Combinatorial Press

Article

Graceful Labellings of Variable Windmills Using Skolem-type sequences

Ahmad H. Alkasasbeh[1,*]**, Danny Dyer**[1]**, and Jared Howell**[2]

[1] Department of Mathematics and Statistics, St. John's Campus, Memorial University of Newfoundland, St. John's, Newfoundland, Canada

[2] School of Science and the Environment, Grenfell Campus, Memorial University of Newfoundland, Corner Brook, Newfoundland, Canada

* Correspondence: ahmad84@mun.ca

Abstract: In this paper, we introduce graceful and near graceful labellings of several families of windmills. In particular, we use Skolem-type sequences to prove (near) graceful labellings exist for windmills with C_3 and C_4 vanes, and infinite families of 3,5-windmills and 3,6-windmills. Furthermore, we offer a new solution showing that the graph obtained from the union of t 5-cycles with one vertex in common (C_5^t) is graceful if and only if $t \equiv 0, 3 \pmod 4$ and near graceful when $t \equiv 1, 2 \pmod 4$.

---

1. Introduction

In [1], Rosa introduced a new type of graph labelling known as a β-labelling, or graceful labelling, as it was renamed later. Let $G = (V, E)$ be a graph with m edges. Let $f : V(G) \to \{0, 1, 2, \dots, m\}$ be a labelling of V of G and let $g : E(G) \to \{1, 2, \dots, m\}$ be the induced edge labelling defined by $g(uv) = |f(u) - f(v)|$, for all $uv \in E$. The labelling f is said to be a *graceful labelling* if and only if f is an injective mapping and g is a bijection. If a graph G has a graceful labelling then we say G is graceful.

A near graceful labelling of a graph $G = (V, E)$ with m edges is defined in a similar way. Let $f : V(G) \to \{0, 1, 2, \dots, m + 1\}$ be a labelling of V of G and let $g : E(G) \to A$ be the induced edge labelling defined by $g(uv) = |f(u) - f(v)|$, for all $uv \in E$, where A is $\{1, 2, \dots, m - 1, m\}$ or $\{1, 2, \dots, m - 1, m + 1\}$. The labelling f is said to be a *near graceful labelling* if and only if f is an injective mapping and g is a bijection. If a graph G has a near graceful labelling then we say G is near graceful. In this paper, all near graceful labellings constructed will omit the vertex label m and the edge label m.

In this paper, we adopt the convention that 0 is a natural number. So, when we write $[a, b]$ with $a, b \in \mathbb{N}$ and $a < b$, we are indicating the set $\{x \in \mathbb{N} | a \le x \le b\}$.

Bermond, in [2], proved that Dutch windmills (the graphs consisting of t copies of K_3 with one vertex in common) are graceful. Let C_n be a cycle of length $n \ge 3$, and C_n^t be the graph obtained from the union of t n-cycles with one vertex in common that we will call the *central vertex*. In [3], the authors stated the following conjecture.

Conjecture 1. [3] C_n^t is graceful if and only if $nt \equiv 0, 3 \pmod 4$.

This conjecture has been shown to hold for $n = 3, n = 4, n = 5$, and $n = 6$ and $n = 4k$ (k any positive integer) in [4–6], and [3], respectively. Also, in [7–10], the authors show that the graceful labellings exist for the C_n^t with $n = 7, 9, 11, 13$, respectively. In 2012, Dyer, et al. [11], use Skolem sequences to prove that all Dutch windmills with zero, one or two pendant triangles are (near) graceful. A comprehensive survey of graceful labelling can be found in [12].

We define an m, n-windmill to be a graph $G = C_m^s C_n^t$ obtained from identifying the central vertices of C_m^s and C_n^t, where $m \neq n$. In other words, the graph $C_m^s C_n^t$ is a windmill with s m-cycle vanes and t n-cycle vanes. More generally, we call any windmill made up of two or more cycle lengths a *variable windmill*.

