

*Article*

A Note on Acyclic Token Sliding Reconfiguration Graphs of Independent Sets

David Avis^{1,2}, and Duc A. Hoang^{3,*}¹ Graduate School of Informatics, Kyoto University, Japan² School of Computer Science, McGill University, Canada³ VNU University of Science, Vietnam National University, Hanoi, Vietnam* **Correspondence:** hoanganhduc@hus.edu.vn

Abstract: We continue the study of Token Sliding (reconfiguration) graphs of independent sets initiated by the authors in an earlier paper [*Graphs Comb.* 39.3, 59, 2023]. Two of the topics in that paper were to study which graphs G are Token Sliding graphs and which properties of a graph are inherited by a Token Sliding graph. In this paper we continue this study specializing on the case of when G and/or its Token Sliding graph $\text{TS}_k(G)$ is a tree or forest, where k is the size of the independent sets considered. We consider two problems. The first is to find necessary and sufficient conditions on G for $\text{TS}_k(G)$ to be a forest. The second is to find necessary and sufficient conditions for a tree or forest to be a Token Sliding graph. For the first problem we give a forbidden subgraph characterization for the cases of $k = 2, 3$. For the second problem we show that for every k -ary tree T there is a graph G for which $\text{TS}_{k+1}(G)$ is isomorphic to T . A number of other results are given along with a join operation that aids in the construction of TS_k -graphs.

Keywords: Reconfiguration graph, Independent set, Token sliding, Forest, Realizability, Acyclic property

2020 Mathematics Subject Classification: 05C05, 05C99

1. Introduction

In a *reconfiguration variant* of a computational problem (e.g., SATISFIABILITY, INDEPENDENT SET, VERTEX-COLORING, etc.), a *transformation rule* that describes an *adjacency relation* between *feasible solutions* (e.g., satisfying truth assignments, independent sets, proper vertex-colorings, etc.) of the problem is given. One of the main goals is to decide whether there is a sequence of adjacent feasible solutions that “reconfigures” one given solution into another. Another way of looking at these reconfiguration problems is via the so-called *reconfiguration graph*—a graph whose nodes are feasible solutions and two nodes are adjacent if one can be obtained from the other by applying the given rule exactly once. The mentioned question now becomes deciding whether there is a path between two given nodes in the reconfiguration graph. Recently, *reconfiguration problems* have been intensively studied from different perspectives [1–4].

One of the most well-studied reconfiguration variants of INDEPENDENT SET is the so-called TOKEN SLIDING problem, which was first introduced by Hearn and Demaine [5] in 2005. We refer readers to [1, 2, 4] and the references therein for more details. Surprisingly, though TOKEN SLIDING has been well-

investigated, the realizability and structural properties of its corresponding reconfiguration graph—the one which we will refer to as the TS_k -graph (which stands for *Token Sliding (Reconfiguration) graph (of a graph)*)—have not been studied until recently [6]. On the other hand, when considering either general vertex subsets, (maximum) matchings, dominating sets, or proper vertex-colorings of a graph as the “input feasible solutions”, their corresponding reconfiguration graphs have been very well-characterized [3, 7, 8]. A closely related and important graph called the *Fibonacci cube* (which, from the reconfiguration viewpoint, can also be called the *Token Addition/Removal (Reconfiguration) graph of paths*) has indeed been studied since 1993 [9]. (See the survey [10] for more details.) Very recently, research on the diameter of another closely related graph called the *Token Jumping (Reconfiguration) graph* has been systematically initiated by Bousquet et al. [11] (and indeed several results presented in [11] also hold for TS_k -graphs).

For any graph-theoretic terminology and notation not defined here, we refer readers to [12]. Given a graph $G = (V, E)$ and a fixed integer $k \geq 2$. For two sets X, Y , we sometimes use $X + Y$ and $X - Y$ to indicate $X \cup Y$ and $X \setminus Y$. We abbreviate $X \cup \{u\}$ (resp., $X \setminus \{u\}$) by $X + u$ (resp., $X - u$). We use $N_G(u)$, or simply just $N(u)$ when the graph G is clear from the context, to denote the (*open*) *neighborhood* of u , i.e., set of all vertices in G that are adjacent to u . The *closed neighborhood* of u , denoted by $N_G[u]$ or simply $N[u]$, is the set $N_G(u) + u$. The *degree* of u , denoted by $\deg_G(u)$, is nothing but the size of $N_G(u)$. An *independent set* (or *stable set*) of G is a vertex subset I such that for every $u, v \in I$ we have $uv \notin E(G)$. The TS_k -graph of G , denoted by $\text{TS}_k(G)$, takes all size- k independent sets of G as its nodes and two nodes I, J are *adjacent (under Token Sliding (TS))* if there exist two vertices $u, v \in V(G)$ such that $I - J = \{u\}$, $J - I = \{v\}$, and $uv \in E(G)$. Two graphs G and H are *isomorphic*, denoted by $G \simeq H$, if there exists a bijective mapping $f : V(G) \rightarrow V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. A graph G is called a TS_k -graph if there exists a graph H such that $G \simeq \text{TS}_k(H)$. A *forest* is a graph having no cycles (i.e., it is *acyclic*) and a connected forest is a *tree*. A TS_k -*tree/forest* is a TS_k -graph which is also a tree/forest. Figure 1 illustrates a TS_2 -tree on six vertices (right). In [6], the authors studied various properties of the family of TS_k -graphs. For a graph

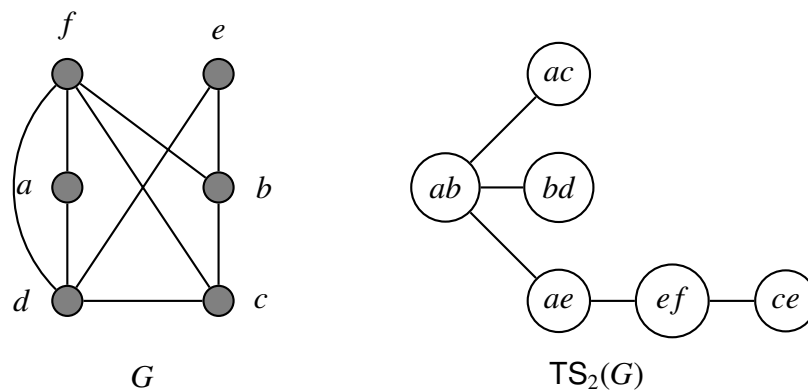


Figure 1. A Graph G with $\text{TS}_2(G) = D_{1,3,2}$. Each Node ab Represents a Size-2 Stable Set of G

G , two of the questions studied were:

- (Q1) What are necessary and sufficient conditions for G so that $\text{TS}_k(G)$ is a forest?
- (Q2) What are necessary and sufficient conditions for G to be a TS_k -graph?

In this paper, we study these two questions for the case when G is a tree or a forest.

The *union* $G \cup H$ of two (labelled) graphs G and H is the graph with $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. When vertices and edges of G and H are considered distinct regardless of their labels, we say that $G \cup H$ is the *disjoint union* of G and H , and write $G + H$ instead of $G \cup H$ to distinguish from their union. We respectively denote by K_n , P_n , and C_n the *complete graph*, *path*,

and *cycle* on n vertices. $K_{m,n}$ ($m \leq n$) is the *complete bipartite graph* whose two partite sets are of sizes m and n respectively. $K_{1,n}$ is also called a *star*—a tree obtained by attaching n leaves to a central vertex. A family of graphs that we will use in the sequel generalizes stars and paths. For fix positive integers n, r, s , let $D_{r,n,s}$ be the tree obtained from P_n by appending r leaves at one end and s leaves at the other. Note that $D_{1,1,s}$ is the star $K_{1,s+1}$ and $D_{1,n,1}$ is the path P_{n+2} . Figure 1 illustrates $D_{1,3,2}$ (right). An n -ary tree is a rooted tree in which each node has at most n children. Any tree with maximum degree at most $n + 1$ can be rooted at a vertex with degree at most n (e.g., a leaf) to produce a n -ary tree. In particular, a 2-ary tree is nothing but the well-known binary tree.

In the next section, we begin by partially answering (Q1) when G is a tree/forest and $k \in \{2, 3\}$ and conclude the section by conjecturing for $k \geq 4$. Then, before addressing (Q2) for some trees/forests, in particular k -ary trees and $D_{r,n,s}$, we define an important graph operation which, under certain conditions, can be used for combining two TS_k -graphs by taking their union to obtain a new one. The final section of the paper gives some concluding remarks.

2. Results on (Q1)

In this section, we prove necessary and sufficient conditions for a tree/forest G such that $\text{TS}_k(G)$ is acyclic for $k \in \{2, 3\}$, partially answering (Q1).

We begin with some definitions and observations. The *complement* \overline{G} of a graph G is the graph with $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{uv : uv \notin E(G)\}$. The *size- m matching*, denoted by mK_2 , is the graph obtained by taking the disjoint union of m copies of K_2 . Observe that $\text{TS}_2(2K_2) \simeq C_4$. We label vertices in a $D_{r,n,s}$ ($r, n, s \geq 1$) as follows: Vertices of P_n are labelled p_1, \dots, p_n . The r leaves attached to p_1 are u_1, \dots, u_r and the s leaves attached to p_n are v_1, \dots, v_s . $D_{2,2,2}$ is shaped like an **H** and $\text{TS}_2(D_{2,2,2})$ contains a cycle C_8 whose vertex-set is $\{u_1v_1, u_1p_2, u_1v_2, p_1v_2, u_2v_2, u_2p_2, u_2v_1, p_1v_1\}$. Indeed, respectively from Lemma 1 of [6] and Figure 2, if a n -vertex graph G is either \overline{C}_n ($n \geq 5$) or a graph in the list \mathcal{G} described in Figure 2 (which includes $2K_2$ and $D_{2,2,2}$), the graph $\text{TS}_2(G)$ contains a cycle. Additionally, we have:

Lemma 1. (a) For $k \geq 2$, $\text{TS}_k(2K_2 + nK_1)$ contains a cycle C_4 if $n \geq k - 2$ otherwise it is acyclic.

(b) For $k \in \{2, 3\}$, $s \geq 1$, $\text{TS}_k(D_{1,n,s})$ contains a cycle C_4 if $n \geq 2k - 1$ otherwise it is acyclic.

(c) For $k \in \{2, 3\}$ and $r, s \geq 2$, $\text{TS}_k(D_{r,n,s})$ contains a cycle C_8 if $n \geq 2k - 2$ otherwise it is acyclic.

Proof. (a) If $1 \leq n < k - 2$, there is no size- k independent set in $2K_2 + nK_1$, thus its TS_k -graph is obviously acyclic. Otherwise, let $I \subseteq V(nK_1)$ be an arbitrary independent set of size $k - 2$, and let $E(2K_2) = \{ab, cd\}$. Then, $\{I + a + c, I + a + d, I + b + c, I + b + d\}$ induces a C_4 in $\text{TS}_k(2K_2 + nK_1)$.

(b) Observe that if $n \geq 2k - 1$, $D_{1,n,s}$ contains an induced $2K_2 + (k - 2)K_1$, which can be obtained by taking u_1p_1 and p_nv_1 as edges of $2K_2$ and the remaining $k - 2$ independent vertices from the path $D_{1,n,s} - \{u_1, p_1, p_2, p_{n-1}, p_n, v_1, \dots, v_s\}$ on $n - 4$ vertices. (Since $n \geq 2k - 1$, this path has an independent set of size at least $\lceil (n - 4)/2 \rceil \geq \lceil (2k - 5)/2 \rceil = k - 2$.) Then, using a similar argument as in (a) we have $\text{TS}_k(D_{1,n,s})$ contains a C_4 .

On the other hand, if $1 \leq n \leq 2k - 2$ for $k \in \{2, 3\}$, since $D_{1,n-1,s}$ is always an induced subgraph of $D_{1,n,s}$ for $n \geq 2$, it follows that if $\text{TS}_2(D_{1,n-1,s})$ has a cycle then so is $\text{TS}_2(D_{1,n,s})$. Therefore, it suffices to show that $\text{TS}_k(D_{1,2k-2,s})$ is acyclic for $k \in \{2, 3\}$. Indeed, based on the number of tokens placed on the path $u_1p_1 \dots p_n$ (which is at most three), one can verify that each component of $\text{TS}_k(D_{1,2k-2,s})$ is either an isolated vertex, a path, or a star.

