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Article

Full Edge-Friendly Index Sets of One Point Union of Cycles

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Abstract: Let $G = (V, E)$ be a graph with vertex set V and edge set E. An edge labeling $f : E \to \mathbf{Z}_2$ induces a vertex labeling $f^+ : V \to \mathbf{Z}_2$ defined by $f^+(v) \equiv \sum_{uv \in E} f(uv) \pmod 2$, for each vertex $v \in V$. For $i \in \mathbf{Z}_2$, let $v_f(i) = |\{v \in V\colon f^+(v) = i\}|$ and $e_f(i) = |\{e \in E : f(e) = i\}|$. An edge labeling f of a graph G is said to be edge-friendly if $|e_f(1) - e_f(0)| \le 1$. The set $\{v_f(1) - v_f(0) :$ f is an edge-friendly labeling of $G\}$ is called the full edge-friendly index set of G. In this paper, we shall determine the full edge-friendly index sets of one point union of cycles.

Keywords: One point union of cycles, Edge labeling vector, Vertex labeling vector, Edge-friendly labeling, Full edge-friendly index set
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1. Introduction

In this paper, all graphs are simple and connected. We refer to [1] for terms and notation that are not defined in this paper.

Since graph labelings was first introduced by Rosa, various labeling concepts have been introduced [2]. For example, cordial labeling, edge-friendly labeling, harmonious labeling, felicitous labeling, odd harmonious labeling and even harmonious labeling, semi-magic labeling, etc. Most graph labeling methods can be traced to the method introduced by Rosa [3] in 1967, or Graham and Sloane [4] in 1980.

Let $G = (V, E)$ be a graph with vertex set V (or $V(G)$) and edge set E (or $E(G)$). A labeling $f : E \to \mathbf{Z}_2$ induces a vertex labeling $f^+ : V \to \mathbf{Z}_2$ defined by $f^+(v) \equiv \sum_{uv \in E} f(uv) \pmod 2$, for each vertex $v \in V$. Here $\mathbf{Z}_2$ is the field of order 2.

For $i \in \mathbf{Z}_2$, let $v_f(i) = |\{v \in V\colon f^+(v) = i\}|$ and $e_f(i) = |\{e \in E : f(e) = i\}|$. The number $v_f(1) - v_f(0)$ is denoted by $I_f(G)$ and is called the *edge-friendly index* of f.

An edge labeled by i is called an *i-edge* (under f). A vertex labeled by i is called an *i-vertex* (under f). In this paper, an *edge labeling* (or a (0, 1)-edge labeling) means an edge labeling whose codomain is $\mathbf{Z}_2$. An edge labeling f of a graph G is said to be *edge-friendly* if $|e_f(1) - e_f(0)| \leq 1$.

Definition 1. *The full edge-friendly index set of G, denoted by FEFI(G), is the set* $\{I_f(G) : f$ *is an edge-friendly labeling of* $G\}$.

The concept of full edge-friendly index set was introduced in [5]. Since $v_f(1) + v_f(0) = |V|$,

$$I_f(G) = 2v_f(1) - |V| = |V| - 2v_f(0). \tag{1}$$

So the edge-friendly index of f of G is determined by the value of $v_f(1)$.

Definition 2. Let f be an edge labeling of a graph G of order p and size q. We fix the sequence of vertices $V = \{v_1, v_2, \dots, v_p\}$ and the sequence of edges $E = \{e_1, e_2, \dots, e_q\}$ of G. The vector $f(E) = (f(e_1), f(e_2), \dots, f(e_q)) \in \mathbf{Z}_2^q$ is called an *edge labeling vector* of f, where $\mathbf{Z}_2^n$ denotes the Cartesian product of n copies of $\mathbf{Z}_2$. Similarly,

$$f^+(V) = (f^+(v_1), f^+(v_2), \dots, f^+(v_p)) \in \mathbf{Z}_2^p$$

is called a *vertex labeling vector* of f.

For any (row) vector $X \in \mathbf{Z}_2^n$, X^T denotes the transpose of X.

Let M be the incidence matrix of a graph G according to fixed sequences of vertices and edges. Let f be an edge-labeling of $G = (V, E)$. It is easy to see that

$$f^+(V)^T = Mf(E)^T,$$

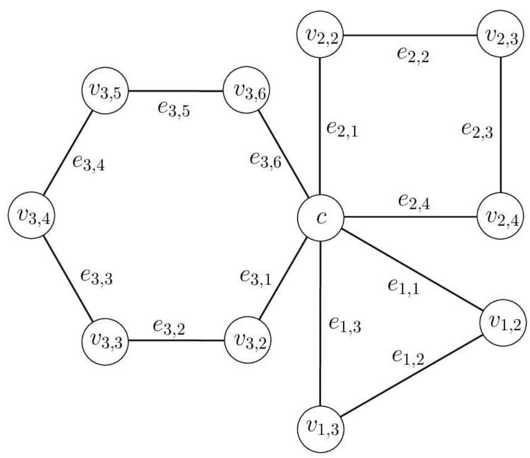
which is a column vector of length p over $\mathbf{Z}_2$.