In this paper we use Skolem-type sequences to show (near) graceful labellings exist for the $G = C_n^t C_m^s$ graphs where $n = 3$ and $m = 4, 5, 6$. An example of a graceful labelling of $C_3^4 C_4^3$ is given in Figure 1.
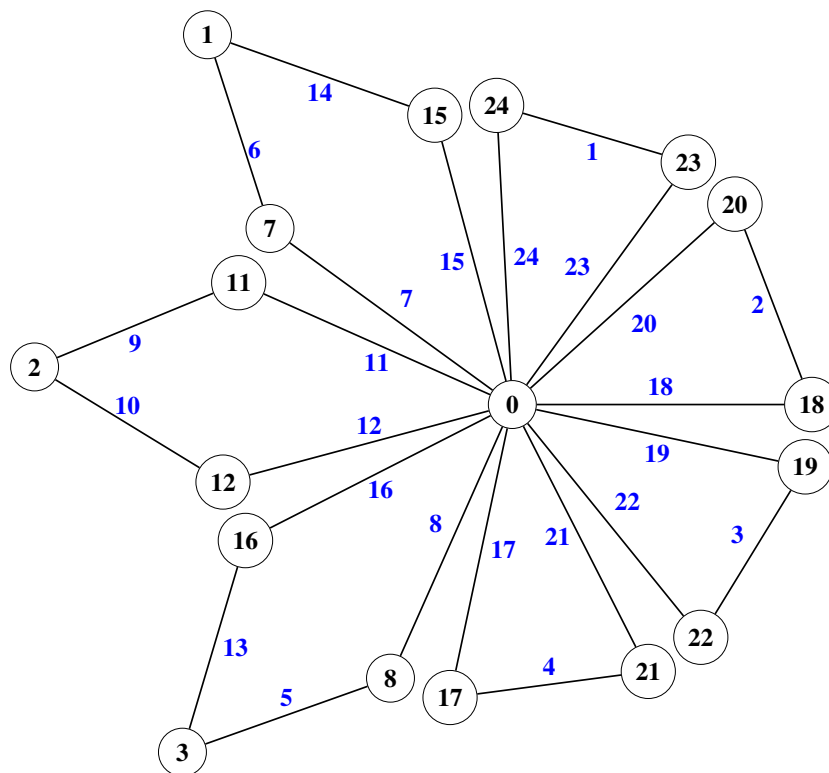

Figure 1. Graceful Labelling of $C_3^4 C_4^3$

2. Skolem-type Sequences

In this section we begin by defining a Skolem-type sequence, and then present several definitions and known results for Skolem and Langford sequences.

A *Skolem-type sequence* of order n is a sequence $(s_1, s_2, \ldots, s_{2n})$ of $2n$ integers satisfying the conditions:

1. for some set H of n distinct positive integers, $\forall k \in H, \exists s_i, s_j$ such that $s_i = s_j = k$;

2. if $s_i = s_j = k$, with $i < j$, then $j - i = k$.

For example, $(6, 4, 1, 1, 3, 4, 6, 3)$ is a Skolem-type sequence of order 4 with $H = \{1, 3, 4, 6\}$. We can also write Skolem-type sequences by specifying the ordered pairs where identical elements of H occur, as follows $\{(a_i, b_i) : i \in H, b_i - a_i = i\}$. So, we could equivalently write the sequence

$\{(3, 4), (5, 8), (2, 6), (1, 7)\}$. In the Skolem-type sequences if $b_i > a_i$ we call b_i *right endpoint*. The definition of the Skolem-type sequence introduced in [13].

A *hooked Skolem-type sequence* of order n is a sequence $(s_1, s_2, \ldots, s_{2n+1})$ of $2n + 1$ integers satisfying the conditions of a Skolem-type sequence with the added condition that $s_{2n} = 0$. For example, $(5, 3, 1, 1, 3, 5, 2, 0, 2)$ is a hooked Skolem-type sequence of order 4 with $H = \{1, 2, 3, 5\}$.

A *(hooked) Skolem sequence* of order n is a (hooked) Skolem-type sequence of order n with $H = [1, n]$. Necessary and sufficient conditions for the existence of Skolem sequences are given in [14].