(c) Observe that if $n \geq 2k - 2$, $D_{r,n,s}$ contains the independent sets $I + u_1 + v_1, I + u_1 + p_n, I + u_1 + v_s, I + p_1 + v_1, I + p_1 + v_s, I + u_r + v_1, I + u_r + p_n, I + u_r + v_s$, where $I = \emptyset$ when $n = 2$ and otherwise I is an independent set of the path $p_2 \dots p_{n-1}$ of size $k - 2$. (Note that $p_2 \dots p_{n-1}$ has an independent set of size at most $\lceil (n - 2)/2 \rceil \geq k - 2$.) They indeed induce a C_8 in $\text{TS}_k(D_{r,n,s})$. On the other hand, if $1 \leq n \leq 2k - 3$ for $k \in \{2, 3\}$, using a similar case-analysis as in (b), one can verify that each component of $\text{TS}_k(D_{r,n,s})$ is either an isolated vertex, a path, or a star, and

G	$TS_2(G)$

Figure 2. A List \mathcal{G} of n -vertex Graphs G ($4 \leq n \leq 7$) such that if $TS_2(H)$ is Acyclic for Some Graph H then H does not Contain any Member of $\mathcal{G} \cup \{C_n : n \geq 5\}$ as an Induced Subgraph

therefore it is acyclic.

□

We are now ready to show the necessary and sufficient conditions for a tree/forest G such that $TS_k(G)$ is acyclic, where $k \in \{2, 3\}$.

Proposition 1. *Let T be a tree. Then $TS_2(T)$ is acyclic if and only if T is $\{2K_2, D_{2,2,2}\}$ -free.*

Proof. (\Rightarrow) Suppose to the contrary that either $2K_2$ or $D_{2,2,2}$ is an induced subgraph of T . In the first case it follows from the discussion above that $TS_2(T)$ contains a C_4 and in the second case that it contains a C_8 .

(\Leftarrow) We assume that $TS_2(T)$ contains a cycle and show that it must contain one of the two forbidden subgraphs. Firstly, suppose that T is a path P_n . Since $TS_2(T)$ contains a cycle, it follows from Lemma 1(b) that $n \geq 5$ and so T contains an induced $2K_2$.

We now assume T has a vertex of at least degree 3. We will construct a copy T' of T by initially choosing a vertex a of maximum degree in T and letting $T' = N[a]$. Note that $TS_2(T')$ is acyclic. We add edges from T to T' and show after each addition that either T' contains a forbidden subgraph, so we are done, or that $TS_2(T')$ remains acyclic so that $T \neq T'$.

Let b be a child of a of highest degree, c be a child of next highest degree, and d be any other child. Since $TS_2(T')$ is acyclic $T \neq T'$ and b must have $r \geq 1$ children. Let e be a child of b with maximum degree. We add $N[b]$ to T' obtaining a copy of $D_{r,2,s}$, where $s = \deg_T(a) - 1 \geq 2$. If $r \geq 2$, we have the required forbidden induced subgraph. If $r = 1$ then by Lemma 1(b) $TS_2(T')$ is acyclic, so there must be extra edges to add to T' . If c has a child y then $\{b, c, e, y\}$ induce a $2K_2$. Otherwise, e must have at least one child g . Adding eg to T' we obtain $2K_2$ as an induced subgraph on $\{a, d, e, g\}$. This completes the proof.

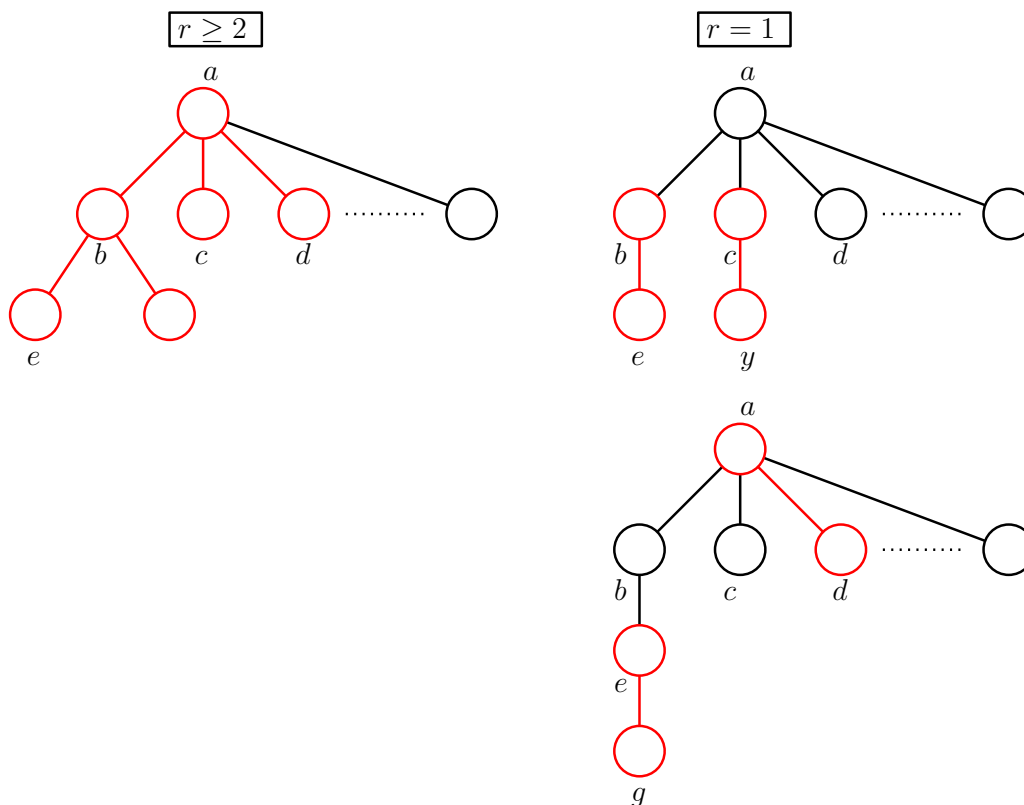


Figure 3. Illustration for Proposition 1: Some trees T' Containing $N[b]$ whose TS_2 -graphs have a Cycle. Here r is the Number of Children of b . Copies of $2K_2$ and $D_{2,2,2}$ are Marked by Red Color

□

Corollary 1. *Let T be a tree. Then $TS_2(T)$ is acyclic if and only if T is either $K_{1,s}$ or $D_{1,2,s}$ for some positive integer s .*

Proof. The proof of Proposition 1 can be viewed as an algorithm that takes a tree T and either terminates with $T = T'$ being one of the trees in the corollary or finds a forbidden induced graph in T . \square

Corollary 2. *Let F be a forest. Then $\text{TS}_2(F)$ is acyclic if and only if F is $\{2K_2, D_{2,2,2}\}$ -free.*

Proof. We prove that $\text{TS}_2(F)$ contains a cycle if and only if F contains one of the graphs in $\{2K_2, D_{2,2,2}\}$ as an induced subgraph.

Suppose that $\text{TS}_2(F)$ contains a cycle. Since the independent sets have size two, both vertices of each independent set must lie in the same connected component T of F . By Proposition 1, the tree T must have either $2K_2$ or $D_{2,2,2}$ as an induced subgraph.

Conversely if F contains $2K_2$ or $D_{2,2,2}$ as an induced subgraph then $\text{TS}_2(F)$ contains respectively a C_4 or a C_8 . \square

Moving to the case of stable sets of size three, the conditions for trees and forests differ slightly. We deal with the tree case first.

Proposition 2. *Let T be a tree. Then $\text{TS}_3(T)$ is acyclic if and only if T is $\{2K_2 + K_1, D_{2,4,2}\}$ -free.*

Proof. The structure of the proof is the same as for Proposition 1. However, there are more cases to consider.

(\Rightarrow) Suppose to the contrary that either $2K_2 + K_1$ or $D_{2,4,2}$ is an induced subgraph of T . In the first case it follows that $\text{TS}_3(T)$ contains a C_4 and in the second case that it contains a C_8 .

(\Leftarrow) We assume that $\text{TS}_3(T)$ contains a cycle and show that it must contain one of the two forbidden subgraphs. The first part of the proof is essentially the same as for Proposition 1 with minor modifications. Firstly suppose that T is a path P_n . Since $\text{TS}_3(T)$ contains a cycle it follows from Lemma 1(b) that $n \geq 7$ and so T contains an induced $2K_2 + K_1$.

We now assume T has a vertex of at least degree 3. We will construct a copy T' of T by initially choosing a vertex a of maximum degree in T and letting $T' = N[a]$. Note that $\text{TS}_3(T')$ is acyclic. We add edges from T to T' showing after each addition that either T' contains a forbidden subgraph, so we are done, or that $\text{TS}_3(T')$ remains acyclic so that $T \neq T'$.

Let b be a child of a of highest degree, c be a child of next highest degree, and d be any other child. Since $\text{TS}_3(T')$ is acyclic $T \neq T'$ and b must have $r \geq 1$ children. Let e be a child of b with maximum degree. If c has a child y then $\{b, c, d, e, y\}$ induce a $2K_2 + K_1$ and we are done. Otherwise we add $N[b]$ to T' obtaining a copy of $D_{r,2,s}$, where $s = \deg_T(a) - 1 \geq 2$. By Lemma 1(c), $\text{TS}_3(T')$ is acyclic and so $T \neq T'$. There are two cases:

($r \geq 2$) Let f be a second child of b and let g be a child of e . Adding eg to T' we obtain $2K_2 + K_1$ as an induced subgraph on $\{a, d, e, f, g\}$.

($r = 1$) Since e is the only child of b it must have children. Let $t \geq 1$ be the number of children of e and let h be the child of e of maximum degree. We add $N[e]$ to T' obtaining a copy of $D_{t,3,s}$ and $\text{TS}_3(T')$ is acyclic by Lemma 1(c). There are two subcases:

($t \geq 2$) Let i be any other child of e . Since $\text{TS}_3(T')$ is acyclic h must have at least one child j . We have now constructed an induced $2K_2 + K_1$ on $\{a, d, h, i, j\}$.

($t = 1$) If h has a single child k add hk to T' which is a copy of $D_{1,4,s}$ and again by Lemma 1(c) $\text{TS}_3(T')$ is acyclic. So k has a child l . Adding kl to T' it contains an induced P_7 and we find the forbidden subgraph $2K_2 + K_1$ on vertices $\{a, d, e, k, l\}$. Otherwise, h has at least two children including vertices k and m . Adding edges hk and hm to T' we obtain the forbidden subgraph $D_{2,4,2}$. This completes the proof. \square