For a vector $X \in \mathbf{Z}_2^p$, the number of 1's in X is denoted by $wt(X)$, the Hamming weight of X. Hence, if f is an edge-friendly labeling of a graph G, then $wt(f^+(V))$ is $v_f(1)$. For convenience, we will identify f with $f(E)^T$. Hence $v_f(1) = wt(f^+(V)) = wt((Mf)^T)$.

Let H_i be a graph and $v_i \in V(H_i)$ be fixed, $1 \leq i \leq t$. A *one point union* of H_i, $1 \leq i \leq t$, is the graph obtained from the disjoint union of H_i by merging all v_i into a single vertex which is called the *merged vertex* or *core vertex*. So there will be many different one point unions of H_i's.

If $H_i = C_{n_i}$ is a cycle of order $n_i \geq 3$, then all one point unions of H_i, $1 \leq i \leq t$, are isomorphic. So we denote the one point union of C_{n_i}, $1 \leq i \leq t$, by $\biguplus_{k=1}^{t} C_{n_k}$ for $t \geq 2$. In this paper, we shall study the full edge-friendly index set of the graph $\biguplus_{k=1}^{t} C_{n_k}$.

For $1 \leq k \leq t$, let $C_{n_k} = v_{k,1}v_{k,2}\cdots v_{k,n_k-1}v_{k,n_k}v_{k,1}$. Let $e_{k,j} = v_{k,j}v_{k,j+1}$ for $1 \leq j \leq n_k$, where $v_{k,n_k+1} = v_{k,1}$. Let $v_{k,1} \in V(C_{n_k})$ be the chosen vertex to be merged. We denote the core vertex of the graph $G_t = \biguplus_{k=1}^{t} C_{n_k}$ by c (or $v_{0,0}$). Note that G_t is of order $\left(\sum_{k=1}^{t} n_k\right) - t + 1$ and of size $\sum_{k=1}^{t} n_k$. Following is the one point union of C_3, C_4 and C_6, i.e., $C_3 \uplus C_4 \uplus C_6$.



In the rest of the paper, we will use the notation defined above.

2. Some Extrema of Edge-Friendly Indices

In this section, necessary and sufficient conditions that give the extrema values of $v_f(0)$ or $v_f(1)$ are obtained. Following is Lemma 2.1 of [5,6].

Lemma 1. *Let f be any edge labeling of a graph $G = (V, E)$, then $v_f(1)$ must be even.*

Suppose there are s odd numbers among $n_1, \ldots, n_t$, where $0 \le s \le t$. Without loss of generality, we may assume that $n_1, \ldots, n_s$ are odd and the others are even. Note that, $s = 0$ means that there is no odd number; $s = t$ means that there is no even number. Now

$$v_f(1) + v_f(0) = |V(G_t)| = \sum_{k=1}^{s} n_k + \sum_{k=s+1}^{t} n_k - t + 1 \equiv s - t + 1 \pmod 2. \tag{2}$$

It is obvious that $|V(G_t)| \ge v_f(0) \ge 0$ for any edge labeling f of G_t. By Lemma 1 and the equation above, we have

$$v_f(0) \equiv s - t + 1 \pmod 2. \tag{3}$$

Hence $v_f(0) \ge 1$ if $t - s$ is even for any edge labeling f of G_t. A natural question is that whether there is an edge-friendly labeling of G_t attaining the lower bound and whether there is an edge-friendly labeling of G_t attaining the upper bound.

Lemma 2. *Suppose there are s odd numbers among $n_1, \ldots, n_t$. There is an edge-friendly labeling f of G_t such that*

(1) $v_f(0) = 1$ if and only if $t - s$ is even;

(2) $v_f(0) = 0$ if and only if $t - s$ is odd.

Proof. The necessity of (1) and (2) come from Eq. (3).

Note that G_t is Eulerian. Let R be an Euler tour of G_t. We label the edge of G_t by 0 and 1 alternatively along R. Clearly the induced label of each vertex except the core is 1 and the label of the core c is $(t - s) \mod 2$. Hence we have the sufficiency of (1) and (2). □

Obtaining the maximum value of $v_f(0)$ is equivalent to obtaining the minimum value of $v_f(1)$. □

Remark 1. If there are r numbers among $n_1, \ldots, n_t$ such that the sum of these numbers is $\lfloor |E(G_t)|/2 \rfloor$, then the sum of the remaining numbers is $\lceil |E(G_t)|/2 \rceil$. So, the statement 'there are r numbers among $n_1, \ldots, n_t$ such that the sum of these numbers is $\lfloor |E(G_t)|/2 \rfloor$' is equivalent to the statement 'there are r numbers among $n_1, \ldots, n_t$ such that the sum of these numbers is $\lceil |E(G_t)|/2 \rceil$'.

