G is called the girth of G. We write $g(G)$ for the girth of G. Suppose $G=(V,E)$ is a graph and p,q $(p \geq 2q)$ are positive integers. A (p,q)-*coloring* of a graph G is a mapping $c: V \rightarrow \{0,1,,...,p-1\}$, such that for any edge uv of G, $q \leq |c(u)-c(v)| \leq p-q$. Note that a $(p,1)$-coloring of

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a graph G is the same as a p-coloring of G. The *circular chromatic number* is defined as

$$\chi_c(G) = \inf\{\ \frac{p}{q} :\ \text{there is a } (p,q)\text{-coloring of } G\}.$$

It is known [7, 12] that for any graph G, $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$.

Let C be a set of integers (called colors). A *list assignment* L is a mapping L which assigns to each vertex v of G a subset $L(v)$ of C. An *L-coloring* of G is a mapping $c : V \to C$ such that for each vertex v, $c(v) \in L(v)$ and for each edge $uv, c(u) \neq c(v)$. The *chooosability* [1] or *list chromatic number* $\chi_l(G)$ of G is the least integer k such that for any list assignment L for which $|L(v)| = k$ for every vertex v of G, there is an L-coloring of G.

Zhu [11] gave the circular version of the list coloring as follows.

Suppose that p, q ($p \geq 2q$) are positive integers and $t \geq 1$ is a real number. A *t-(p,q)-list assignment* L of G is a list assignment such that, for every vertex v, $L(v) \subseteq \{0, 1, \cdots, p-1\}$ and $|L(v)| \geq tq$. An *L-(p,q)-coloring* of G is a (p,q)-coloring c of G such that for any vertex v, $c(v) \in L(v)$. We say that G is *circular t-(p,q)-choosable* if for any t-(p,q)-list assignment L, G has an L-(p,q)-coloring. We say that G is *circular t-choosable* if G is circular t-(p,q)-choosable for any positive integers p, q with $p \geq 2q$. The *circular choosability* (or the *circular list chromatic number*) of G is defined as

$$\chi_{c,l}(G) = \inf\{\ t \geq 1 :\ G \text{ is circular } t\text{-choosable}\}.$$

Observe that for an integer k, G is k-choosable means that G is circular k-$(p,1)$-choosable for any p. Hence $\chi_l(G) = \min\{\ k :\ G$ is circular k-$(p,1)$-choosable for any $p\}$.

The following propositions are taken from [11] and can be easily derived from the definitions.

Proposition 1 [11] For any graph G, $\chi_c(G) \leq \chi_{c,l}(G)$.

Proposition 2 [11] For any graph G, $\chi_l(G) - 1 \leq \chi_{c,l}(G)$.

Zhu [11] proved that for finite k-degenerate graph G, $\chi_{c,l}(G) \leq 2k$. On the other hand it was proved by Zhu [11] that for each $\varepsilon > 0$, there is a k-degenerate graph which is not circular $(2k - \varepsilon)$-choosable. Since every k-degenerate graph G satisfies $\chi_l(G) \leq k + 1$, it follows that the difference $\chi_{c,l}(G) - \chi_l(G)$ can be arbitrarily large.

Mohar [5] proposed a problem: What is the smallest real number t such that every planar graph is circularly t-choosable? Inspired by Thomassen's proof for 5-choosability of planar graphs, it was proved that every planar graph is circular 8-choosable in [8], also see [4]. Moreover, the following theorem was given in [4].

Theorem 1 [4] *For any $n \geq 2$, there exists a planar graph G_n with $\chi_{c,l}(G) \geq 6 - \frac{1}{n}$.*

A natural question is to ask what happens when we restrict ourselves to planar graphs with high girth. The circular choosability of planar graphs and some special planar graphs of large girth was studied in [4], [8] and [10].

Theorem 2 [8] *Let G be a planar graph with $g(G) \geq 8n+2$, then $\chi_{c,l}(G) \leq 2 + \frac{1}{n}$.*

In [4], the following theorem was given, which is slightly stronger than Theorem 2.

Theorem 3 [4] *Let G be a planar graph with $g(G) \geq 4n+2$, then $\chi_{c,l}(G) \leq 2 + \frac{2}{n}$.*

The main result of this paper is as follows.