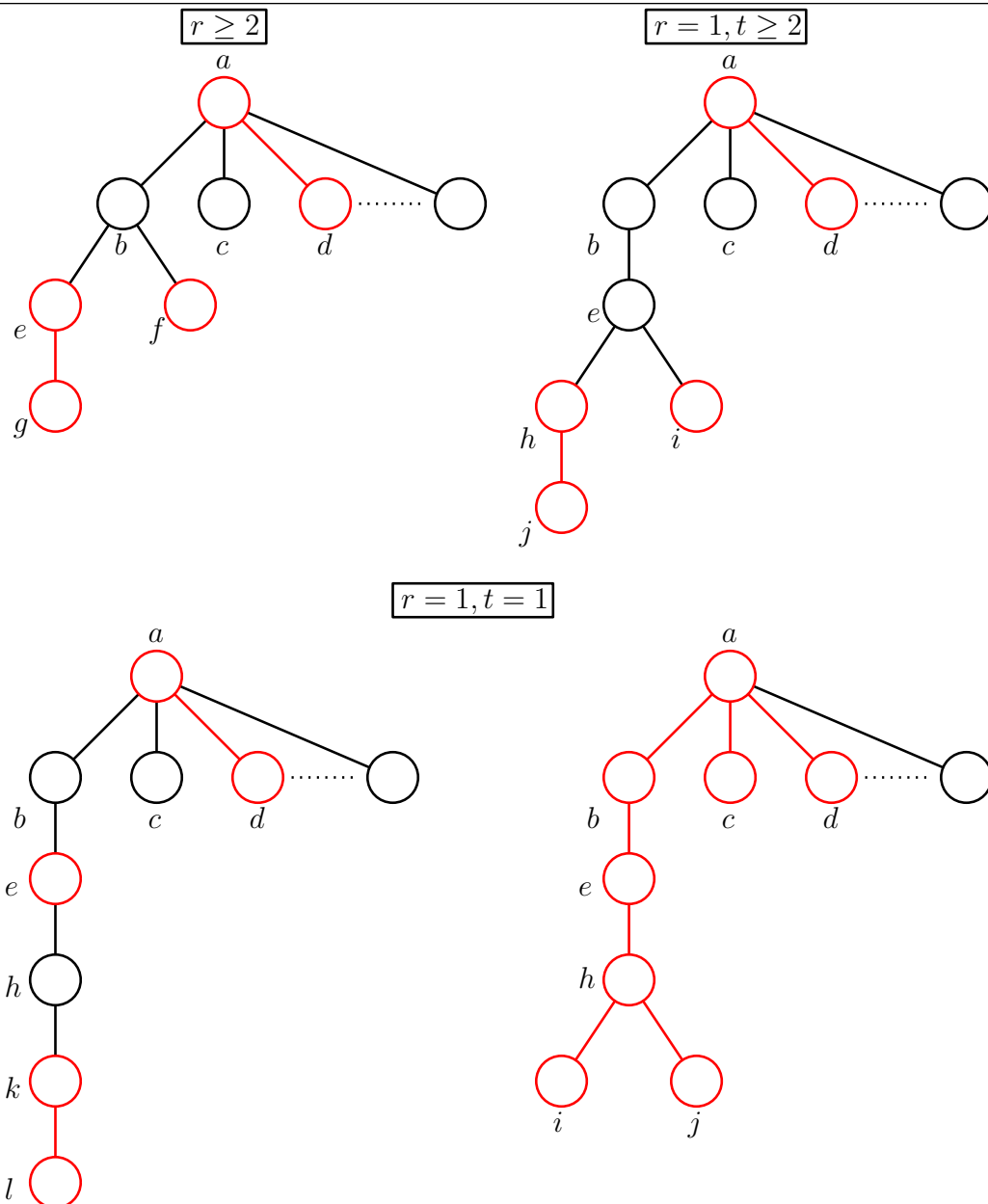


Figure 4. Illustration for Proposition 2: Some trees T' Containing $N[b]$ whose TS_3 -graphs have a cycle. Here r and t are Respectively the Number of Children of B and its Child E . Copies of $2K_2 + K_1$ and $D_{2,4,2}$ are Marked by Red Color

Corollary 3. *Let T be a tree. Then $TS_3(T)$ is acyclic if and only if for some positive integer s , T is either $K_{1,s}$, $D_{1,n,s}$ where $n \leq 4$, or $D_{r,n,s}$ where $r \geq 2$ and $n \leq 3$.*

Proof. The proof of Proposition 2 can be viewed as an algorithm that takes a tree T and either terminates with $T = T'$ being one of the trees in the corollary or finds a forbidden induced graph in T showing that $TS_3(T)$ has a cycle. □

Corollary 4. *Let F be a forest. Then $TS_3(F)$ is a forest if and only if F is $\{2K_2 + K_1, D_{2,2,2} + K_1, D_{2,4,2}\}$ -free.*

Proof. We prove that $TS_3(F)$ contains a cycle if and only if F contains one of the graphs in $\{2K_2 + K_1, D_{2,2,2} + K_1, D_{2,4,2}\}$ as an induced subgraph.

Suppose that $TS_3(F)$ contains a cycle C . Since the independent sets have size three, there are three cases to consider. Firstly, if the three vertices of each independent set in C lie in the same connected component T of F , by Proposition 2, the tree T must have either $2K_2 + K_1$ or $D_{2,4,2}$ as an

induced subgraph. Secondly, suppose two of the vertices of each stable set lie in the same connected component T of F , which must have at least two connected components. Thus, C induces a cycle in $\text{TS}_2(T)$. So by Proposition 1, the tree T must have either $2K_2$ or $D_{2,2,2}$ as an induced subgraph. Since F has at least two components, F contains $2K_2 + K_1$ or $D_{2,2,2} + K_1$. Finally, suppose each vertex of each stable set lies in a different component of F , which therefore has at least three components. At least two of these components must be non-trivial, i.e., contain an edge. Therefore, F contains an induced $2K_2 + K_1$.

Conversely, suppose F contains $2K_2 + K_1$, $D_{2,2,2} + K_1$ or $D_{2,4,2}$ as an induced subgraph. Then $\text{TS}_3(F)$ contains a C_4 in the first instance or a C_8 in the other two. \square

For $k \geq 4$, we have the following proposition.

Proposition 3. *Let F be a forest. For $k \geq 4$, if F contains either $2K_2 + (k-2)K_1$, or $D_{2,2,2} + (k-2)K_1$, or $D_{2,4,2} + (k-3)K_1$ as an induced subgraph, $\text{TS}_k(F)$ has a cycle.*

Proof. One can verify that $\text{TS}_2(2K_2)$ contains a C_4 , and $\text{TS}_2(D_{2,2,2})$ and $\text{TS}_3(D_{2,4,2})$ both contain a C_8 . As a result, so do $\text{TS}_k(2K_2 + (k-2)K_1)$, $\text{TS}_k(D_{2,2,2} + (k-2)K_1)$, and $\text{TS}_k(D_{2,4,2} + (k-3)K_1)$, respectively. Consequently, $\text{TS}_k(F)$ has a cycle, as desired. \square

We conclude this section with the following conjecture for $k \geq 4$.

Conjecture 1. *Let F be a forest. For $k \geq 4$, if $\text{TS}_k(F)$ is a forest, F is $\{2K_2 + (k-2)K_1, D_{2,2,2} + (k-2)K_1, D_{2,4,2} + (k-3)K_1\}$ -free.*

3. H -join and H -decomposition

Before considering (Q2), in this section, we describe an operation for combining TS_k -graphs to produce new ones. We first define a family of *base graphs* as follows. Let V be a set of $k+1$ vertices including two vertices labelled u and v . Then $B_k(V, uv)$ is the graph with vertex set V and single edge uv . We have $\text{TS}_k(B_k(V, uv)) = K_2$ whose two vertices are labelled by the independent sets $V-u$ and $V-v$. Next, we define the H -join operation and its inverse.

Definition 1. *Vertex-labelled graphs G_1 and G_2 are H -consistent if the (possibly empty) intersection of their vertex sets define the same (possibly empty) common induced subgraph H . The H -join of H -consistent graphs G_1 and G_2 is the graph $H(G_1, G_2)$ with $V(H(G_1, G_2)) = V(G_1) \cup V(G_2)$. The edges $E(H(G_1, G_2))$ consist of $E(G_1) \cup E(G_2)$ plus all edges vw with $v \in V(G_1) \setminus V(H)$ and $w \in V(G_2) \setminus V(H)$.*

Recall that a (*vertex*) *cut-set* in a connected graph G is a vertex set W such that $G - W$ is disconnected. We extend this definition to the case where G is disconnected by allowing $W = \emptyset$. We say that W decomposes G into two (not necessarily connected) induced subgraphs G_1 and G_2 for which $V(G_1) \cap V(G_2) = W$ and $V(G_1) \cup V(G_2) = V(G)$. If $G - W$ has more than two (connected) components, the decomposition is not unique.

Definition 2. *Let G be a vertex-labelled graph. Let $W \subset V(\overline{G}) = V(G)$ decompose the complement \overline{G} into \overline{G}_1 and \overline{G}_2 . Let H be the subgraph of G induced by W . We say that G can be H -decomposed into G_1 and G_2 .*

It follows from the definitions that if $G = H(G_1, G_2)$ then G can be H -decomposed into G_1 and G_2 , and vice versa. It is easy to verify that the size- k independent sets of $H(G_1, G_2)$ are the union of those of G_1 and those of G_2 .

As an example consider the two 4-vertex graphs G_1 and G_2 that are paths with edge sets $E(G_1) = \{ad, bc, cd\}$ and $E(G_2) = \{ad, ae, eb\}$. These share a common induced subgraph H with $V(H) = \{a, b, d\}$ and $E(H) = \{ad\}$. We have $V(H(G_1, G_2)) = \{a, b, c, d, e\}$ and $E(H(G_1, G_2)) = \{ad, ae, bc, cd, ce, be\}$. Note that $\text{TS}_2(G_1)$ is the path with edges $\{ac - ab, ab - bd\}$ and that $\text{TS}_2(G_2)$ is

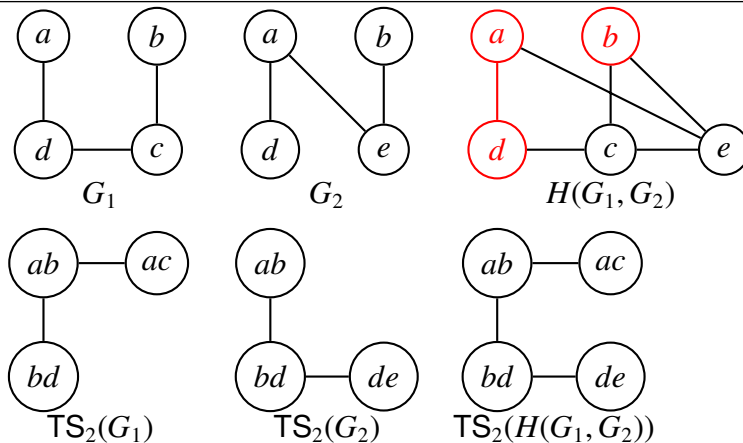


Figure 5. The Graphs G_1 , G_2 , $H(G_1, G_2)$, and their Corresponding TS_2 -graphs. Here $TS_2(H(G_1, G_2)) = TS_2(G_1) \cup TS_2(G_2)$

the path with edges $\{ab - bd, bd - de\}$. It can be verified that $TS_2(H(G_1, G_2))$ is the path with edges $\{ac - ab, ab - bd, bd - de\}$ which is the union of two paths $TS_2(G_1)$ and $TS_2(G_2)$. (See Figure 5.)

Now consider the graph G_3 which is the path with edges $\{ad, cd, ce\}$. G_1 and G_3 share a common induced subgraph H with $V(H) = \{a, c, d\}$ and $E(H) = \{ad, cd\}$. We have $E(H(G_1, G_3)) = \{ad, bc, be, cd, ce\}$. Note that $TS_2(G_3)$ is the path with edges $\{ac - ae, ae - de\}$. In this case, $TS_2(H(G_1, G_3))$ is the graph with edges $\{ab - ac, ac - ae, ae - de, de - bd, bd - ab, ab - ae\}$ which is the union of $TS_2(G_1)$, $TS_2(G_3)$, and the two additional edges $de - bd, ab - ae$. (See Figure 6.)

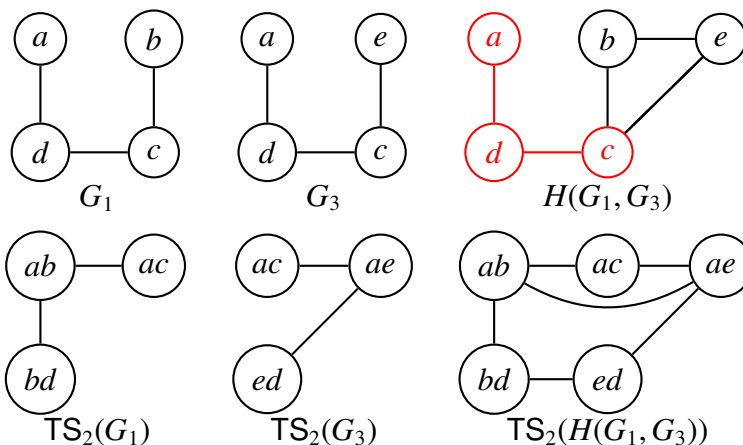


Figure 6. The Graphs G_1 , G_3 , $H(G_1, G_3)$, and their Corresponding TS_2 -graphs. Here $TS_2(H(G_1, G_3)) \neq TS_2(G_1) \cup TS_2(G_3)$

As the last example in this section, consider the graphs G_4 and G_5 as follows. G_4 is the cycle with edges $\{ae, eb, bc, cd, ad\}$ and G_5 is the graph with edges $\{ae, eb, bc, ag, eg, bg\}$. G_4 and G_5 shares a common induced subgraph H with $V(H) = \{a, e, b, c\}$ and $E(H) = \{ae, eb, bc\}$. We have $E(H(G_4, G_5)) = \{ae, eb, bc, cd, ad, ag, eg, bg, dg\}$. In this case, $TS_2(H(G_4, G_5))$ is the (non-acyclic) graph with edges $\{ab - ac, ac - ce, ce - de, de - bd, ab - bd, ac - cg, ce - cg\}$ which is the union of $TS_2(G_4)$ and $TS_2(G_5)$. (See Figure 7.)