Lemma 3. *There is an edge-friendly labeling f of G_t such that $v_f(1) = 0$ if and only if there are r numbers among $n_1, \ldots, n_t$ such that the sum of these numbers is $\lfloor |E(G_t)|/2 \rfloor$.*

Necessity. Consider the subgraph E_i induced by all i-edges of G_t under f, $i = 0, 1$. Since there is no 1-vertex, each cycle of G_t is entirely edge-labelled by 1's or entirely edge-labelled by 0's. Hence E_1 is a one point union of some subcycles of G_t. Hence E_0 is also a one point union of some subcycles of G_t, namely E_0 is the one point union of r subcycles. Since f is edge-friendly, $|E(E_0)| = \lfloor |E(G_t)|/2 \rfloor$ or $\lceil |E(G_t)|/2 \rceil$. Thus we have the necessary condition by Remark 1.

[Sufficiency] Suppose there are r numbers among $n_1, \ldots, n_t$ such that the sum of such numbers is $\lfloor |E(G_t)|/2 \rfloor$. We label the edges of the corresponding cycles by 0 and label the other edges of G_t by 1. Thus this labeling is an edge-friendly labeling and the labels of all vertices are 0's. □

3. Full Edge-Friendly Index Sets

The main results are given in this section. For a given one point union of cycles $G_t = \biguplus_{i=1}^{t} C_{n_i}$, we fix the sequences of vertices and edges with respect to the lexicographic order. Thus the incident matrix of G_t is

$$M = \begin{pmatrix} Z_1 & Z_2 & \cdots & Z_{t-1} & Z_t \\ B_1 & O & \cdots & \cdots & O \\ O & B_2 & \ddots & \vdots & O \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ O & O & \cdots & B_{t-1} & O \\ O & O & \cdots & O & B_t \end{pmatrix},$$

where $B_k = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{pmatrix}_{(n_k-1)\times n_k}$, $Z_k = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \end{pmatrix}_{1\times n_k}$, and each O is a zero matrix of certain size, $1 \le k \le t$.

We define some notation and recall some known results first. For positive l, let 1_l be the row vector of length l whose entries are 1's, and 0_l be the row vector of length l whose entries are 0's. Let $\alpha_{2l} = (1,0,1,0,\ldots,1,0) \in \mathbf{Z}_2^{2l}$ and $\beta_{2l} = (0,1,0,1\ldots,0,1) \in \mathbf{Z}_2^{2l}$. For convenience, we let 1_0, 0_0, α_0 and β_0 be the empty rows.

Let $p = |V(G_t)| = \left(\sum_{k=1}^{t} n_k\right) - t + 1$. From (2), we have $p \equiv s - t + 1 \pmod 2$, where s is defined in Section 2. So $t - s$ is odd if and only if p is even. From now on, we shall use $q = \sum_{k=1}^{t} n_k = |E(G_t)|$. For each edge labeling $f = (Y_1, Y_2, \ldots, Y_t)^T$, where $Y_k \in \mathbf{Z}_2^{n_k}$, $1 \le k \le t$, we have

$$Mf = \left(\sum_{k=1}^{t} Z_k Y_k^T, \quad B_1 Y_1^T, \quad \ldots, \quad B_t Y_t^T\right)^T.$$

From (1) we have

$$\begin{aligned} \mathrm{FEFI}(G_t) &= \{2v_f(1) - p : f \text{ is an edge-friendly labeling of } G_t\} \\ &\subseteq \{4j - p : 0 \le j \le \lfloor p/2 \rfloor\} \end{aligned}$$

For each possible value $2j$ of the number of 1-vertices, we want to find an edge-friendly labeling f such that $v_f(1) = 2j$.

We deal with some special cases first.

Theorem 1. *Suppose all $n_1, \ldots, n_t$ are even.*

1. *Suppose there are r numbers among $n_1, \ldots, n_t$ such that the sum of these numbers is $q/2$, then*

   (1a) $\mathrm{FEFI}(G_t) = \{4j - p : 0 \le j \le p/2\}$ *if t is odd;*

   (1b) $\mathrm{FEFI}(G_t) = \{4j - p : 0 \le j \le (p-1)/2\}$ *if t is even.*

2. *Suppose the sum of any combination of integers $n_1, \ldots, n_t$ is not equal to $q/2$.*