Theorem 4 *Let G be a graph with $g(G) \geq 3n+1$. If every subgraphs of G have average degree less than $2 + \frac{6}{5n+1}$, then $\chi_{c,l}(G) \leq 2 + \frac{2}{n}$.*

Corollary 5 *If G is a planar graph with girth at least $\frac{10n+8}{3}$, or a graph embeddable on the torus or Klein bottle with girth greater than $\frac{10n+8}{3}$, then $\chi_{c,l}(G) \leq 2 + \frac{2}{n}$.*

Proof. Clearly, $\frac{10n+8}{3} \geq 3n+1$. And every subgraph of G has girth at least as large as the girth of G, thus, if the conclusion fails, by Theorem 4, we have that the average degree of G is at least $2 + \frac{6}{5n+1}$. It follows that $|E(G)| \geq \frac{5n+4}{5n+1}|V(G)|$. Let ν, e be the number of vertices and edges of G, respectively. Let f be the number of faces in an embedding of G on a surface of Euler characteristic N. By Euler's Formula,

$$2 - N = \nu - e + f \leq e(\frac{5n+1}{5n+4} - 1 + \frac{2}{g(G)}) = e(\frac{2}{g(G)} - \frac{3}{5n+4}).$$

For the surface mentioned, $N \le 2$. Hence $\frac{2}{g(G)} \ge \frac{3}{5n+4}$, that is, $g(G) \le \frac{10n+8}{3}$, and equality holds only when $N = 2$. This contradiction completes the proof. $\square$.

Immediately, from Corollary 5 and Proposition 1, we obtain the following corollary, which improves the result of Theorem 3.

Corollary 6 *Let G be a planar graph with $g(G) \ge \frac{10n+8}{3}$, then $\chi_c(G) \le \chi_{c,l}(G) \le 2 + \frac{2}{n}$.*

2 Proof of Theorem 4

Now we give the idea of the proof of Theorem 4. If the conclusion fails, that is, G is not circular $(2+\frac{2}{n})$-choosable; We will show that the average degree of G is large (at least $2+\frac{6}{5n+1}$), which gives a contradiction. The method we will use is the discharging method, which is inspired by Borodin et al. [3].

Firstly, we give a basic lemma in [4]. As in [4], in the following Lemma 1, G is given, integers $p \ge 2q$ are given, $t \ge 1$ is also given, as is a t-(p,q)-list assignment L. Moreover, some vertices $u_1, u_2, \cdots, u_k$ are L-(p,q)-precolored. Now we want to extend this L-(p,q)-coloring to G according to some ordering of the vertices $(v_1 = u_1, \cdots, v_k = u_k, v_{k+1}, \cdots, v_{|V(G)|})$. Moreover, we require that in this ordering every non-precolored vertex has at most one neighbor with higher index. We say a color a in $L(v_j)$ is *extendable* if there exists some L-(p,q)-coloring c of the subgraph induced by $\{v_1, v_2, \cdots, v_j\}$ such that $c(v_j) = a$ and c respects the precoloring. Note that if every vertex of G has at least one extendable color, then G has an L-(p,q)-coloring.

Lemma 1[4] *Suppose $F = \{w_1, \cdots, w_k\}$ is the set of neighbors of v_j with smaller index in the ordering. If w_i has at least $x_i \ge 1$ extendable colors for each i, then v_j has at least $|L(v_j)| - \sum_{i:x_i<2q}(2q - x_i)$ extendable colors.*

We define a graph G to be *t-critical* if $\chi_{c,l}(G) > t$ and $\chi_{c,l}(H) \le t$ for every proper subgraph H of G. Let $L_e(v)$ be the extendable colors of v. In the following, G is a $(2+\frac{2}{n})$-critical graph. Clearly, G has minimum degree at least two.

A *thread* in a graph G is a path whose internal vertices have degree 2 in G. Its *order* is the number of its internal vertices. Two vertices are *weak neighbors* or *weakly adjacent* if they are the endpoints of a thread (this includes adjacent vertices, since threads may have order 0). In [4], the following claim was proved.

Claim 1 [4] *Every thread in G has order at most $n-1$.*

Clearly, we have the following claim.

Claim 2 *No three vertices of G with degree at least 3 are pairwise weakly adjacent, and no two threads have the same set of endpoints.*

Proof. Otherwise, by Claim 1, G has a cycle of length at most $3n$, which contradicts with $g(G) \geq 3n+1$. □

When u and v are weakly adjacent, let l_{uv} denote the order of a shortest u,v-thread. (Note that if u, v are adjacent, then $l_{uv} = 0$). Let $Y = \{v \in V(G) : d(v) \geq 3\}$. A weak neighbor u of v is a *weak Y-neighbor* of v if $u \in Y$; Otherwise it is a *weak 2-neighbor* of v.