In the next proposition, we show how to compute the TS_k -graph of an H -join, generalizing the examples given above.

Proposition 4. Let $k \geq 2$ and let G_1 and G_2 be two H -consistent graphs. $TS_k(H(G_1, G_2))$ is the union of $TS_k(G_1)$, $TS_k(G_2)$ and for every pair of k -element independent sets S_1 in G_1 and S_2 in G_2 satisfying

$$|S_1 \cap V(H)| = |S_2 \cap V(H)| = |S_1 \cap S_2| = k - 1, \tag{1}$$

the edge between S_1 and S_2 .

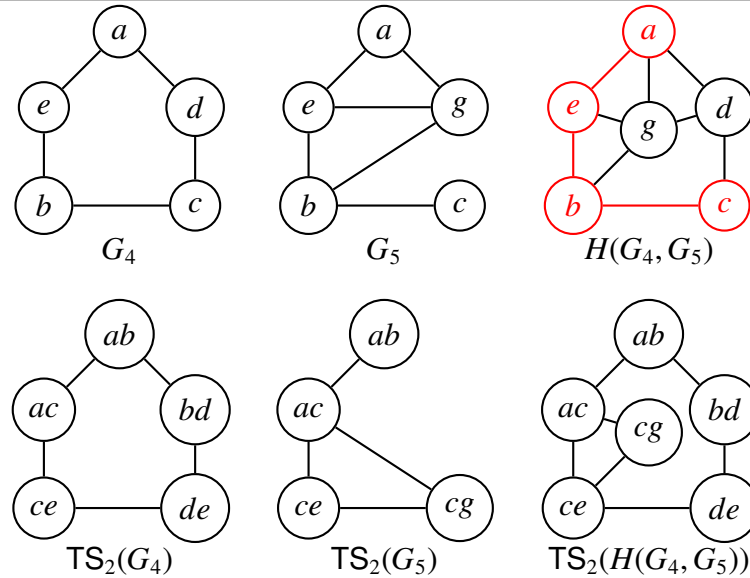


Figure 7. The Graphs G_4 , G_5 , $H(G_4, G_5)$ and their Corresponding (non-acyclic) TS_2 -graphs. Here $\text{TS}_2(G_4, G_5) = \text{TS}_2(G_4) \cup \text{TS}_2(G_5)$.

Proof. As remarked, the k -element independent sets of $H(G_1, G_2)$ are the same as the union of those of G_1 and G_2 . Therefore, $V(\text{TS}_k(H(G_1, G_2))) = V(\text{TS}_k(G_1)) \cup V(\text{TS}_k(G_2))$. Next, consider an edge in $E(\text{TS}_k(G_1))$ (respectively, $E(\text{TS}_k(G_2))$). It is a token-slide between two independent sets S_1 and S_2 in G_1 (respectively, G_2). This remains as a token-slide in $H(G_1, G_2)$. Therefore, $E(\text{TS}_k(G_1)) \cup E(\text{TS}_k(G_2)) \subseteq E(\text{TS}_k(H(G_1, G_2)))$. Now, consider an edge in $E(\text{TS}_k(H(G_1, G_2)))$ between two independent sets S_1 and S_2 . If both of these are independent sets are in G_1 (respectively, G_2) then the edge is also present in $E(\text{TS}_k(G_1))$ (respectively, $E(\text{TS}_k(G_2))$). Otherwise, we may assume the edge in $E(\text{TS}_k(H(G_1, G_2)))$ has as endpoints an independent set S_1 in G_1 (but not G_2) and an independent set S_2 in G_2 (but not G_1). We have $S_1 \cap S_2 \subset V(H)$ and since S_1 and S_2 are adjacent $|S_1 \cap S_2| = k - 1$. It follows that $|S_1 \cap V(H)| = |S_2 \cap V(H)| = k - 1$ and so condition (1) is satisfied. We have shown that each edge in $E(\text{TS}_k(H(G_1, G_2)))$ is either in $\text{TS}_k(G_1)$, $\text{TS}_k(G_2)$ or satisfies condition (1), proving the proposition. \square

For two H -consistent graphs G_1 and G_2 , we say that $H(G_1, G_2)$ is k -crossing free if there are no k -element independent sets satisfying condition (1) of Proposition 4. For example, one can verify that the graphs $H(G_1, G_2)$ in Figure 5 and $H(G_4, G_5)$ in Figure 7 are both k -crossing free, while the graph $H(G_1, G_3)$ in Figure 6 is not. The following result will be used for constructing TS_k -trees/forests.

Corollary 5. *Let $k \geq 2$ and let G_1 and G_2 be two H -consistent graphs. $H(G_1, G_2)$ is k -crossing free if and only if*

$$\text{TS}_k(H(G_1, G_2)) = \text{TS}_k(G_1) \cup \text{TS}_k(G_2). \quad (2)$$

Proof. If $H(G_1, G_2)$ is k -crossing free then (2) follows from Proposition 4. Otherwise, there exist k -element independent sets S_1 in G_1 and S_2 in G_2 satisfying condition (1) of Proposition 4. This implies that $\text{TS}_k(H(G_1, G_2))$ contains an additional edge between S_1 and S_2 . \square

Therefore, if $H(G_1, G_2)$ is k -crossing free and both $\text{TS}_k(G_1)$ and $\text{TS}_k(G_2)$ are acyclic, then so is $\text{TS}_k(H(G_1, G_2))$. The reason for allowing H to be empty in defining an H -join is that the corollary then applies to vertex disjoint graphs G_1 and G_2 , since in this case $H(G_1, G_2)$ is trivially k -crossing free. Therefore, we can create new reconfiguration graphs that are forests from those that are trees (or forests).

The following result follows from the relationship between H -join and H -decomposition discussed above.

Corollary 6. *If G can be H -decomposed into G_1 and G_2 and $H(G_1, G_2)$ is k -crossing free then $\text{TS}_k(G)$ can be decomposed into $\text{TS}_k(G_1) \cup \text{TS}_k(G_2)$.*

4. Results on (Q2)

We currently have no general necessary and sufficient conditions for when a forest F is a TS_k -graph, but we present some partial results in this section. Firstly, we recall that in [6] it is shown that P_n is a TS_k -graph for all $n \geq 1$ and $k \geq 2$ and $K_{1,n}$ is a TS_k -graph if and only if $n \leq k$. In this section, we show how to construct acyclic TS_k -graphs from graphs that have a single edge using the join operation that was introduced in Section 3. We show that it gives an alternate method of constructing TS_k -graphs which are paths and stars. Moreover, this operation can also be applied to construct more general TS_k -trees/forests, especially members of the classes k -ary trees and $D_{r,n,s}$.

4.1. Paths and Stars Revisited

Using just the base graphs and the H -join operation defined in Section 3, we can obtain large families of TS_k -trees/forests. We begin with paths. For any $k \geq 2$, let $J_k = \{b_1, \dots, b_k\}$ be an independent set of size k and define the base graph $B_k^i = B_k(J_{k-2} \cup \{a_i, a_{i+1}, a_{i+2}\}, a_i a_{i+2})$ and let $G_2 = B_k^i$.

Proposition 5. *For $i \geq 2$, G_i and B_k^i are H -consistent with H being the independent set $J_{k-2} \cup \{a_i, a_{i+1}\}$. Define $G_{i+1} := H(G_i, B_k^i)$. Then*

$$\text{TS}_k(G_{i+1}) = \text{TS}_k(G_i) \cup \text{TS}_k(B_k^i) \simeq P_{i+1}.$$

Proof. We will prove by induction, for $i \geq 2$, that $\text{TS}_k(G_i)$ is the path P_i with vertices labelled $J_{k-2} \cup \{a_j, a_{j+1}\}, j = 1, \dots, i$. For the base case $i = 2$, we observe that indeed $\text{TS}_k(B_k^i)$ is a P_2 with vertices labelled $J_{k-2} \cup \{a_1, a_2\}$ and $J_{k-2} \cup \{a_2, a_3\}$.

For the inductive step we observe that, for $i \geq 2$, G_i and B_k^i are H -consistent with H the independent set $J_{k-2} \cup \{a_i, a_{i+1}\}$. To verify that $H(G_i, B_k^i)$ is k -crossing free, note that the only independent set we need to consider in B_k^i is $J_{k-2} \cup \{a_{i+1}, a_{i+2}\}$. In the path P_i which is $\text{TS}_k(G_i)$, the candidate independent sets are $J_{k-2} \cup \{a_j, a_{j+1}\}, j = 1, \dots, i$. Their intersection with B_k^i is J_{k-2} which has cardinality $k - 2$. Therefore, condition (1) of Proposition 4 is not satisfied, which indeed confirms that $H(G_i, B_k^i)$ is k -crossing free. We define $G_{i+1} := H(G_i, B_k^i)$. By Corollary 5, $\text{TS}_k(G_{i+1})$ is the union of the above labelled P_i with a P_2 with endpoints $J_{k-2} \cup \{a_i, a_{i+1}\}$ and $J_{k-2} \cup \{a_{i+1}, a_{i+2}\}$. This is the required P_{i+1} . \square

An easy inductive argument based on the H -join in the proposition shows that, for $i \geq 2$, G_i is isomorphic to $\overline{P}_{n+1} \cup J_{k-2}$, a result proved in Corollary 5(a) of [6]. (Observe that the vertex a_{i+1} in G_i is adjacent to every a_j for $1 \leq j \leq i - 1$.)

Next we consider graphs G_i such that $\text{TS}_k(G_i)$ is the star $K_{1,i}$. For $k \geq 2$ and $1 \leq i \leq k$, let $I_k = \{a_1, \dots, a_k\}$ be an independent set of size k , define the base graph $C_k^i = B_k(I_k + b_i, a_i b_i)$ and let $G_1 = C_k^1$.

Proposition 6. *For $k \geq 2$ and $1 \leq i \leq k$, G_i and C_k^{i+1} are H -consistent with H being the independent set I_k . Define $G_{i+1} := H(G_i, C_k^{i+1})$. Then*

$$\text{TS}_k(G_{i+1}) = \text{TS}_k(G_i) \cup \text{TS}_k(C_k^{i+1}) \simeq K_{1,i+1}.$$

Proof. We will prove by induction, for $i \geq 1$, that $\text{TS}_k(G_i)$ is the star $K_{1,i}$ with centre labelled I_k and leaves labelled $I_k + b_j - a_j, j = 1, \dots, i$. For the base case $i = 1$, we observe that indeed $\text{TS}_k(C_k^1)$ is a $K_{1,1}$ with centre labelled I_k and leaf labelled $I_k + b_1 - a_1$.