   (2a) $\mathrm{FEFI}(G_t) = \{4j - p : 1 \le j \le p/2\}$ *if t is odd;*

(2b) $\mathrm{FEFI}(G_t) = \{4j - p : 1 \le j \le (p-1)/2\}$ *if t is even.*

Proof. For each i and l, $1 \le i \le t$ and $1 \le l \le n_i/2$, let $Y_{i,l} = (1_l, 0_l, \alpha_{n_i-2l}) \in \mathbf{Z}_2^{n_i}$. Thus, $B_i Y_{i,l}^T = (0_{l-1}, 1, 0_{l-1}, 1_{n_i-2l})^T$ and $Z_i Y_{i,l}^T = 1$. Hence

$$wt(B_i Y_{i,l}^T) = n_i - 2l + 1. \tag{4}$$

For $1 \le k \le t$, let $f = (Y_{1,n_1/2}, \dots, Y_{k-1,n_{k-1}/2}, Y_{k,l}, Y_{k+1,1}, \dots, Y_{t,1})^T$. Thus,
$Mf = (t \mod 2, B_1 Y_{1,n_1/2}^T, \dots, B_{k-1}^T Y_{k-1,n_{k-1}/2}, B_k^T Y_{k,l}, B_{k+1}^T Y_{k+1,1}, \dots, B_t^T Y_{t,1})^T$. By (4)

$$v_f(1) = wt(Mf) = (t \mod 2) + \sum_{i=1}^{k-1} 1 + (n_k - 2l + 1) + \sum_{i=k+1}^{t} (n_i - 1) \tag{5}$$

1. Without loss of generality, we assume $\sum_{k=1}^{r} n_k = q/2$. Thus $\sum_{k=r+1}^{t} n_k = q/2$. Since either r or $t - r$ is at most $t/2$. We may assume that $r \le \lfloor t/2 \rfloor$ and $t - r \ge \lceil t/2 \rceil$.

   (1a) Suppose t is odd.
   Step 1: For a fixed k, $1 \le k \le t$, let l be an integer such that $1 \le l \le n_k/2$.
   Let f be defined above, then (5) becomes

   $$v_f(1) = wt(Mf) = 1 + \sum_{i=k+1}^{t} n_i + 2k + n_k - 2l - t = p - \sum_{i=1}^{k-1} n_i - 2l + 2k.$$

   For a fixed k, when l runs through from 1 to $n_k/2$, the range of $v_f(1)$ is an increasing arithmetic sequence with common difference 2 from $p - \sum_{i=1}^{k} n_i + 2k$ to $p - \sum_{i=1}^{k-1} n_i - 2 + 2k$. Thus when k runs through from 1 to t, the range of $v_f(1)$ is an increasing arithmetic sequence with common difference 2 from $p - \sum_{i=1}^{t} n_i + 2t = t + 1$ to p.
   Step 2: Let R be the Euler tour starts from the core c and travels the cycles C_{n_1}, C_{n_2}, ..., C_{n_t} in order. Denote $R = u_1 u_2 u_3 \cdots u_{\frac{q}{2}} u_{\frac{q}{2}+1} u_{\frac{q}{2}+2} \cdots u_q u_1$, where $u_1 = c$. In this case $u_{\frac{q}{2}+1} = c$. There may have some u_i's equal to c. Let $g = (1_{\frac{q}{2}}, 0_{\frac{q}{2}})^T$, then $v_g(1) = 0$.
   Step 3: Now we swap the labels of $u_i u_{i+1}$ and $u_{\frac{q}{2}+2i-1} u_{\frac{q}{2}+2i}$, $1 \le i \le \lfloor \frac{q}{2} \rfloor$, where the indices are taken in modulo q. For each case, the number of 1-vertices increases by 2 if $i = 1$ or c is not incident with neither $u_i u_{i+1}$ nor $u_{\frac{q}{2}+2i-1} u_{\frac{q}{2}+2i}$; and may not change if c is incident with either $u_i u_{i+1}$ or $u_{\frac{q}{2}+2i-1} u_{\frac{q}{2}+2i}$ for $i \ne 1$. The last case can occur at most $2r - 1$ times. So the number of 1-vertices increases by 2 at least $\frac{q}{2} - 2r + 1 \ge 2t - 2\lfloor t/2 \rfloor + 1 \ge t + 1$ times. Thus, the number of 1-vertices runs through each even integers from 0 to $t + 1$ under the above swapping.
   So we obtain all the possible values of the number of 1-vertices.

   (1b) Suppose t is even.
   Perform Step 1 as Case (1a). Now (5) becomes

   $$v_f(1) = wt(Mf) = 0 + \sum_{i=k+1}^{t} n_i + 2k + n_k - 2l - t = p - \sum_{i=1}^{k-1} n_i - 2l + 2k - 1.$$

   Similar to Case (1a), when k and l run through all possible values, $v_f(1)$ runs through $p - 1$, $p - 3, \dots, t$.
   Perform Steps 2 and 3 as Case (1a). We obtain that the number of 1-vertices runs through each even integers from 0 to t.
   So we obtain all the possible values of the number of 1-vertices.

