Geodesic Graphs

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Abstract

In this paper we introduce the concept of geodesic graph at a vertex of a connected graph and investigate its properties. We determine the bounds for the number of edges of the geodesic graph. We prove that an edge of a graph is a cut edge if and only if it is a cut edge of each of its geodesic graphs. Also we characterize a bipartite graph as well as a geodetic graph in terms of its geodesic graph.

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1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to West [1]. For vertices x and y in a connected graph G, the distance $d(x, y)$ is the length of a shortest x-y path in G. An x-y path of length $d(x, y)$ is called an x-y *geodesic*. A vertex v is said to lie on an x-y geodesic P if v is a vertex of P including the vertices x and y. The *diameter* $d(G)$ of a connected graph G is the length of any longest geodesic. Two vertices u and v in G are *antipodal* if $d(u, v) = d(G)$. A graph G is *geodetic* if every pair of vertices u and v are joined by a unique path of length $d(u, v)$.

We need the following theorems in the sequel.

Theorem 1. [3] *The maximum number of edges among all p vertex graphs with no triangles is* $\left\lfloor \frac{p^2}{4} \right\rfloor$.

Theorem 2. [1] *An edge is a cut edge if and only if it belongs to no cycle.*

2. Geodesic Graphs

Definition 1. *Let G be a connected graph. Let x be any vertex in G. For any two vertices u and v, define a relation $\leq_x$ on V by $u \leq_x v$ if u lies on an x-v geodesic.*

In the following theorem, we prove that the relation $\leq_x$ is a partial order on V.

Theorem 3. *Let G be a connected graph. Then for any vertex x in G, $(V, \leq_x)$ is a poset.*

Proof. Since G is connected, it is clear that any vertex u lies on a shortest x-u path so that $u \leq_x u$. Hence $\leq_x$ is reflexive.

Let $u \leq_x v$ and $v \leq_x u$. Since $u \leq_x v$, the vertex u lies on an x-v geodesic so that $d(x,u) \leq d(x,v)$. Similarly, since $v \leq_x u$, the vertex v lies on an x-u geodesic so that $d(x,v) \leq d(x,u)$. Hence $d(x,u) = d(x,v)$ and it follows that $u = v$. Thus $\leq_x$ is antisymmetric.

Let $u \leq_x v$ and $v \leq_x w$. Since $u \leq_x v$, the vertex u lies on an x-v geodesic P and since $v \leq_x w$, the vertex v lies on an x-w geodesic Q. Then clearly the union of the geodesic P from x to v and the $(v - w)$-section of the geodesic Q is an x-w geodesic so that $u \leq_x w$. Hence $\leq_x$ is transitive. Thus $(V, \leq_x)$ is a poset. □

We will represent the poset by its *Hasse diagram*, the graph with vertex set $V(G)$ and an edge between x and y whenever $x < y$ and there is no vertex between x and y.

Definition 2. *Let G be a connected graph. For any vertex x in G, the geodesic graph at x of G, denoted by $P_x(G)$ is defined to be the Hasse diagram of the poset $(V, \leq_x)$.*

Remark 1.

(i) *$P_x(G)$ is simply denoted by P_x.*

(ii) *Since P_x is defined to be the Hasse diagram of $(V, \leq_x)$, P_x is a spanning subgraph of G.*

(iii) *Every edge incident with x in G will also be an edge in P_x.*

Example 1. *For the graph G given in Figure 1, all the geodesic graphs are given in Figure 2.*