For $v \in V(G)$, let $N_Y(v)$ denote the set of weak Y-neighbors of v in G. For $v \in Y$, let $f(v) = -n - 1 + \sum_{u \in N_Y(v)}(n - l_{vu})$.

The next two claims place lower bounds on $f(v)$ and on $\sum_{u \in N_Y(v)} f(u)$.

Claim 3 *If $v \in Y$, then $f(v) \geq 1$.*

Proof. As $\chi_{c,l}(G) > 2 + \frac{2}{n}$, there exist $\varepsilon > 0$, two integers p, q with $p \geq 2q$ and a $(2 + \frac{2}{n} + \varepsilon)$-list-assignment L such that G has no L-(p,q)-coloring. Observe that every proper subgraph H of G has an L-(p,q)-coloring since, by the criticality of G, $\chi_{c,l}(H) \leq 2 + \frac{2}{n}$. Let $t = 2 + \frac{2}{n} + \varepsilon$.

Let H be the graph obtained from G by deleting v and all its weak 2-neighbors. By the criticality of G, H has an L-(p,q)-coloring.

Consider $u \in N_Y(v)$. Applying the Lemma 1 for each u,v-thread. If there exist vertex $u \in N_Y(v)$ and an integer j, $1 \leq j \leq l_{uv}$, such that $1 + j\frac{2q}{n} \geq 2q$, then assume that the vertex ordering in u,v-thread is $u, v_1, v_2, \cdots, v_{l_{uv}}, v$. We choose the minimum j in this vertex ordering such that $1 + j\frac{2q}{n} \geq 2q$. Let $H' = G - \{v_1, \cdots, v_j\}$ and consider the L-(p,q)-coloring of H'. By Lemma 1, we have $|L_e'(v_{j-1})| \geq 1 + (j-1)\frac{2q}{n} \geq 2q - \frac{2q}{n}$ and $|L_e'(v_{j+1})| \geq 1 (v_{l_{uv}+1} = v)$. Then $|L_e'(v_j)| \geq tq - (2q-1) - (2q - 1 - (j-1)\frac{2q}{n}) \geq tq - 2q + 1 - (2q - (2q - \frac{2q}{n})) \geq 1 + \varepsilon q \geq 1$. Thus this L-(p,q)-coloring of H' can be extended to G, which is a contradiction. So assume that for $1 \leq j \leq \max\{l_{uv} : u \in N_Y(v)\}$, it holds that $1 + j\frac{2q}{n} < 2q$.

Then, by Lemma 1, if u_j is a neighbor of v belonging to a u,v-thread of order l_{uv}, then $|L_e(u_j)| \geq 1 + l_{uv}\frac{2q}{n}$. Therefore,

$$|L_e(v)| \quad \geq tq - \sum_{u \in N_Y(v)} \left(2q - (l_{uv}\frac{2q}{n} + 1)\right)$$

$$
\begin{aligned}
&\geq n\frac{2q}{n}+\frac{2q}{n}+\varepsilon q-\sum_{u\in N_Y(v)}(n-l_{uv})\frac{2q}{n}+\sum_{u\in N_Y(v)}1\\
&\geq d(v)+\varepsilon q-f(v)\frac{2q}{n}\\
&>1-f(v)\frac{2q}{n}.
\end{aligned}
$$

If $f(v)\leq 0$, then v has at least 1 extendable color; It follows that G has an L-(p,q)-coloring, a contradiction. So $f(v)\geq 1$. □

Claim 4 *If $v\in Y$, then $\sum_{u\in N_Y(v)} f(u)\geq n+2$.*

Proof. Similarly to previous claim, we begin with a t-(p,q)-list-assignment L, where $t=2+\frac{2}{n}+\varepsilon$, such that G is not L-(p,q)-colorable. Say that a vertex $u\in N_Y(v)$ is *v-free* if $f(u)\leq n-l_{uv}$. Let H be obtained from G by deleting the vertex v, the v-free neighbors, and all their weak 2-neighbors. By the criticality of G, there is an L-(p,q)-coloring of H.