For the inductive step we observe that, for $i \geq 1$, G_i and C_k^{i+1} are H -consistent with H the independent set I_k . To verify that $H(G_i, C_k^{i+1})$ is k -crossing free, note that the only independent set we

need to consider in C_k^{i+1} is $I_k + b_{i+1} - a_{i+1}$. In the above labelled $K_{1,i}$ which is $\text{TS}_k(G_i)$, the candidate independent sets for condition (1) of Proposition 4 are $I_k + b_j - a_j, j = 1, \dots, i$. Their intersection with $I_k + b_{i+1} - a_{i+1}$ has cardinality $k - 2$. Therefore, condition (1) is not satisfied. We define $G_{i+1} := H(G_i, C_k^{i+1})$. By Corollary 5, $\text{TS}_k(G_{i+1})$ is the union of the above labelled $K_{1,i}$ and a $K_{1,1}$ with centre also labelled I_k and leaf labelled $I_k + b_{i+1} - a_{i+1}$. This is the required $K_{1,i+1}$. \square

4.2. k -ary Trees

In this section, we show that for each $k \geq 2$, every k -ary tree is a TS_{k+1} -graph (Proposition 7). Next, we show that any tree T is an induced subgraph of some TS_2 -forest (Proposition 8). Moreover, we state and prove necessary and sufficient conditions for T to be an induced subgraph of some TS_2 -tree (Proposition 9). Additionally, when $T = K_{1,n}$, we describe a sufficient condition for T to be an induced subgraph of some TS_k -tree (Proposition 10).

We begin by defining a canonical vertex labelling. In this subsection, for any integer n , define $I_n := \{a_1, \dots, a_n\}$ and $J_n := \{b_1, \dots, b_n\}$.

Definition 3. Let $k \geq 2$ and G be a graph for which $T := \text{TS}_{k+1}(G)$ is a k -ary tree. We say that G and T are canonically labelled if

- (a) the root of T is labelled I_{k+1} ,
- (b) the $d \leq k$ children of the root are labelled $I_{k+1} - a_i + b_i, i = 1, \dots, d$,
- (c) the labels $b_j, j = d + 1, \dots, k$ (if any) are not used, and
- (d) all other nodes in T receive a label S such that $|I_{k+1} \cap S| \leq k - 1$.

It is clear that labelling $K_{1,d}, d \leq k$ according to (a) and (b) with root the centre of the star is a canonical labelling. In this subsection, we will show that every k -ary tree has canonical labelling hence proving it is a TS_{k+1} -graph. First, we give a lemma that shows how to combine canonically labelled k -ary trees to get a larger k -ary tree that is canonically labelled.

Lemma 2. For integers $k \geq 2$ and $1 \leq i \leq d \leq k$, let G_i be a graph for which $\text{TS}_{k+1}(G_i)$ a canonically labelled k -ary tree. We can construct a canonically labelled k -ary tree T isomorphic to the tree formed by choosing a new root and adjoining it to the root of each T_i .

Proof. The proof consists of showing that we can make a series of H -joins between the leaves of a canonically labelled $K_{1,d}$ and the roots of the canonically labelled trees $T_i, i = 1, \dots, d$, after a suitable relabelling. Suppose the root of T_i has $n_i \leq k$ children. We relabel the vertices in the underlying graphs as follows:

- (i) relabel vertices of the G_i not in $I_{k+1} \cup J_k$ to be distinct, ie, for $1 \leq i \leq j \leq d$, we have $V(G_i) \cap V(G_j) \subseteq I_{k+1} \cup J_k$,
- (ii) for $i = 1, \dots, d, j = 1, \dots, n_i$ set $b_j \leftarrow b_j^i$, where the b_j^i were previously unused, and
- (iii) for $i = 1, \dots, d$, set $a_i \leftarrow a_{k+1}$ and $a_{k+1} \leftarrow b_i$.

By an abuse of notation, for simplicity we let for $i = 1, \dots, d, G_i$ and T_i refer to the relabelled graphs and trees. Item (i) ensures that the only labels shared between two trees are in $I_{k+1} \cup J_k$, (ii) ensures that all labels from J_k in the T_i are given unique labels to avoid clashes, and (iii) gives the root of T_i a correct label to be a child of a new root labelled I_k . We note that after relabelling b_i only appears in T_i, a_i does not appear in T_i and the only labels shared between the T_i are in I_k . Furthermore all tree vertices have unique labels.

Next take a canonically labelled graph G^0 such that $\text{TS}_{k+1}(G^0) \simeq K_{1,d}$, with the centre of the star labelled I_{k+1} . For $i = 1, \dots, d$, we claim that the H -join $G^i := H(G^{i-1}, G_i)$ is well-defined, k -crossing free, and $\text{TS}_{k+1}(G^i)$ is canonically labelled. To see this, note at that iteration $i, V(G^{i-1}) \cap V(G_i) = I_{k+1} - a_i + b_i$ which is the label of the root of T_i and a leaf of $\text{TS}_{k+1}(G^{i-1})$. Definition 3(d) implies that condition (1) of Proposition 4 is not satisfied. Therefore by Corollary 5, $\text{TS}_{k+1}(G^i)$ is obtained from

$TS_{k+1}(G^{i-1})$ by appending T_i to the corresponding leaf in $TS_{k+1}(G^{i-1})$. The conditions of Definition 3 are satisfied so $TS_{k+1}(G^i)$ is canonically labelled. At the end of iteration d , $T := TS_{k+1}(G^d)$ is the required tree. \square

The construction described in the proof is illustrated in Figure 8. We may now prove the main

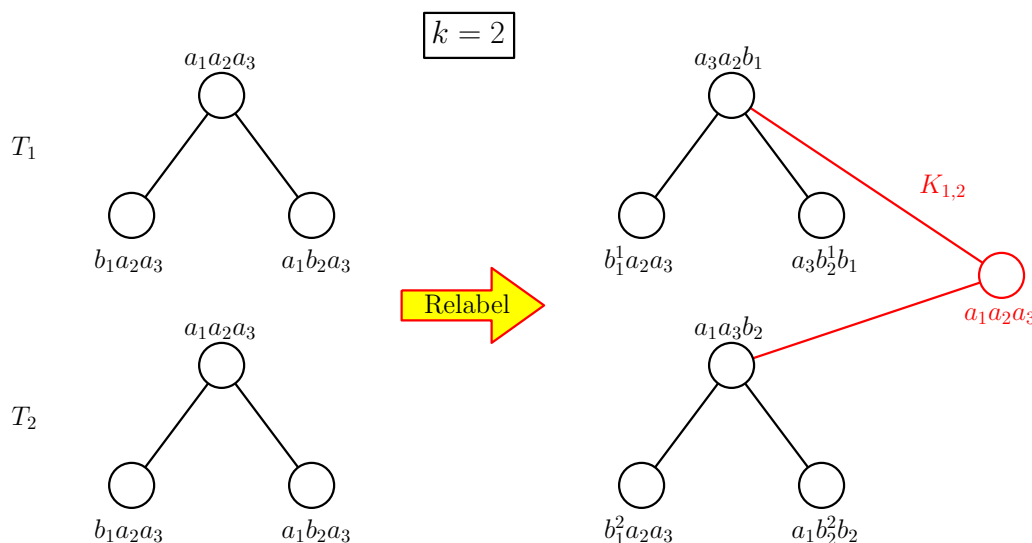


Figure 8. Construction of $D_{2,3,2}$ from two $K_{1,2}$ s

result of this section.

Proposition 7. For every k -ary tree T , there is a canonically labelled graph G such that $T \simeq TS_{k+1}(G)$.

Proof. Suppose that the root r of T has $d \leq k$ children. We prove the proposition by induction on the height t of T , which is the length of the longest path to a leaf from the root. If $t = 1$ then $T \simeq K_{1,d}$ and so has a canonically representation as described following Definition 3. Otherwise, by deleting r we obtain d subtrees $T_i, i = 1, \dots, d$, which are also k -ary trees, with height less than t . Therefore, by induction each T_i can be represented by a canonically labelled graph G_i . It follows from Proposition 2 that we can perform d H -joins to obtain a canonically labelled graph G for which $T \simeq TS_{k+1}(G)$. \square

As noted in Section 4 of [6], $K_{1,k+1}$ is an example of a k -ary tree that is not an TS_k -graph so the proposition is tight. Nevertheless, if we add a sufficient number of isolated vertices to $K_{1,t}$, for $t > k$, it becomes a TS_2 -graph—a result we will now prove in general. We will need a special labelling of a tree that will be defined next.

Definition 4. A tree T is well-labelled if

- (a) the root r of T is labelled ab ,
- (b) the d children of r have roots labelled $r_i = bc_i, i = 1, \dots, d - 1$ and $r_d = ac_d$,
- (c) the only labels containing a and b are $ab, ac_d, bc_i, 1 \leq i \leq d - 1$, and
- (d) for $i = 1, \dots, d$ label c_i only occurs in the subtree with root r_i .

We note that there is nothing special about the ordering of the subtrees of r . The subtree rooted at r_i can play the role of r_d by relabelling those two subtrees with the exchanges $a \leftrightarrow b$ and $c_i \leftrightarrow c_d$, which leaves T well-labelled. As an example, for $d \geq 1$ we can well-label $K_{1,d}$ simply by using (a) and (b). Consider the graph G defined by $V(G) = \{a, b\} \cup \{c_i : 1 \leq i \leq d\}$ and $E(G) = \{ac_i, c_ic_d : 1 \leq i \leq d - 1\} \cup \{bc_d\}$. Furthermore let $J = \{c_i c_j : 1 \leq i < j \leq d - 1\}$. Then it is not hard to verify that $TS_2(G) \simeq K_{1,d} + (d - 1)(d - 2)K_1$, where the $K_{1,d}$ is well-labelled and the K_1 are labelled by the set J . This motivates the following definition.

Definition 5. A tree T is well-labelled by a labelled graph G if there is an integer n such that $\text{TS}_2(G) \simeq T + nK_1$ and T is well-labelled.

We now show the following general result.

Proposition 8. For every tree T there is a graph G and integer n such that T is well-labelled by G and $\text{TS}_2(G) \simeq T + nK_1$.

Proof. The proof is by induction on N , the number of nodes in a given tree T . As noted above, the proposition is true for all stars $K_{1,t}$ and these act as base cases. For the inductive step, assume the proposition is true for all trees on N nodes and consider a tree T with $N + 1$ nodes. If T is a star we are done. Otherwise, let r be the root of T and assume r has degree d with its children r_i being roots of subtrees T_i , $1, \dots, d$. We may also assume that T_d is a subtree of T with height at least one. We now construct two trees from T . The first, T^1 consists of T with subtree T_d deleted and a pendant vertex added to its root r . The second, T^2 consists of T_d with a pendant vertex added to its root r_d . By induction, there are integers n_1, n_2 and graphs G^1, G^2 which well-label T^1 and T^2 such that $\text{TS}_2(G^1) \simeq T^1 + n_1K_1$ and $\text{TS}_2(G^2) \simeq T^2 + n_2K_1$. Apart from the vertex labels used in Definition 4, we may assume the vertex labels in G^1 and G^2 are different.

We will show that G^1 and a relabelled G^2 can be H -joined and that this will identify the pendant edges added to T^1 and T^2 to give us back T . In T^1 we note that root r is labelled ab , and by relabelling subtree roots if necessary, that the added pendent vertex can be labelled ac_d . In T^2 the root r_d is also labelled ab and we can again assume the added pendant vertex is labelled ac_d . In T^2 we interchange the labels $b \leftrightarrow c_d$ and set $c_i \leftarrow c'_i, i = 1, \dots, d - 1$, for labels c'_i that are unused in either T^1 or T^2 . Let G^3 and T^3 denote the relabelled G^2 and T^2 . Setting $H = \{a, b, c_d\}$, we have $V(G^1) \cap V(G^3) = H$. H induces the same subgraph, containing the single edge bc_d , in both G^1 and G^3 . G^1 and G^3 are H -consistent and since $k = 2$ and their vertex sets are otherwise disjoint, condition (1) of Proposition 4 is not satisfied. Let $G^4 = H(G^1, G^3)$. Applying Corollary 5 we have that

$$\begin{aligned} T^4 &:= \text{TS}_2(G^4) \simeq \text{TS}_2(G^1) \cup \text{TS}_2(G^3) \\ &\simeq \{T^1 + n_1K_1\} \cup \{T^3 + n_2K_1\} \simeq T + (n_1 + n_2)K_1. \end{aligned}$$

is well-labelled by G^4 . This proves the proposition. \square

The proof of the proposition is illustrated in Figure 9. The proposition tells us that for every tree T there is a graph G for which $\text{TS}_2(G)$ is forest containing T as an induced subgraph. Therefore, there can be no forbidden induced subgraph characterization of which *forests* are TS_2 -graphs. However, this does not imply that there can be no forbidden induced subgraph characterization of which *trees* are TS_2 -graphs. Indeed, in the next propositions, we present some of such characterizations.