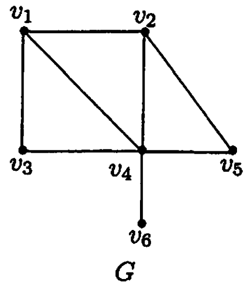


Figure 1

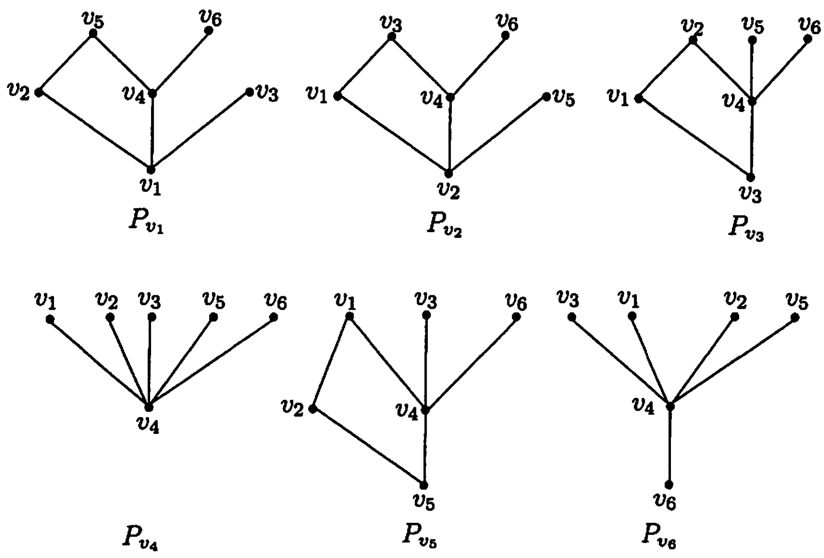


Figure 2

Definition 3. *Let G be a connected graph and x any vertex of G. Then S_x is the spanning subgraph of G in which uv is an edge if and only if uv lies on either an x-u geodesic or an x-v geodesic in G.*

In the following theorem, we prove that the subgraph S_x and the geodesic graph P_x of G are isomorphic.

Theorem 4. *For a connected graph $G, S_x \simeq P_x$.*

Proof. We have $V(S_x) = V(P_x) = V$. Define $f : V \to V$ by $f(v) = v$ for all $v \in V$. Let uv be an edge of S_x and hence uv is an edge of G and uv lies on either an x-u geodesic or an x-v geodesic in G. Hence $d(u, v) = 1$ in G and either $v \leq_x u$ or $u \leq_x v$ so that either u is a cover of v or v is a cover of u. Hence u and v are adjacent in P_x. The proof of the converse is similar. Hence $S_x \simeq P_x$. □

Because of Theorem 4, we are free to make use of the Definition 3 for P_x. Hence

Remark 2. *If $(x\ x_1\ \ldots x_n)$ is a geodesic in G, then it is a geodesic in P_x. In general, a geodesic $(v_1 v_2 \ldots v_n)$, where $v_1 \neq x$ in G need not be a geodesic in P_x.*

For the graph G in Figure 3, $(u\ u_1\ u_2\ v)$ is an u-v geodesic in G but it is not an u-v geodesic in P_x.

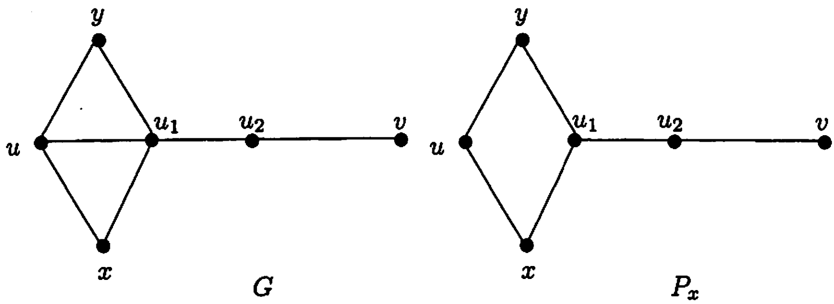


Figure 3

If uv is an edge of G, then clearly uv is an edge of an u-v geodesic so that uv is an edge of P_u. Hence $E(G) \subseteq \cup_{x \in V} E(P_x)$. Since the edges of P_x are in G, we make the following remark.

Remark 3. *For any connected graph G, $E(G) = \cup_{x \in V} E(P_x)$.*

Theorem 5. *For every vertex x in a connected graph G, P_x is a connected bipartite graph.*

Proof. Let u, v be any two distinct vertices of P_x. Then u and v are also vertices of G. Since G is connected, there exits a shortest path from x to u, say $P = (x\ u_1\ u_2 \ldots u_n\ u)$. Then by Remark 2, it is a shortest path from x to u in P_x. Similarly there is a shortest path $Q = (x\ v_1\ v_2 \ldots v_m\ v)$ from x to v. Let $v_j = u_i$ be the last vertex common to both P and Q. Then

$(u\ u_n\ u_{n-1} \ldots v_j\ v_{j+1} \ldots v_m\ v)$ is a path connecting u and v in P_x. Thus P_x is connected.