If u is v-free, applying the Lemma 1 to each thread from u except u,v-thread. Similarly to previous lemmas, if $w\in N_Y(u)-\{v\}$, we assume that, for $1\leq j\leq \max\{l_{uw}: w\in N_Y(u), u\in N_Y(v)\}$, $1+j\frac{2q}{n}\leq 2q$. Then by induction, if u_j is a neighbor of u belonging to a thread of order l_{uw}, then $|L_e(u_j)|\geq 1+l_{uw}\frac{2q}{n}$. Therefore,

$$
\begin{aligned}
|L_e(u)| \quad &\geq tq-\sum_{w\in N_Y(u)-\{v\}}\left(2q-(l_{uw}\frac{2q}{n}+1)\right)\\
&\geq n\frac{2q}{n}+\frac{2q}{n}+\varepsilon q-\sum_{w\in N_Y(u)-\{v\}}(n-l_{uw})\frac{2q}{n}+\sum_{w\in N_Y(u)-\{v\}}1\\
&\geq d(u)-1+\varepsilon q+(n-l_{vu}-f(u))\frac{2q}{n}.
\end{aligned}
$$

As $f(u)\leq n-l_{vu}$, then $|L_e(u)|\geq 1$. Applying the Lemma 1 to u-v thread: $uv_1^u v_2^u\cdots v_{l_{uv}}^u$. By induction, it holds that $|L_e(v_i^u)|\geq d(u)-1+\varepsilon q+(n-l_{vu}-f(u))\frac{2q}{n}+i\frac{2q}{n}$. If $d(u)-1+\varepsilon q+(n-l_{vu}-f(u))\frac{2q}{n}+i\frac{2q}{n}\geq 2q$, for $1\leq i\leq l_{uv}$, then let H' be obtained from G by deleting $\{u\}$ and the weak neighbors of u and their 2-neighbors except $\{v_{i+1}^u,\cdots,v_{l_{uv}}^u\}$. We get that $|L_e'(v_{i+1}^u)|=1$. Thus $|L_e'(v_i^u)|\geq d(u)-1+\varepsilon q+(n-l_{vu}+i-f(u))\frac{2q}{n}-(2q-1)\geq 2q-(2q-1)=1$, then G has an L-(p,q)-coloring, which obtains a contradiction. Thus we assume that for each v-free vertex u and $1\leq i\leq l_{uv}$, it holds that $d(u)-1+\varepsilon q+(n-l_{vu}-f(u))\frac{2q}{n}+i\frac{2q}{n}<2q$. Thus if $v_{l_{uv}}^u$ is a neighbor of v belonging to P_{uv}, then

$$
|L_e(v_{l_{uv}}^u)| \quad \geq d(u)-1+\varepsilon q+(n-l_{vu}-f(u))\frac{2q}{n}+l_{uv}\frac{2q}{n}
$$

$$\geq d(u) - 1 + \varepsilon q + (n - f(u))\frac{2q}{n}$$
$$> (n - f(u))\,\frac{2q}{n}.$$

If $u \in N_Y(v)$ is not v-free, then $|L_e(u)| \geq 1$. Applying Lemma 1 to u-v thread and by the same method, if $v^u_{l_{uv}}$ is a neighbor of v belonging to P_{uv}, we can get $|L_e(v^u_{l_{uv}})| \geq 1 + l_{uv}\frac{2q}{n} \geq 1 + (n - f(u))\frac{2q}{n} > (n - f(u))\frac{2q}{n}$.

Now consider v. We have

$$\begin{aligned}
|L_e(v)| \quad & \geq tq - \sum_{u \in N_Y(v)} \left(2q - (n - f(u))\frac{2q}{n}\right) \\
& \geq n\frac{2q}{n} + \frac{2q}{n} + \varepsilon q - \sum_{u \in N_Y(v)} f(u)\frac{2q}{n} \\
& \geq \left(n + 1 - \sum_{u \in N_Y(v)} f(u)\right)\frac{2q}{n} + \varepsilon q \\
& > \left(n + 1 - \sum_{u \in N_Y(v)} f(u)\right)\frac{2q}{n}.
\end{aligned}$$

So we have $\sum_{u \in N_Y(v)} f(u) \geq n + 2$. Otherwise, $\sum_{u \in N_Y(v)} f(u) \leq n + 1$. Then $|L_e(v)| \geq 1$ and G has an L-(p, q)-coloring, a contradiction. □