2. By Lemma 3, $1 \le v_f(1) \le p$ for any edge-friendly labeling f. Without loss of generality, we assume that $n_1 \ge n_2 \ge \cdots \ge n_t$ and $\sum_{k=1}^{r} n_k < q/2 < \sum_{k=r+1}^{t} n_k$. Here $r \le \lfloor t/2 \rfloor$.
   - (2a) Suppose t is odd. Do the same Step 1 as Case (1a) to obtain the edge labeling f. Thus, we obtain that $v_f(1)$ runs through $p, p-2, \dots, p - \sum_{i=1}^{t} n_i + 2t = t + 1$.
     Let R be an Euler tour as in Case (1a). In this case, $u_{\frac{q}{2}+1} \ne c$. Constructing the edge labeling g as in Case (1a), we get that $v_g(1) = 2$, since $g^+(u_1) = g^+(c) = 1$.
     Do the same Step 3 as in Case (1a). Here we obtain that the number of 1-vertices runs through each even integers from 2 to $t + 1$.
     So we obtain all the possible values of the number of 1-vertices.
   - (2b) Suppose t is even. By a similar procedure and argument, we obtain all possible values of the number of 1-vertices.

Hence this completes the proof. □

Theorem 2. *Suppose all $n_1, \dots, n_t$ are odd.*

1. *Suppose there are r numbers among $n_1, \dots, n_t$ such that the sum of these numbers is $\lfloor q/2 \rfloor$, then* $\mathrm{FEFI}(G_t) = \{4j - p : 0 \le j \le (p-1)/2\}$.
2. *Suppose the sum of any combination of integers $n_1, \dots, n_t$ is not equal to $\lfloor q/2 \rfloor$, then* $\mathrm{FEFI}(G_t) = \{4j - p : 1 \le j \le (p-1)/2\}$.

Proof. Recall that $\alpha_{2l} = (1, 0, \dots, 1, 0) \in \mathbf{Z}_2^{2l}$ and $\beta_{2l} = (0, 1, \dots, 0, 1) \in \mathbf{Z}_2^{2l}$; 1_0, 0_0, α_0 and β_0 are the empty rows (see the beginning of this section).

For a fixed i, $1 \le i \le \lceil t/2 \rceil$, let l be an integer such that $1 \le l \le (n_i - 1)/2$. Let $Y_{i,l} = (1_l, 0_{l-1}, \beta_{n_i+1-2l}) \in \mathbf{Z}_2^{n_i}$, then $B_i Y_{i,l}^T = (0_{l-1}, 1, 0_{l-1}, 1_{n_i-2l})^T$ and $Z_i Y_{i,l}^T = 0$. Hence $wt(B_i Y_{i,l}^T) = n_i - 2l + 1$.

For a fixed i, $\lceil t/2 \rceil + 1 \le i \le t$, let l be an integer such that $1 \le l \le (n_i - 1)/2$. Let $Y_{i,l} = (0, 1_l, 0_l, \alpha_{n_i-1-2l}) \in \mathbf{Z}_2^{n_i}$, then $B_i Y_{i,l}^T = (1, 0_{l-1}, 1, 0_{l-1}, 1_{n_i-1-2l})^T$ and $Z_i Y_{i,l}^T = 0$. Hence $wt(B_i Y_{i,l}^T) = n_i - 2l + 1$.

Step 1: Let $f = (Y_{1,(n_1-1)/2}, \dots, Y_{k-1,(n_{k-1}-1)/2}, Y_{k,l}, Y_{k+1,1}, \dots, Y_{t,1})^T$, where $1 \le k \le t$ and $1 \le l \le (n_k - 1)/2$. Clearly f is edge-friendly.

$$\begin{aligned} v_f(1) = wt(Mf) &= \sum_{i=1}^{k-1} 2 + (n_k - 2l + 1) + \sum_{i=k+1}^{t} (n_i - 1) \\ &= p - \sum_{i=1}^{k-1} n_i - 2l + 3k - 2. \end{aligned}$$

One may check that when k and l run through all possible values, $v_f(1)$ runs through $p-1, p-3, \dots, p - \sum_{i=1}^{t} n_i + 3t - 1 = 2t$.

Step 2: For $1 \le i \le \lceil t/2 \rceil$, we define $Y_{i,(n_i+1)/2} = (1_{(n_i+1)/2}, 0_{(n_i-1)/2})$, then $B_i Y_{i,(n_i+1)/2}^T = (0_{(n_i-1)/2}, 1, 0_{(n_i-3)/2})^T$ and $Z_i Y_{i,(n_i+1)/2}^T = 1$. Hence $wt(B_i Y_{i,(n_i+1)/2}^T) = 1$.

For $\lceil t/2 \rceil + 1 \le i \le t$, we define $Y_{i,(n_i+1)/2} = (1_{(n_i-1)/2}, 0_{(n_i+1)/2})$. Here $B_i Y_{i,(n_i+1)/2}^T = (0_{(n_i-3)/2}, 1, 0_{(n_i-1)/2})^T$ and $Z_i Y_{i,(n_i+1)/2}^T = 1$. Hence $wt(B_i Y_{i,(n_i+1)/2}^T) = 1$.