Let m and n be positive integers, with $m \le n$. A *(hooked) near-Skolem sequence* of order n and defect m is a sequence $(s_1, s_2, \ldots, s_{2n-2})$ is a (hooked) Skolem-type sequence of order $n - 1$ with $H = [1, m - 1] \cup [m + 1, n]$. For example, $(1, 1, 6, 3, 7, 5, 3, 2, 6, 2, 5, 7)$ is a 4-near-Skolem sequence of order 7 and $(2, 5, 2, 4, 6, 7, 5, 4, 1, 1, 6, 0, 7)$ is a hooked 3-near-Skolem sequence of order 7. Necessary and sufficient conditions for the existence of near-Skolem sequences are given in [15].

A *(hooked) Langford sequence* with defect d and order l is a (hooked) Skolem-type sequence with $H = [d, d + l - 1]$. Necessary and sufficient conditions for the existence of Langford sequences are given in [16].

Note that in this paper, in the constructions, a_i and b_i or c_j and d_j or e_j and f_j represent the two positions in the Skolem-type sequence of the element i and j, with $a_i < b_i$, $c_j < d_j$ and $e_j < f_j$ with $1 \le i, j \le n$. When written in this form, we say that a_i, c_j, and e_j are left endpoints, and b_i, d_j, and f_j are right endpoints.

Construction 1. *From Table 1, we can construct a Langford sequence with defect $d \ge 1$ and order $2d - 1$, (omitting row 2 when $d = 1$). We define L_d^{2d-1} to be exactly this sequence for $d \ge 1$.*

i	a_i	b_i	
$d + 2r$	$d - r$	$2d + r$	$0 \le r \le d - 1$
$d + 2r + 1$	$2d - 1 - r$	$3d + r$	$0 \le r \le d - 2$

Table 1. Langford Sequence, L_d^{2d-1}

Construction 1 is new and it is straightforward to check that Construction 1 gives a Langford sequence.

Notice that there is a different use of defect in Langford and near-Skolem sequences. In Langford sequences the defect (d) is the smallest integer in the sequence, but in near-Skolem sequences the defect (m) is the integer omitted from the sequence.

Construction 2. *From Table 2, we can construct a hooked near-Skolem sequence of order $n = 4m + 1$ with $m \ge 3$ and defect $n - 1$, (omitting row 5 when $m = 3$).*

Construction 3. *From Table 3, we can construct a near-Skolem sequence of order $n = 4m + 3$ with $m \ge 2$ and defect $n - 1$, (omitting row 4, 5 when $m = 2$).*

It is straightforward to check that Constructions 2 and 3 both produce the desired sequences. These constructions establish Lemma 1, which will be useful in Section 3 and 5 when labelling C_5^p and $C_3^t C_5^p$.

Lemma 1. *If $n = 2k + 1$, with $k \ge 5$, then there exists a (hooked) near-Skolem sequence of order n, with defect $n - 1$, which has no right endpoints in the positions $[1, k + 2]$.*

We now extend our definition of Skolem-type sequences to sequences where every pair of elements in H occurs exactly twice. A *two-fold Skolem-type sequence* of order n is a sequence $(s_1, s_2, \ldots, s_{4n})$ of $4n$ positive integers such that the following conditions hold:

i	a_i	b_i	
$2r + 1$	$2m + 1 - r$	$2m + 2 + r$	$1 \le r \le 2m$
$4m - 4$	$2m + 2$	$6m - 2$	$-$
$4m - 2$	$2m + 1$	$6m - 1$	$-$
$2m + 2r$	$5m - r$	$7m + r$	$0 \le r \le m - 3$
$2r + 4$	$6m - r - 3$	$6m + r + 1$	$0 \le r \le m - 4$
$2m - 2$	$6m$	$8m - 2$	$-$
1	$7m - 2$	$7m - 1$	$-$
2	$8m - 1$	$8m + 1$	$-$

Table 2. Hooked near-Skolem Sequence from Construction 2

i	a_i	b_i	
$2r + 1$	$2m + 2 - r$	$2m + 3 + r$	$1 \le r \le 2m + 1$
$4m$	$2m + 2$	$6m + 2$	$-$
$4m - 2$	$2m + 3$	$6m + 1$	$-$
$2m + 2 + 2r$	$5m + 2 - r$	$7m + 4 + r$	$0 \le r \le m - 3$
$2r + 4$	$6m - r$	$6m + 4 + r$	$0 \le r \le m - 3$
1	$7m + 2$	$7m + 3$	$-$
$2m$	$6m + 3$	$8m + 3$	$-$
2	$8m + 2$	$8m + 4$	$-$