We complete the proof using a discharging method. Let $d(v)$ be the initial charge on the vertex $v \in V(G)$. We will move charge from vertex to vertex, without changing the total. By the following two claims, we will show that after discharging, it holds that $d^*(v) \geq 2 + \frac{4d(v)-2}{5n+5}$, for all $v \in V(G)$. Then

$$\begin{aligned}
2|E(G)| \quad & = \sum_{v \in V(G)} d^*(v) \geq \sum_{v \in V(G)} (2 + \frac{4d(v) - 2}{5n + 5}) \\
& = 2(1 - \frac{1}{5n + 5})|V(G)| + \frac{8}{5n + 5}|E(G)|,
\end{aligned}$$

and hence $\frac{5n+4}{5n+5}|V(G)| \leq \frac{5n+1}{5n+5}|E(G)|$. Thus, the average degree of G is at least $2 + \frac{6}{5n+1}$, which gives a contradiction.

Discharging rules.

a. Every $v \in Y$ gives each weak 2-neighbor the amount $\frac{3}{5n+5}$.

b. Every $v \in Y$ gives each weak Y-neighbor the amount $\frac{3f(v)+(n+2)(d(v)-3)}{(5n+5)d(v)}$.

Claim 5 *Every $v \in Y$ receives from its weak Y-neighbors at least $\frac{n+2}{5n+5}$.*

Proof. If every $u \in N_Y(v)$ sends v at least $\frac{f(u)}{5n+5}$, then v receives from $N_Y(v)$ at least $\frac{1}{5n+5}\sum_{u\in N_Y(v)} f(u) \geq \frac{n+2}{5n+5}$, by Claim 4.

Otherwise, for some $u \in N_Y(v)$, it holds that $\frac{3f(u)+(n+2)(d(u)-3)}{(5n+5)d(u)} < \frac{f(u)}{5n+5}$. That is $(n+2)(d(u)-3) < f(u)(d(u)-3)$. Then we conclude that $d(u) \geq 4$ and $f(u) > n+2$. Thus, u by itself gives at v at least $\frac{3f(u)+(n+2)(d(u)-3)}{(5n+5)d(u)} \geq \frac{3(n+2)+(n+2)(d(u)-3)}{(5n+5)d(u)} = \frac{n+2}{5n+5}$. Moreover, all other amounts to v are nonnegative, since if $y \in N_Y(v)$, then $d(y) \geq 3$ and $f(y) \geq 1$. □

Claim 6 *After the discharging, it holds that $d^*(v) \geq 2 + \frac{4d(v)-2}{5n+5}$, for all $v \in V(G)$.*

Proof. If $d(v) = 2$, then v sends out nothing and receives $\frac{3}{5n+5}$ from each of its two weak Y-neighbors. So $d^*(v) = 2 + \frac{6}{5n+5} = 2 + \frac{4d(v)-2}{5n+5}$.

Now consider $v \in Y$. By the discharging rule, vertex v sends out $\frac{3}{5n+5}\sum_{w\in N_Y(v)} l_{vw}$ to its weak 2-neighbors and $\frac{3f(v)+(n+2)(d(v)-3)}{5n+5}$ to its weak Y-neighbors. By Claim 5, v also receives at least $\frac{n+2}{5n+5}$ from its weak Y-neighbors. Then

$$\begin{aligned}
d^*(v) \quad & \geq d(v) - \frac{3}{5n+5}\sum_{w\in N_Y(v)} l_{wv} - \frac{3f(v)+(n+2)(d(v)-3)}{5n+5} \\
& \quad + \frac{n+2}{5n+5} \\
& = d(v) - \frac{3}{5n+5}[-n-1+\sum_{w\in N_Y(v)}(n-l_{wv}+l_{wv})] \\
& \quad - \frac{(n+2)(d(v)-4)}{5n+5} \\
& = \frac{d(v)}{5n+5}[5n+5-3n-(n+2)] + \frac{1}{5n+5}[3n+3+4(n+2)] \\
& = \frac{(n+3)d(v)+7n+11}{5n+5}.
\end{aligned}$$

Since $d(v) \geq 3$, we have

$$(n+3)d(v)+7n+11 = (d(v)-3)n+3d(v)+3+10n+8 \geq 4d(v)+10n+8.$$

Therefore, $\frac{(n+3)d(v)+7n+11}{5n+5} \geq 2 + \frac{4d(v)-2}{5n+5}$. Then we complete the proof of Claim 6 and so the Theorem 4. □

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