Proposition 9. Let T be a tree. Then there exists a TS_2 -tree containing T if and only if T is a 3-ary tree.

Proof. (\Leftarrow) In the proof of Proposition 8, we see that isolated vertices are only added when the base case of a star appears as a subproblem. Therefore, it suffices to consider only the case $T = K_{1,t}, 1 \leq t \leq 4$. As we have noted, neither $K_{1,3}$ nor $K_{1,4}$ are TS_2 -graphs. It is not hard to see that there is a G^1 such that $\text{TS}_2(G^1) \simeq K_{1,3} + K_1$. However, by adding an extra vertex to G^1 , we can construct a graph G^2 such that $\text{TS}_2(G^2) \simeq D_{1,3,2}$. Furthermore, we can construct a graph G^3 by applying H -join to two copies of G^2 with slightly different vertex-labellings such that $\text{TS}_2(G^3)$ is isomorphic to a P_7 with two pendant vertices attached to the midpoint of the path. (See Figure 10.) Thus, it follows that when $T = K_{1,t}, 1 \leq t \leq 4$, we can embed it as an induced subgraph of a tree $T' = \text{TS}_2(G)$, for some graph G (see Figure 10). Our proof of the if direction is complete.

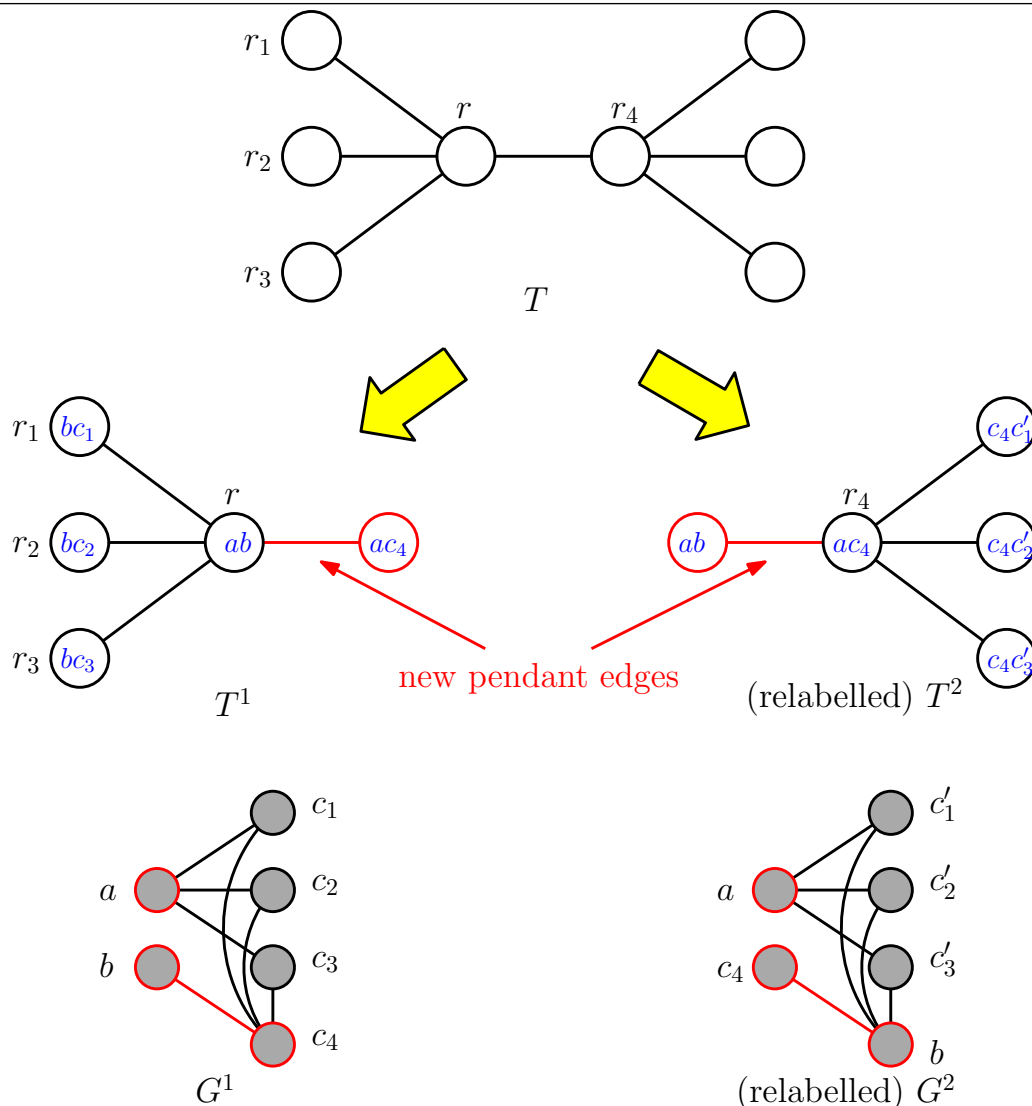


Figure 9. Illustrating Proposition 8

(\Rightarrow) We show that if T is a k -ary tree but not a 3-ary tree for $k \geq 4$ then there does not exist any TS_2 -tree T' containing T (as an induced subgraph). (By definition, any k -ary tree is also a ℓ -ary tree for $\ell \geq k$.) Let x be a vertex of T whose degree is at least five. (Since T is a k -ary tree but not a 3-ary tree, such a vertex x exists.)

Suppose to the contrary that T' exists, i.e., there exists a graph G' such that $T' \simeq \text{TS}_2(G')$ contains T . Without loss of generality, assume that x is labelled by ab , where $\{a, b\}$ is a size-2 stable set of G' . By the pigeonhole principle, we may further assume that three neighbors x_1, x_2 , and x_3 of x are labelled ac, ad , and ae , respectively. Since T' is a tree, it follows that cd, ce , and de are respectively the labels of y_1, y_2 , and y_3 where y_i is not adjacent to any of $\bigcup_j \{x_j\} + x + \bigcup_{j \neq i} \{y_j\}$ for $1 \leq i, j \leq 3$. It follows that T' contains the labelled graph $F \simeq K_{1,3} + 3K_1$ and therefore G' must contain the labelled graph $G \simeq K_{1,3} + K_1$, both described in Figure 11, as an induced subgraph.

Since $T' \simeq \text{TS}_2(G')$ is a tree and G' contains G , it follows that G' has exactly one non-trivial component C (having more than two vertices) and C contains G , otherwise G' must contain an induced $2K_2$ and by Proposition 1 its TS_2 -graph is not a tree, a contradiction.

- **Case 1:** $a \in V(C)$. By definition, the distance from a to any of b, c, d, e in G' must be at least two. If there is a path of length at least two between a and one of c, d, e not passing through b , the graph G' contains a $2K_2$, a contradiction. Thus, any path between a and one of c, d, e must go through b . Moreover, if there is a path of length at least three between a

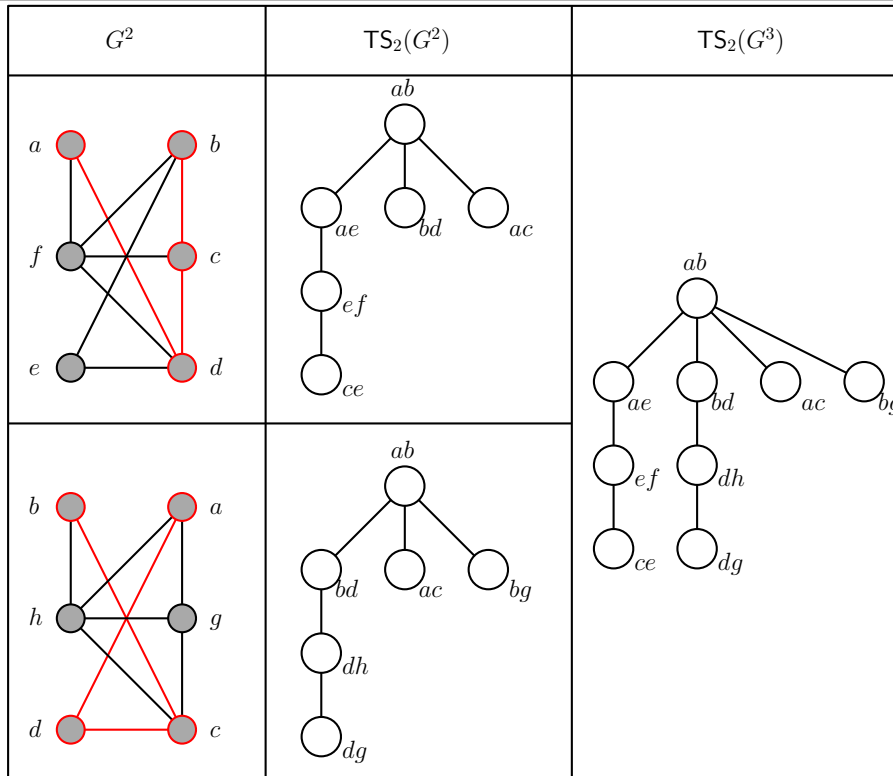


Figure 10. Taking H -join of Two Copies of G^2 , where H is the Path $adcb$, Results a Graph G^3 such that $TS_2(G^3)$ is Isomorphic to a P_7 with Two Pendant Vertices Attached to the Midpoint of the Path

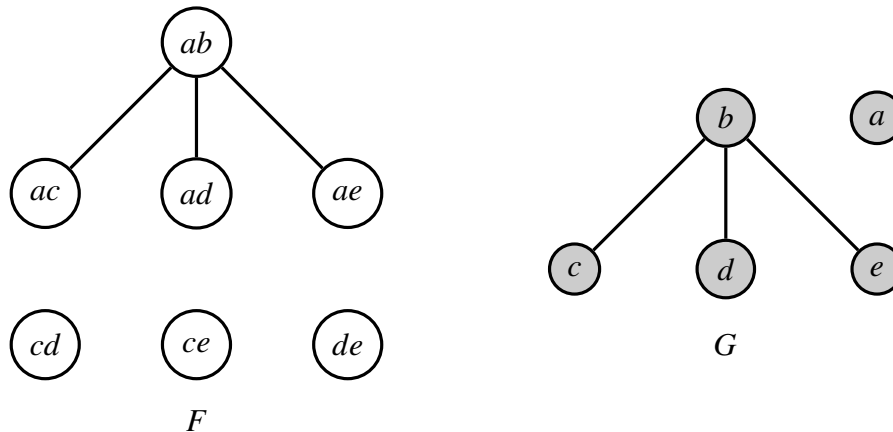


Figure 11. The Graphs F and G in the proof of Proposition 9

and b not passing through any of c, d, e , again the graph G' contains a $2K_2$, a contradiction. Since $a \in V(C)$, it follows that a and b must have a common neighbor in G' , say f . Observe that for each $y \in V(C) - \{a, b, c, d, e, f\}$, y must be adjacent to b in G' , otherwise G' either contains $2K_2$ or $D_{2,2,2}$ and again by Proposition 1 its TS_2 -graph is not a tree, a contradiction. However, this implies that $TS_2(C)$ must be a forest and since G' has exactly one non-trivial component C , we have $TS_2(G')$ is also a forest, a contradiction.

- **Case 2:** $a \notin V(C)$. In this case, there are two types of size-2 stable sets of G' : those containing a and those do not. Since G' contains G , each type has at least one member. Moreover, since a is isolated (the only non-trivial component is C and a is not in it), no member from one type is adjacent to a member from another type in $TS_2(G')$, which means $TS_2(G')$ is indeed disconnected, a contradiction.

In the above cases, we proved that some contradiction must happen. Our proof is complete.



Indeed, for $K_{1,n}$, in general we have

Proposition 10. *There exists a TS_k -tree T containing $K_{1,n}$ if $n \leq 2k$.*

Proof. From either [6] or Proposition 6, the proposition holds for $n \leq k$. (Indeed, in this case, $T = K_{1,n}$.) Thus, it suffices to consider $k+1 \leq n \leq 2k$. For each $i \in \{1, \dots, n-k\}$, let $A_i = \{1, \dots, k\} - i$.