For each k, $1 \le k \le t$, let $f_k = (Y_{1,(n_1+1)/2}, \dots, Y_{k,(n_k+1)/2}, Y_{k+1,(n_{k+1}-1)/2}, \dots, Y_{t,(n_t-1)/2})^T$. Note that, $f_t = (Y_{1,(n_1+1)/2}, \dots, Y_{t,(n_t+1)/2})^T$. Clearly f_k is edge-friendly. Suppose k is even. We have $v_{f_k}(1) = wt(Mf_k) = (\{\sum_{i=1}^{k} 1 \mod 2\} + k) + 2(t-k) = (0 + k) + 2(t-k) = 2t - k$.

One may check that when k runs through all even numbers from 2 to t, $v_{f_k}(1)$ runs through all even numbers from $2t-2$ down to t.

Now we shall perform some steps similar to Steps 2 and 3 in the proof of Theorem 1 to obtain the remaining possible values of the number of 1-vertices.

1. Without loss of generality, we assume that $\sum_{k=1}^{r} n_k = \lfloor q/2 \rfloor$. Here $r \le \lfloor t/2 \rfloor$.

   Let $R = u_1 u_2 u_3 \cdots u_{\lfloor q/2 \rfloor} u_{\lfloor q/2 \rfloor+1} u_{\lfloor q/2 \rfloor+2} \cdots u_q u_1$ be the Euler tour starts from the core $u_1 = c$ and travels the cycles $C_{n_1}, C_{n_2}, \ldots, C_{n_t}$ in order. By convention $u_{q+1} = u_1$.

   Step 3: Define $g = (1_{\lfloor q/2 \rfloor}, 0_{\lceil q/2 \rceil})^T$. Now $v_g(1) = 0$.

   Step 4:

   (1a) When $q = 4m + 1$, where $m \ge 2$. Note that $u_{2m+1} = c$, $4m + 1 = \sum_{i=1}^{t} n_i \ge 3t$ and t is odd. Swap the labels of $u_i u_{i+1}$ and $u_{2m+2i-1} u_{2m+2i}$, $1 \le i \le m + 1$. The number of 1-vertices increases by 2 at least $(m + 1) - (r - 1)$ times, since the first swapping increases $v_g(1)$ by 2.

   Next, swap the labels of $u_{m+1+i} u_{m+2+i}$ and $u_{2i-1} u_{2i}$, $1 \le i \le m - 1$. So the number of 1-vertices increases by 2 at least $m - 1 - r$ times.

   By the same argument as Step 3 in the proof of Theorem 1, the number of 1-vertices increases by 2 at least $2m + 1 - 2r \ge (3t - 1)/2 + 1 - t = (t + 1)/2$ times totally. Thus, the number of 1-vertices runs through each even integer from 0 to $t + 1$ under the above swapping.

   (1b) When $q = 4m + 3$, where $m \ge 2$ (in this case q cannot be 7). Note that $u_{2m+2} = c$, $4m + 3 = \sum_{i=1}^{t} n_i \ge 3t$ and t is odd.

   Swap the labels of $u_i u_{i+1}$ and $u_{2m+2i} u_{2m+2i+1}$, $1 \le i \le m + 1$. Next, swap the labels of $u_{m+1+i} u_{m+2+i}$ and $u_{2i-1} u_{2i}$, $1 \le i \le m$.

   Totally, the number of 1-vertices increases by 2 at least $(2m+1)-2r+1 \ge (3t-3)/2+2-t = (t + 1)/2$ times. Thus, the number of 1-vertices runs through each even integer from 0 to $t + 1$ under the above swapping.

   (1c) When $q = 4m$, where $m \ge 3$ (in this case q cannot be 8). Note that $u_{2m} = c$, $4m = \sum_{i=1}^{t} n_i \ge 3t$ and t is even.

   Swap the labels of $u_i u_{i+1}$ and $u_{2m+2i-1} u_{2m+2i}$, $1 \le i \le m$. Next, swap the labels of $u_{m+i} u_{m+i+1}$ and $u_{2i-1} u_{2i}$, $1 \le i \le m - 1$.

   Totally, the number of 1-vertices increases by 2 at least $(2m - 1) - 2r + 1 \ge 3t/2 - t = t/2$ times. Thus, the number of 1-vertices runs through each even integer from 0 to t under the above swapping.

   (1d) When $q = 4m + 2$, where $m \ge 1$. Note that $u_{2m+1} = c$, $4m + 2 = \sum_{i=1}^{t} n_i \ge 3t$ and t is even.

   Swap the labels of $u_i u_{i+1}$ and $u_{2m+2i} u_{2m+2i+1}$, $1 \le i \le m + 1$. Next, swap the labels of $u_{m+1+i} u_{m+i+2}$ and $u_{2i-1} u_{2i}$, $1 \le i \le m$.