Table 3. Near-Skolem Sequence from Construction 3

1. for a set H of n distinct positive integers, for every $p \in H$, there exist exactly 2 disjoint pairs $\left(s_i, s_j\right)$ and $(s_{i'}, s_{j'})$ such that $s_i = s_j = s_{i'} = s_{j'} = p$,

2. if $s_i = s_j = p$ and $s_{i'} = s_{j'} = p$ with $i < j$ and $i' < j'$, then $j - i = p$ and $j' - i' = p$.

For example, $(2, 3, 2, 3, 3, 2, 3, 2)$ is a two-fold Skolem-type sequence of order 2 with $H = \{2, 3\}$.

Construction 4. *From Table 4, we can construct a two-fold Skolem-type sequence construction of order $n \ge 1$, where $H = \{1\} \cup \{4i | 1 \le i \le n - 1\}$.*

j	$\left(c_j, d_j\right)$	$\left(e_j, f_j\right)$	
1	$(2n - 1, 2n)$	$(4n - 1, 4n)$	$-$
$4r$	$(2n - 2r - 1, 2n + 2r - 1)$	$(2n - 2r, 2n + 2r)$	$1 \le r \le n - 1$

Table 4. Two-fold Skolem-type Sequence from Construction 4

For example, $(8, 8, 4, 4, 1, 1, 4, 4, 8, 8, 1, 1)$ is a two-fold Skolem-type sequence of order 3 with $H = \{1, 4, 8\}$, created using Construction 4.

A $(2n + 1, 2n + 2)$-*extended two-fold Skolem-type sequence* of order n is a sequence $(\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_{4n+2})$ of $4n + 2$ positive integers that satisfies the above conditions as well as $\hat{s}_{2n+1} = \hat{s}_{2n+2} = 0$.

Proposition 1. *The following sequences are all two-fold Skolem-type sequences:*
$C^0 = \emptyset$, *(the sequence of length 0),*
$C^1 = (2, 2, 2, 2)$, *where $H = \{2\}$,*
$C^2 = (2, 3, 2, 3, 3, 2, 3, 2)$, *where $H = \{2, 3\}$,*
$C^3 = (2, 2, 2, 2, 5, 3, 5, 3, 3, 5, 3, 5)$, *where $H = \{2, 3, 5\}$, and*
$C^4 = (6, 6, 2, 2, 2, 2, 6, 6, 5, 3, 5, 3, 3, 5, 3, 5)$, *where $H = \{2, 3, 5, 6\}$.*

By placing restrictions on the nature of H, we can define two-fold Skolem and Langford sequences, similar to the way we defined the Skolem and Langford sequences.

A *double Skolem-type sequence* of order l and defect d is a sequence obtained by concatenating two existing Skolem-type sequences with the same defect and order, or by interlacing a hooked Skolem-type sequence with its reverse. A *two-fold Skolem sequence* of order n is a two-fold Skolem-type sequence with $H = [1, n]$. It is straightforward to show that two-fold Skolem sequences exist for any order n by concatenating Skolem sequences (possibly interlacing their hooks). However, in Tables 5 and 6 we introduce a different construction for two-fold Skolem sequences which will be useful later.

j	(c_j, d_j)	(e_j, f_j)	
$2r + 1$	$\left(\frac{n+1}{2} - r, \frac{n+3}{2} + r\right)$	$\left(\frac{3n+3}{2} - r, \frac{3n+5}{2} + r\right)$	$0 \leq r \leq \frac{n-1}{2}$
$n - 1$	$(2n + 3, 3n + 2)$	$\left(\frac{5n+5}{2}, \frac{7n+3}{2}\right)$	$-$
$2r + 2$	$\left(\frac{5n+3}{2} - r, \frac{5n+3}{2} + r + 2\right)$	$\left(\frac{7n+1}{2} - r, \frac{7n+1}{2} + r + 2\right)$	$0 \leq r \leq \frac{n-5}{2}$