Let $I_k = \{a_1, \dots, a_k\}$ and $B_n = \{b_1, \dots, b_n\}$. We construct a graph G^0 such that $TS_k(G^0) \simeq K_{1,n} + (n-k)(k-1)K_1$. Let $I_k = \{a_1, \dots, a_k\}$ and $B_n = \{b_1, \dots, b_n\}$. Let $V(G) = I_k + B_n$. Vertices in B_n form a graph $K_n - M$ where M is the matching that contains $b_i b_{k+i}$ for $1 \leq i \leq n-k$. Additionally, for each $i \in \{1, \dots, k\}$, we add an edge in G^0 between a_i and both b_i and b_{k+i} . Observe that $V(TS_k(G^0))$ consists of I_k , the sets $I_k - a_i + b_i$ ($1 \leq i \leq k$), $I_k - a_i + b_{k+i}$ ($1 \leq i \leq n-k$), and $(I_k - a_i + b_i) - a_j + b_{k+i}$ ($1 \leq i \leq n-k$ and $j \in A_i$). Moreover, one can verify that the independent sets $(I_k - a_i + b_i) - a_j + b_{k+i}$ are isolated in $TS_k(G^0)$ and the remaining independent sets form a $K_{1,n}$ in which I_k is adjacent to every other set. In short, G^0 is indeed our desired graph.

For each $i \in \{1, \dots, n-k\}$, we construct a graph G^i whose TS_k -graph is a star $K_{1,k-1}$ as follows. Let $V(G^i) = (I_k - a_i + b_i) + \bigcup_{j \in \{1, \dots, k\} - i} \{c_j^i\}$. Vertices in $\bigcup_{j \in A_i} \{c_j^i\}$ form a clique in G^i of size $k-1$. We also add an edge in G^i between a_j and c_j^i for each $j \in A_i$. From either [6] or Proposition 6, one can verify that $TS_k(G^i) \simeq K_{1,k-1}$ as desired. For each $i \in \{1, \dots, n-k\}$ and $j \in A_i$, we construct a graph G_j^i whose TS_k -graph is a K_2 as follows. Let $V(G_j^i) = (I_k - a_i + b_i) - a_j + b_{k+i} + c_j^i$. The only edge in G_j^i is the one joining c_j^i and b_{k+i} . From either [6] or Proposition 5, one can verify that $TS_k(G_j^i) \simeq K_2$ as desired.

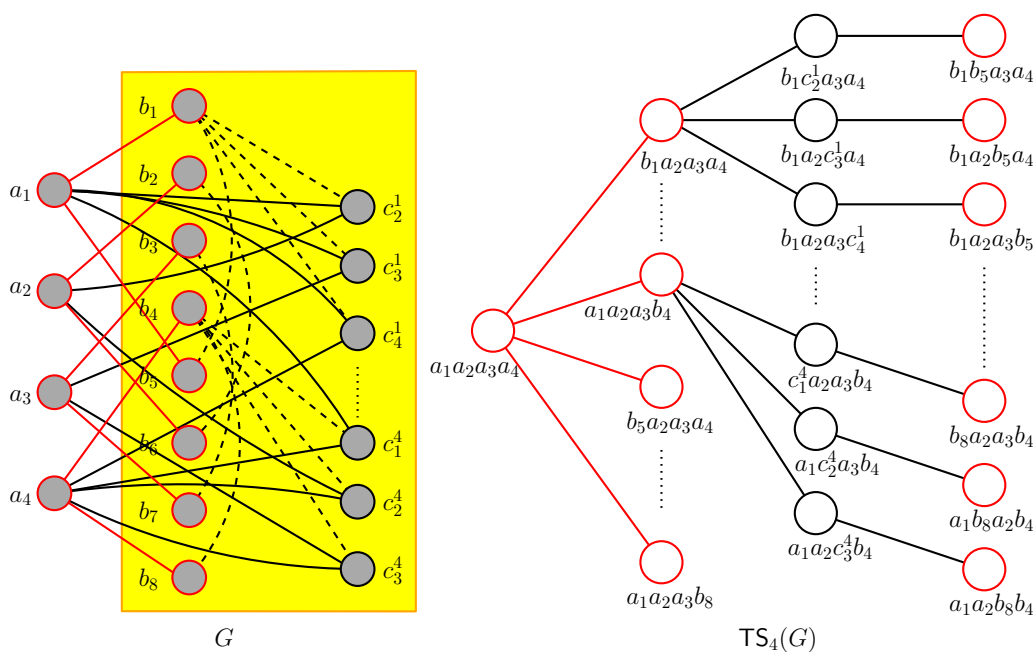


Figure 12. Construction of a Graph G such that $TS_4(G)$ is a Tree Containing $K_{1,8}$. Vertices of G in the Yellow Box form a Clique Having all Dashed Edges Removed. The Red Induced Subgraph of G forms a Graph G^0 whose $TS_4(G^0) \simeq K_{1,8} + 12K_1$

Now, we construct a graph G whose TS_k -graph is a tree containing $K_{1,n}$ as follows. For convenience, we assume that for each $i \in \{1, \dots, n-k\}$ the set $A_i = \{1, \dots, k\} - i$ can be enumerated as $\{j_1, \dots, j_{k-1}\}$. We define $\mathcal{K}_{j_0}^i = G^i$ and $\mathcal{K}_{j_p}^i = H_{j_p}(\mathcal{K}_{j_{p-1}}^i, G_{j_p}^i)$ for $j_p \in A_i$ where H_{j_p} is the stable set $(I_k - a_i + b_i) - a_{j_p} + c_{j_p}^i$ for $p \in \{1, \dots, k-1\}$. Observe that the graphs $\mathcal{K}_{j_{p-1}}^i$ and $G_{j_p}^i$ are H_{j_p} -consistent, which implies that $\mathcal{K}_{j_p}^i$ are well-defined. Moreover, one can also directly verify that the sets $(I_k - a_i + b_i) - a_j + c_j^i$ and $(I_k - a_{i'} + b_{i'}) - a_{j'} + c_{j'}^{i'}$ always differ in at least two members, which means the condition (1) of Proposition 4 is not satisfied. In short, for each $i \in \{1, \dots, n-k\}$, we obtain the

graph \mathcal{K}_{jk-1}^i whose TS_k -graph is isomorphic to the one obtained from $K_{1,k-1}$ by replacing each edge with a P_3 . Next, we define $\mathcal{K}^0 = G^0$ and $\mathcal{K}^i = H^i(\mathcal{K}^{i-1}, G^i)$ where $i \in \{1, \dots, n-k\}$ and H^i is the subgraph induced by $(I_k - a_i + b_i) + b_{k+i}$. Observe that the graphs \mathcal{K}^i are well-defined because \mathcal{K}^{i-1} and G^i are H^i -consistent. Moreover, we have I_k and each $(I_k - a_i + b_i) - a_j + c_j^i$ for $1 \leq i \leq n-k$ and $j \in A_i$ always differ in at least two members. It follows that the condition (1) of Proposition 4 is not satisfied. In short, we finally obtain the graph $G = \mathcal{K}^{n-k}$ whose TS_k -graph is indeed a tree containing $K_{1,n}$ as desired. \square

Unfortunately, we have not been able to show whether the reverse statement of Proposition 10 also holds. We conclude this section with the following open problems:

Problem 1. For every $k \geq 3$ and tree T , is there a graph G such that $\text{TS}_k(G)$ is a forest containing T as an induced subgraph?

Problem 2. For every $k \geq 3$ and $(k+1)$ -ary tree T , is there a graph G such that $\text{TS}_k(G)$ is a tree containing T as an induced subgraph?

Problem 3. Does there exist a TS_k -tree T containing $K_{1,n}$ for $n > 2k$?

4.3. $D_{r,n,s}$

We now consider graphs in the $D_{r,n,s}$ family for whose TS_k -graphs are trees and show how they can be constructed by the H -join operation. We remark that when $n = 1$, $D_{r,n,s}$ is nothing but a star $K_{1,r+s}$ and this case was considered in [6] and revisited in Proposition 6. Furthermore, it follows from Proposition 7 that for $n, k \geq 2$ and $1 \leq r \leq s \leq k-1$, $D_{r,n,s}$ is a $(k-1)$ -ary tree and so by Proposition 7 it is a TS_k -graph. The reverse statement does not hold in general: there exists a TS_k -graph $D_{r,n,s}$ even when $s \geq k$. For example, one of such graphs, as already proved in [6], is $D_{1,3,2}$ ($r = 1$, $s = k = 2$, and $n = 3$). (See also Figure 1.) Indeed, as we will see in Proposition 11, it is the unique TS_2 -graph among all trees $D_{1,n,2}$ for $n \geq 1$. Additionally, for the sake of completeness, we will also show in Proposition 12 that the reverse statement indeed holds when $n = 2$.

We are now characterizing which $D_{1,n,2}$ -graphs are TS_2 -graphs and show that this property is non-hereditary for this simple class of trees. We then consider the $D_{r,2,s}$ -graphs characterizing those that are TS_k -graphs.

Assume for some G , $\text{TS}_2(G)$ is a forest containing a $K_{1,3}$. There are four stable sets in G corresponding to the vertices of the $K_{1,3}$. There are two ways of labelling the $K_{1,3}$ but in each case there are five vertices, say a, \dots, e , of G involved. Up to permutations of the labels, the corresponding stable sets in G are either $\{ab, ac, bd, ae\}$ or $\{ab, ac, ad, ae\}$. Using these definitions we have the following lemma.

Lemma 3. Let H be the subgraph of G induced by a, b, \dots, e . The edges of H are

- (a) ad, de, eb, bc, cd , if the $K_{1,3}$ is labelled $\{ab, ac, bd, ae\}$, or
- (b) bc, bd, be if the $K_{1,3}$ is labelled $\{ab, ac, ad, ae\}$.

Proof. (a) This labelling of $K_{1,3}$ immediately gives edges ad, bc, be and non-edges ab, ac, ae, bd . That leaves three edges of H to be decided:

- (i) ce must be a non-edge else there is an edge ae, ac in the $K_{1,3}$.
- (ii) cd is an edge else there is a cycle ab, bd, cd, ad in $\text{TS}_2(G)$, so it is not a tree.
- (iii) de is an edge else there is a cycle de, bd, ab, ae in $\text{TS}_2(G)$.

Note that ce must also be a vertex in $\text{TS}_2(G)$.

- (b) This labelling of $K_{1,3}$ immediately gives edges bc, bd, be and non-edges ab, ac, ad, ae . There are no other edges in H as c, d, e form a stable set. This implies that $\text{TS}_2(G)$ must also contain vertices cd, ce and de .

\square

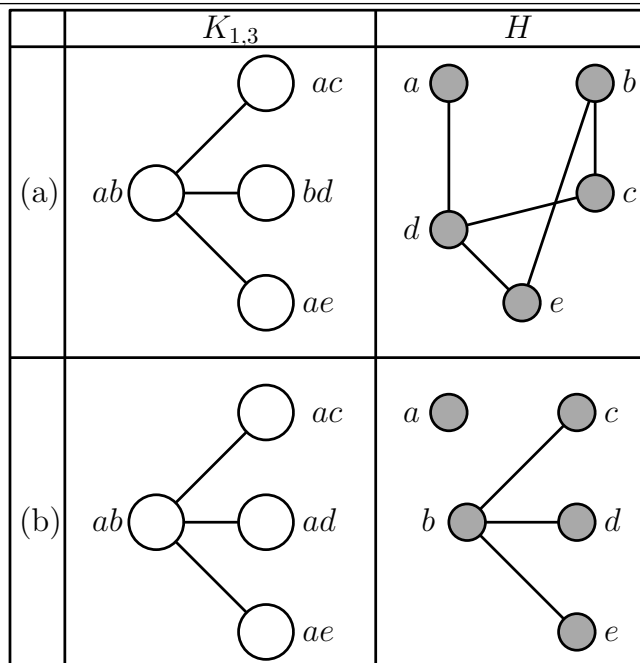


Figure 13. If $TS_2(G)$ is a Forest Containing a $K_{1,3}$ then G must Contain One of the Induced Subgraphs H

Using the lemma we show that precisely one of the $D_{1,n,2}$ -graphs is a TS_2 -graph, incidentally proving the non-hereditary property mentioned above for this class of graphs.