   Totally, the number of 1-vertices increases by 2 at least $(2m+1)-2r+1 \ge 3t/2+1-t = t/2+1$ times. Thus, the number of 1-vertices runs through each even integer from 0 to $t + 2$ under the above swapping.

2. Without loss of generality, we assume that $n_1 \ge n_2 \ge \cdots \ge n_t$ and $\sum_{k=1}^{r} n_k < q/2 < \sum_{k=r+1}^{t} n_k$. Here $r \le \lfloor t/2 \rfloor$. We do the same procedure as Case 1. The only difference is $v_g(1) = 2$. Hence the number of 1-vertices at least runs through each even integer from 2 to t under the above swapping.

This completes the proof. □

Example 1. *Consider the graph $C_5 \uplus C_3 \uplus C_3 \uplus C_3 \uplus C_3$. Here $t = 5$, $p = 13$, $q = 17$, $m = 4$ and $r = 2$. Let $R = u_1 u_2 \cdots u_{17} u_1$ be an Euler tour of the graph, where $u_1 = u_6 = u_9 = u_{12} = u_{15} = c$. For $1 \le i \le 17$, let $e_i = u_i u_{i+1}$, where $u_{18} = u_1$.*

The procedure of Steps 2 listed below:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	e_{16}	e_{17}	$v(1)$
f_1	1	1	1	0	0	1	0	1	1	0	1	0	1	0	0	1	0	10
f_2	1	1	1	0	0	1	1	0	1	0	1	0	1	0	0	1	0	8
f_3	1	1	1	0	0	1	1	0	1	1	0	0	1	0	0	1	0	8
f_4	1	1	1	0	0	1	1	0	1	1	0	1	0	0	0	1	0	6
f_5	1	1	1	0	0	1	1	0	1	1	0	1	0	0	1	0	0	6

The procedure of swapping (Steps 3 and 4) listed below start from $g = (1_8, 0_9)$.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	e_{16}	e_{17}	$v(1)$
g	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	2
	0	0	1	1	1	1	1	1	1	0	1	0	0	0	0	0	0	4
	0	0	0	1	1	1	1	1	1	0	1	0	1	0	0	0	0	6
	0	0	0	0	1	1	1	1	1	0	1	0	1	0	1	0	0	6
	0	0	0	0	0	1	1	1	1	0	1	0	1	0	1	0	1	6
	1	0	0	0	0	0	1	1	1	0	1	0	1	0	1	0	1	8
	1	0	1	0	0	0	0	1	1	0	1	0	1	0	1	0	1	10
	1	0	1	0	1	0	0	0	1	0	1	0	1	0	1	0	1	10

Example 2. *Consider the graph $C_3 \uplus C_3$. Here $t = 2$, $p = 5$, $q = 6$, $m = 1$ and $r = 1$. Let $R = u_1 u_2 \cdots u_6 u_1$ be an Euler tour of the graph, where $u_1 = u_4 = c$. For $1 \le i \le 6$, let $e_i = u_i u_{i+1}$, where $u_7 = u_1$.*

The procedure of Steps 3 and 4 listed below start from $g = (1_3, 0_3)$.

	e_1	e_2	e_3	e_4	e_5	e_6	$v(1)$
g	1	1	1	0	0	0	0
	0	1	1	1	0	0	2
	0	0	1	1	0	1	4
	1	0	0	1	0	1	4

Lemma 4. *Let G and H be two graphs with only one common vertex u. Suppose f_G and f_H be two edge-friendly labelings of G and H, respectively. If $(f_G^+(u), f_H^+(u)) \ne (1, 1)$, then there is an edge-friendly labeling h of $G \uplus H$ such that $v_h(1) = v_{f_G}(1) + v_{f_H}(1)$. If $(f_G^+(u), f_H^+(u)) = (1, 1)$, then there is an edge-friendly labeling h of $G \uplus H$ such that $v_h(1) = v_{f_G}(1) + v_{f_H}(1) - 2$.*

Proof. Let h be the combined labeling of f_G and f_H. We obtain the lemma easily. □

Theorem 3. *Let p and q be the order and the size of G_t, respectively.*

1. *Suppose there are r numbers among $n_1, \ldots, n_t$ such that the sum of these numbers is $\lfloor q/2 \rfloor$, then* $\mathrm{FEFI}(G_t) = \{4j - p : 0 \le j \le \lfloor p/2 \rfloor\}$.
2. *Suppose there are no r numbers among $n_1, \ldots, n_t$ such that the sum of these numbers are $\lfloor q/2 \rfloor$, then* $\mathrm{FEFI}(G_t) = \{4j - p : 1 \le j \le \lfloor p/2 \rfloor\}$.

Proof. Without loss of generality, we assume that $n_1, \ldots, n_s$ are even and $n_{s+1}, \ldots, n_t$ are odd, where $0 \le s \le t$. When $s = 0$ or $s = t$, we get the results using Theorem 1 and Theorem 2.