Let $V_1 = \{u \in V : d(x,u)$ is even in $P_x\}$ and $V_2 = \{u \in V : d(x,u)$ is odd in $P_x\}$. Suppose there is an edge uv joining two vertices u, v of V_1. Then either uv lies on an x-u geodesic or an x-v geodesic. In either case $d(x,u) = d(x,v) + 1$ or $d(x,v) = d(x,u) + 1$, which is not possible since both $d(x,u)$ and $d(x,v)$ are even. Similarly there cannot be an edge joining two vertices of V_2. Hence P_x is bipartite. □

Corollary 1. *If x is a vertex of a connected graph G such that $P_x = G$, then G is bipartite.*

Corollary 2. *For any vertex x in a connected graph G, $p-1 \leq |E(P_x)| \leq \left\lfloor \frac{p^2}{4} \right\rfloor$.*

Proof. Since P_x is a connected graph for every vertex x in G, $p-1 \leq |E(P_x)|$. Also, since P_x is a bipartite graph, it is triangle free so that by Theorem 1, $|E(P_x)| \leq \left\lfloor \frac{p^2}{4} \right\rfloor$. □

The bounds in Corollary 2 are sharp. For any odd cycle G on p vertices, the graph P_x is the path on p vertices for any vertex x so that $|E(P_x)| = p-1$. For the graph $G = C_4$, the cycle on four vertices, $P_x = G$ so that $|E(P_x)| = 4 = \left\lfloor \frac{p^2}{4} \right\rfloor$. Also, for the geodesic graph P_x given in Figure 3, both the inequalities are strict.

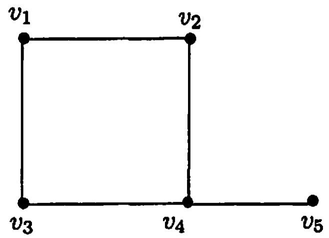
Theorem 6. *Let G be a connected graph. A vertex v in G is a pendent vertex of G if and only if v is a pendent vertex of P_x for every vertex x in G.*

Proof. Let v be a pendent vertex of G. Then P_x is a spanning subgraph of G and is connected, by Theorem 5, so v is a pendent vertex of P_x for every vertex x in G. Let v be a pendent vertex of P_x for every vertex x in G. If v is not a pendent vertex of G, then $deg\ (v) \geq 2$. It follows from Remark 1 (iii) that v is not a pendent vertex of P_v, which is a contradiction. □

Theorem 7. *Let e be a cut edge of G. Then e is an edge of P_x for every vertex x in G.*

Proof. Let $e = uv$ be a cut edge of G. Let G_1 and G_2 be the components of G if the edge uv were removed. Since uv is the only edge between vertices in G_1 and G_2, any path (in particular, any geodesic) from a vertex in G_1 to a vertex in G_2 must contain the edge uv. Therefore $uv \in P_x$ for any vertex x in G. □

The converse of Theorem 7 is false. For the graph G in Figure 4, the edge v_1v_2 is an edge in every P_x. However v_1v_2 is not a cut edge of G.



G

Figure 4

In the following theorem, we give the condition for which the converse of Theorem 7 is true.

Theorem 8. *An edge of G is a cut edge of G if and only if it is a cut edge of P_x for every vertex x in G.*

Proof. If e is a cut edge of G, then by Theorem 7, e is an edge of P_x for every vertex x in G. If e is not a cut edge of P_x for some vertex x in G, then by Theorem 2, the edge e lies on a cycle C in P_x and since P_x is a spanning subgraph of G, the edge e lies on the cycle C in G, which is a contradiction to e is a cut edge of G. Thus e is a cut edge of P_x for every x.

Conversely, let $e = uv$ be a cut edge of P_x for every vertex x in G. If uv is not a cut edge of G, then by Theorem 2, uv is an edge of some cycle in G. Let C be a smallest cycle in G containing the edge uv. Now we consider two cases.

Case i. Let C be an even cycle. Then every edge of C is an edge of P_u and hence C is also in P_u. Thus uv is an edge of the cycle C in P_u and so uv is not a cut edge of Pu, which is a contradiction to the assumption.