Proposition 11. $D_{1,n,2}$ is a TS_2 -graph if and only if $n = 3$.

Proof. We first consider $1 \leq n \leq 3$ and show that $D_{1,3,2}$ is a TS_2 -graph while $D_{1,1,2} = K_{1,3}$ and $D_{1,2,2}$ are not. (We note that the results for the first two graphs have also been proved in [6].) According to Lemma 3, if $D_{1,n,2}$ is a TS_2 -graph of some graph G , the unique star $K_{1,3}$ in $D_{1,n,2}$ can be labelled in one of two ways. However, we may immediately eliminate the possibility of the labelling in Lemma 3(b). This is because, as pointed out in the proof, there must be additional vertices in $D_{1,n,2} = TS_2(G)$ labelled cd, ce and de which are non-adjacent since c, d, e form a stable set in G . This implies that $n \geq 6$. So we may assume that if $D_{1,n,2}$ is a TS_2 -graph, the $K_{1,3}$ must be labelled as in Lemma 3(a) with corresponding induced subgraph H of $D_{1,n,2}$. From the proof of Lemma 3(a) there must be an additional vertex ce in $D_{1,n,2}$ however this cannot be adjacent to any of the other four vertices. This implies that $n \geq 3$ and so neither $D_{1,1,2}$ nor $D_{1,2,2}$ can be TS_2 -graphs. However, we may extend H to G by adding a vertex f adjacent to all vertices except e , as illustrated in Figure 1. This introduces the new stable set ef which is adjacent to both ae and ce . Therefore, $D_{1,3,2}$ is isomorphic to $TS_2(G)$. We note that G is the unique graph (up to label permutations) for which this is true, due to the uniqueness of the labelling of $K_{1,3}$.

It remains to consider $n \geq 4$ and show that $D_{1,n,2}$ is not a TS_2 -graph. Suppose to the contrary that there exists a graph G such that $D_{1,n,2} = TS_2(G)$. Again, $D_{1,n,2}$ must contain a copy of $K_{1,3}$ with exactly two ways of labelling (up to label permutations) by size-2 independent sets of G .

- **Case 1: $K_{1,3}$ is labelled $\{ab, ac, bd, ae\}$.** Since ac and ae are not adjacent, ce must be a vertex of $D_{1,n,2} = TS_2(G)$. We consider the following cases:
 - **Case 1.1: the distance between ce and any vertex of $\{ac, bd, ae\}$ is at least three.** Since the roles of c and e are equal, we assume without loss of generality that ce is adjacent to some vertex cf . Observe that a and f are not adjacent in G , otherwise ac and cf are adjacent, which means the distance between ac and ce is two, a contradiction. Since ce and cf are adjacent, so are ae and af . Moreover, bf must be a vertex, otherwise there is an edge between ab and af in $D_{1,n,2} = TS_2(G)$ which creates a C_3 having $\{ab, ae, af\}$ as its

vertex-set, a contradiction. Since ab and ac are adjacent, so are cf and bf . Now, df must be a vertex, otherwise bd and bf are adjacent which contradicts $D_{1,n,2} = \text{TS}_2(G)$. Since ab and bd are adjacent, so are af and df . From the proof of Lemma 3(a)(ii) c and d are adjacent in G , so df and cf are adjacent, which again contradicts $D_{1,n,2} = \text{TS}_2(G)$.

– **Case 1.2: the distance between ce and one of $\{ac, bd, ae\}$ is exactly two.** Observe that bd and ce has no common neighbor, otherwise that neighbor must be labelled as one of $\{bc, be, dc, de\}$: the first two can be ignored because ab and ac (resp., ab and ae) are adjacent, the last two can be ignored because ab and bd are adjacent. Again, since the roles of c and e are equal, we assume without loss of generality that ae and ce has a common neighbor ef . Since $n \geq 4$, ce must have another neighbor which is different from ef , which can be either cg or eg for some vertex g of G .

* If it is cg then ag must be a vertex, otherwise cg and ac must be adjacent, which creates a C_6 whose vertex-set is $\{ac, ab, ae, ef, ce, cg\}$, a contradiction. Since ce and cg are adjacent, so are ae and ag , which contradicts $D_{1,n,2} = \text{TS}_2(G)$.

* If it is eg then ag must be a vertex, otherwise eg and ae must be adjacent, which creates a C_4 whose vertex-set is $\{ae, ef, ce, eg\}$, a contradiction. Since ce and eg are adjacent, so are ag and ac , which contradicts $D_{1,n,2} = \text{TS}_2(G)$.

• **Case 2: $K_{1,3}$ is labelled $\{ab, ac, ad, ae\}$.** As before, cd , ce , and de must be vertices in $D_{1,n,2}$. Without loss of generality, since the roles of c, d, e are equal, we may assume that only ae is adjacent to another vertex of $D_{1,n,2}$. As shown in the proof of Lemma 3(b), $D_{1,n,2}$ must also contain vertices cd, ce, de . Let P be the path between ae and cd . Since the roles of c and d are equal, we can assume without loss of generality that cd is adjacent to a vertex cf in P . Observe that if af is not a vertex ac and cf are adjacent contradicting the choice of ae . So af is a vertex and since cd and cf are adjacent so are ad and af , which contradicts $D_{1,n,2} = \text{TS}_2(G)$.

□

We remark that if we add a vertex g to G in Figure 1 joining it to all vertices except d the corresponding TS_2 -graph is obtained by adding the edge between bd and dg to $\text{TS}_2(G)$. Note that this tree is not in the class $D_{r,n,s}$.

In the next proposition we consider two arbitrary stars whose centers are connected by an edge.

Proposition 12. $D_{r,2,s}$ ($1 \leq r \leq s$) is a TS_k -graph if and only if $s \leq k - 1$.

Proof. (\Leftarrow) It follows directly from Proposition 7.

(\Rightarrow) Suppose that $D_{r,2,s}$ ($r \leq s$) is obtained from $P_2 = p_1p_2$ by attaching r leaves u_1, \dots, u_r at p_1 and s leaves v_1, \dots, v_s at p_2 for some $s \geq k$. We show that this graph is not a TS_k -graph for any fixed $k \geq 2$. Suppose to the contrary that there exists a graph G such that $D_{r,2,s} \simeq \text{TS}_k(G)$, i.e., there exists a bijective mapping $f : V(D_{r,2,s}) \rightarrow V(\text{TS}_k(G))$ such that $uv \in E(D_{r,2,s})$ if and only if $f(u)f(v) \in E(\text{TS}_k(G))$. Without loss of generality, let $f(p_2) = I = \{a_1, \dots, a_k\}$, where I is a size- k independent set of G . Since p_2 has $s + 1$ neighbors, from the pigeonhole principle, it follows that there must be some $i \in \{1, \dots, k\}$ such that $f(u) = I - a_i + x$ and $f(v) = I - a_i + y$, where $u, v \in N(p_2)$. Observe that $J = (I - a_i - a_j) + x + y \notin \{f(p_2), f(u), f(v)\}$ must be a size- k independent set of G , where $j \in \{1, \dots, k\} - i$ and therefore there exists $z \in V(D_{r,2,s}) - \{p_2, u, v\}$ such that $f(z) = J$. We consider the following cases:

– **Neither u nor v is p_1 .** In this case, we must have $z \notin N(p_2)$, otherwise it must be adjacent to p_2 , but then $f(z) = J$ and $f(p_2) = I$ must be adjacent in $\text{TS}_k(G)$, a contradiction. It follows that $z \in N(p_1) - p_2$ and thus $f(p_1)$ must be in $\{I - a_i + x, I - a_i + y, I - a_j + x, I - a_j + y\}$. Since neither u nor v is p_1 , the first two can be ignored. Now, if $f(p_1) = I - a_j + x$, the vertices x and a_j must be adjacent in G , which contradicts the fact that $f(u) \in \text{TS}_k(G)$. A similar contradiction can be derived for the case $f(p_1) = I - a_j + y$. Thus, $f(p_1)$ cannot be defined.

– u is p_1 . Again, $z \notin N(p_2)$. Thus, $z \in N(p_1) - p_2$, which implies that y and a_j must be adjacent in G . This contradicts $f(v) \in \text{TS}_k(G)$. Thus, $f(z)$ cannot be defined.

In both cases, we showed that some contradiction must occur. Our proof is complete. \square

5. Conclusions

In this paper, we considered two token sliding problems for trees and forests. The two questions studied seem remarkably complicated, even for this simple class of graphs. For the first question, finding necessary and sufficient conditions on G for $\text{TS}_k(G)$ to be a forest, we could only get a complete solution for $k = 2, 3$. For the second question, finding necessary and sufficient conditions for a tree or forest to be a token sliding graph, we could get more general results. Nevertheless, as noted in Section 4 several interesting important questions remain. We expect the join and decomposition operations introduced there will be of use for similar questions for more general graphs. Finally, we remark that all the acyclic TS_k -graphs described in this paper can be obtained by successive H -joins starting only with the simple base graphs described in Section 3. Is it true that every acyclic TS_k -graph can be obtained in this way?

Acknowledgments

Avis' research is partially supported by the Japan Society for the Promotion of Science (JSPS) KAKENHI Grants JP18H05291, JP20H00579, and JP20H05965 (AFSA). The majority of this work was done when Duc A. Hoang was affiliated with Kyoto University and supported by the Japan Society for the Promotion of Science (JSPS) KAKENHI Grant JP20H05964 (AFSA). Part of this work was done when he was working at the Vietnam Institute for Advanced Study in Mathematics (VIASM) and he would like to thank VIASM for their support and hospitality.

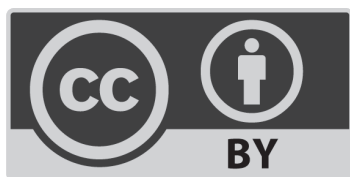
Conflict of Interest

The authors declare no conflict of interest

References

1. van den Heuvel, J., 2013. The complexity of change. In *Surveys in Combinatorics* (Vol. 409, pp. 127-160). Cambridge University Press.
2. Nishimura, N., 2018. Introduction to reconfiguration. *Algorithms*, 11(4), 52.
3. Mynhardt, C.M. and Nasserar, S., 2019. Reconfiguration of colourings and dominating sets in graphs. In *50 Years of Combinatorics, Graph Theory, and Computing* (pp. 171-191). Chapman and Hall/CRC.
4. Bousquet, N., Mouawad, A. E., Nishimura, N. and Siebertz, S., 2022. A survey on the parameterized complexity of the independent set and (connected) dominating set reconfiguration problems. *arXiv preprint*. arXiv:2204.10526.
5. Hearn, R. A. and Demaine, E. D., 2005. PSPACE-completeness of sliding-block puzzles and other problems through the nondeterministic constraint logic model of computation. *Theoretical Computer Science*, 343(1-2), pp.72-96.
6. Avis, D. and Hoang, D.A., 2023. On reconfiguration graphs of independent sets under token sliding. *Graphs and Combinatorics*, 39(3), p.59.

7. Monroy, R. F., Flores-Peñaloza, D., Huemer, C., Hurtado, F., Urrutia, J., and Wood, D. R., 2012. Token graphs. *Graphs and Combinatorics*, 28(3), pp.365-380.
8. Eroh, L. and Schultz, M., 1998. Matching graphs. *Journal of Graph Theory*, 29(2), pp.73-86.
9. Hsu, W.-J., 1993. Fibonacci cubes-a new interconnection topology. *IEEE Transactions on Parallel and Distributed Systems*, 4(1), pp.3-12.
10. Klavžar, S., 2013. Structure of Fibonacci cubes: A survey. *Journal of Combinatorial Optimization*, 25(4), pp.505-522.
11. Bousquet, N., Durain, B., Pierron, T. and Thomassé, S., 2023. Extremal Independent Set Reconfiguration. *The Electronic Journal of Combinatorics*, p.P3.8.
12. Diestel, R. (2017). *Graph Theory* (5th ed., Vol. 173). Springer.



©2024 the Author(s), licensee Combinatorial Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<http://creativecommons.org/licenses/by/4.0>)