By Lemma 3, it suffices to find an edge-friendly labeling f of G_t such that $v_f(1)$ runs through all even numbers in $[2, p]$.

Let $G = \biguplus_{i=1}^{s} C_{n_i}$ and $H = \biguplus_{i=s+1}^{t} C_{n_i}$. Let p_G and p_H be orders of G and H, respectively. Note that p_H is odd and $p = p_G + p_H - 1$.

By Theorem 1 there is an edge-friendly labeling f_G of G such that $v_{f_G}(1)$ at least runs through all even numbers of $[2, p_G]$, and by Theorem 2 there is an edge-friendly labeling f_H of H such that $v_{f_H}(1)$ at least runs through all even numbers of $[2, p_H - 1]$.

By Lemma 4, in the worst case, there is an edge-friendly labeling h of $G \uplus H$ such that $v_h(1)$ runs through all even numbers of $[4, p_G + p_H - 1 - 2] = [4, p - 2]$.

Now we only need to find an edge-friendly labeling g of $G \uplus H$ such that $v_g(1)$ is p when p is even, or $p-1$ when p is odd, or else, $v_g(1)$ is 2 (see ① and ② below).

Let $R = u_1u_2u_3\cdots u_{\lfloor q/2\rfloor}u_{\lfloor q/2\rfloor+1}u_{\lfloor q/2\rfloor+2}\cdots u_qu_1$ be the Euler tour of G_t starts from the core $u_1 = c$. Note that $q = p+t-1$ and $\deg(c) = 2t$.

① When q is even, $p \equiv t-1 \pmod 2$. Define $g = (1,0,1,0,\dots,1,0)^T \in \mathbf{Z}_2^q$, then $g^+(c) = t \mod 2$. Hence $v_g(1) = \begin{cases} p & \textit{if } t \textit{ is odd,} \\ p-1 & \textit{if } t \textit{ is even,} \end{cases} = \begin{cases} p & \textit{if } p \textit{ is even,} \\ p-1 & \textit{if } p \textit{ is odd.} \end{cases}$

When q is odd, $p \equiv t \pmod 2$. Define $g = (1,0,1,0,\dots,1,0,1)^T \in \mathbf{Z}_2^q$, then $g^+(c) = t+1 \mod 2$. Hence $v_g(1) = \begin{cases} p-1 & \textit{if } t \textit{ is odd,} \\ p & \textit{if } t \textit{ is even,} \end{cases} = \begin{cases} p-1 & \textit{if } p \textit{ is odd,} \\ p & \textit{if } p \textit{ is even.} \end{cases}$

② Suppose $u_{\lfloor q/2\rfloor+1} \neq c$. Define $g = (1_{\lfloor q/2\rfloor}, 0_{\lceil q/2\rceil})^T$. Thus only $g^+(c) = g^+(u_{\lfloor q/2\rfloor+1}) = 1$. Hence $v_g(1) = 2$.

Suppose $u_{\lfloor q/2\rfloor+1} = c$, then $u_2 \neq c$ and $u_{\lfloor q/2\rfloor+2} \neq c$. Define $g = (0, 1_{\lfloor q/2\rfloor}, 0_{\lceil q/2\rceil-1})^T$. Thus only $g^+(u_2) = g^+(u_{\lfloor q/2\rfloor+2}) = 1$. Hence $v_g(1) = 2$.

This completes the proof. □

We denote the graph G_t by $C_n^{(t)}$ when $n_1 = \cdots = n_t = n$. When $n = 3$, $C_3^{(t)}$ is called the *Dutch t-windmill graph*. By Theorem 3, we have the following corollaries.

Corollary 1. *Suppose $n \geq 3$ and $t \geq 2$. Now, the order of $C_n^{(t)}$ is $p = nt - t + 1$.*

1. Suppose n is odd, then

$$\mathrm{FEFI}(C_n^{(t)}) = \begin{cases} \{4j-p : 1 \leq j \leq (p-1)/2\} & \textit{if } t \textit{ is odd;} \\ \{4j-p : 0 \leq j \leq (p-1)/2\} & \textit{if } t \textit{ is even.} \end{cases}$$

2. Suppose n is even, then

$$\mathrm{FEFI}(C_n^{(t)}) = \begin{cases} \{4j-p : 1 \leq j \leq p/2\} & \textit{if } t \textit{ is odd;} \\ \{4j-p : 0 \leq j \leq (p-1)/2\} & \textit{if } t \textit{ is even.} \end{cases}$$

Corollary 2. *For $t \geq 2$,*

$$\mathrm{FEFI}(C_3^{(t)}) = \begin{cases} \{4j-2t-1 : 1 \leq j \leq t\} & \textit{if } t \textit{ is odd;} \\ \{4j-2t-1 : 0 \leq j \leq t\} & \textit{if } t \textit{ is even.} \end{cases}$$

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