Case ii. Let C be an odd cycle of length $2n+1$. Let x be a vertex on the cycle C such that $d(x,u) = d(x,v) = n$. If uv is an edge of P_x, then either uv lies on an x-u geodesic or an x-v geodesic in G. In either case $d(x,v) < d(x,u)$ or $d(x,u) < d(x,v)$, which is a contradiction. Hence uv is not an edge of P_x and so it is not a cut edge of P_x, which is a contradiction to the assumption. Hence the theorem. □

In the following theorems we determine P_x for a few classes of graphs.

Theorem 9. *For any connected graph G, G is bipartite if and only if $P_x = G$ for every vertex x in G.*

Proof. Let G be a bipartite graph. To prove $P_x = G$ for every vertex x in G, it is enough to prove that every edge of G is an edge of P_x for every vertex x in G. If uv is an edge of G and not an edge of P_x for some vertex x, then the edge uv is neither in any x-u geodesic nor in any x-v geodesic in G. Let w be the last vertex common to an x-u geodesic say P and an x-v geodesic say Q in G. Then the union of the $(w-u)$ and $(w-v)$ - sections of the geodesics P and Q respectively and the edge uv is a cycle C in G. We claim that C is an odd cycle in G. Otherwise, C is an even cycle in G. Let z be the antipodal vertex to w in C. Then all the edges of C lie on an x-z geodesic and hence all the edges of C are in P_x, which is a contradiction to the assumption that uv is not an edge of P_x. Thus C is an odd cycle in G, which contradicts that G is a bipartite graph. Hence $P_x = G$ for every vertex x in G. The converse follows from Corollary 1. □

Corollary 3. *If G is a tree or an even cycle, then $P_x = G$ for every vertex x in G.*

Remark 4. *If G is an odd cycle say $(v_1\ v_2 \ldots v_{2n+1}\ v_1)$ of length $2n+1$, it is obvious that P_x is a path for any vertex x in G. In fact $P_{v_1} = (v_{n+1}\ v_n\ v_{n-1} \cdots v_2\ v_1\ \ v_{2n+1} \cdots v_{n+2})$. Thus we see that Corollary 3 is not true for an odd cycle.*

Remark 5. *For a graph G with $\Delta = p-1$, it follows from Remark 1 (iii) and Theorem 5 that the geodesic graph P_x is a star at x if and only if deg $x = p-1$. Thus it follows that a graph G is complete if and only if its geodesic graph P_x is a star at x for every vertex x in G.*

Theorem 10. *A graph G is geodetic if and only if P_x is a tree for every vertex x in G.*

Proof. Let G be a geodetic graph. Then it follows from Theorem 5 that P_x is a connected bipartite graph for every vertex x and hence it contains no odd cycles. Suppose that P_x contains an even cycle, then G also contains that even cycle, since P_x is a spanning subgraph of G. Let C be an even cycle of G of smallest length. Hence there are two different geodesics between any two antipodal vertices of C, which is a contradiction to G is a geodetic graph. Thus P_x is a tree for every vertex x in G.

Conversely, let P_x be a tree for every vertex x in G. Let u and v be two vertices of G and P and Q be any two different geodesics from u to v. Denote by w the last vertex common to both P and Q. Since P and Q are shortest paths, the $(w-v)$- sections of both P and Q are shortest $(w-v)$ paths. Now, since the lengths of both P and Q are same, the lengths of both the $(w-v)$- sections of P and Q are same. Hence both the $(w-v)$- sections of P and Q together form an even cycle C in G and hence C is also an even cycle in P_v, which contradicts that P_v is a tree. Thus G is a geodetic graph. □

It is proved in Theorem 7 that each cut edge of a graph G is an edge of P_x for every vertex x in G. This leads to the question "Which edges in a graph G belong to P_x for every vertex x in G?"

Theorem 11. *An edge e of a graph G is an edge of P_x for every vertex x in G if and only if e is not in any odd cycle of G.*

Proof. Let $e = uv$ be an edge of P_x for every vertex x in G. If uv is an edge of some odd cycle of G, let C be a smallest odd cycle of length $2n + 1$ in G having uv as an edge. Let x be a vertex of C such that $d(x, u) = d(x, v) = n$. Then clearly the edge uv is neither in any x-u geodesic nor in any x-v geodesic so that uv is not an edge of the geodesic graph P_x, which is a contradiction. The converse is proved by an argument as in the proof of Theorem 9. □

Remark 6. *For a bipartite graph G, it follows from Theorem 11 that every edge of G is also an edge of P_x for every vertex x in G.*

3. Acknowledgments

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