Table 5. Two-fold Skolem Sequence Construction of Odd Order n

j	(c_j, d_j)	(e_j, f_j)	
$2r + 1$	$\left(\frac{n}{2} - r, \frac{n+2}{2} + r\right)$	$\left(\frac{3n}{2} - r, \frac{3n+2}{2} + r\right)$	$0 \leq r \leq \frac{n-2}{2}$
n	$(2n + 1, 3n + 1)$	$\left(\frac{5n+2}{2}, \frac{7n+2}{2}\right)$	$-$
$2r + 2$	$\left(\frac{5n}{2} - r, \frac{5n}{2} + r + 2\right)$	$\left(\frac{7n}{2} - r, \frac{7n}{2} + r + 2\right)$	$0 \leq r \leq \frac{n-4}{2}$

Table 6. Two-fold Skolem Sequence Construction of Even Order n

A *two-fold Langford sequence* of order l and defect d is a two-fold Skolem-type sequence of order l, $(l_1, l_2, \ldots, l_{4l})$ with $H = [d, d + l - 1]$. Again, it is straightforward to show that a two-fold Langford sequence can be obtained by concatenating (hooked) Langford sequences. In Construction 5, we give a two-fold Langford sequence that is not constructed by concatenation that will be useful later.

Construction 5. *From Table 7, we can construct a two-fold Langford sequence with defect $6k - 1$ and order $4k - 1$, where $k \geq 1$.*

j	(c_j, d_j)	(e_j, f_j)	
$10k-2-2r$	$(r, 10k-2-r)$	$(2k-1+r, 12k-3-r)$	$1 \leq r \leq 2k-1$
$10k-1-2r$	$(4k-2+r, 14k-3-r)$	$(6k-2+r, 16k-3-r)$	$1 \leq r \leq 2k$

Table 7. Two-fold Langford Sequence with Defect $6k - 1$ and Order $4k - 1$, where $k \geq 1$

We summarize necessary and sufficient conditions for the existence of various useful Skolem-type sequences in Table 8 as an aid for the reader.

3. Labellings from Skolem-type Sequences

In this section, we use Skolem-type sequences to label variable windmills of different orders.

3.1. Constructions Using Skolem-type Sequences

To begin, we discuss how to label C_3^t and C_5^p by using Skolem-type sequences. In [6], Yang et al. proved that the C_5^p is graceful when $p \equiv 0, 3 \pmod 4$. Bermond in [2] proved that the C_3^t is graceful when $t \equiv 0, 1 \pmod 4$.

Sequence	Order/ Defect	Necessary and Sufficient conditions	Ref.
Skolem	n/-	$n \equiv 0, 1 \pmod 4$	[14]
hooked Skolem	n/-	$n \equiv 2, 3 \pmod 4$	[17]
Langford	l/d	$l \geq 2d - 1$, $l \equiv 0, 1 \pmod 4$ and d is odd, or $l \equiv 0, 3 \pmod 4$ and d is even	[16]
hooked Langford	l/d	$l(l - 2d + 1) + 2 \geq 0$, $l \equiv 2, 3 \pmod 4$ and d is odd, or $l \equiv 1, 2 \pmod 4$ and d is even	[16]
m-near-Skolem	n/-	$n \equiv 0, 1 \pmod 4$ and m is odd, or $n \equiv 2, 3 \pmod 4$ and m is even	[15]
hooked m-near-Skolem	n/-	$n \equiv 0, 1 \pmod 4$ and m is even, or $n \equiv 2, 3 \pmod 4$ and m is odd	[15]
m-fold Skolem	n/-	$n \equiv 0, 1 \pmod 4$ and any m, or $n \equiv 2, 3 \pmod 4$ and m is even	[18]
hooked m-fold Skolem	n/-	$n \equiv 2, 3 \pmod 4$ and m is odd	[18]

Table 8. Summary of Necessary and Sufficient Conditions for the Existence of Various Skolem-type Sequences

Construction 6. *From a Skolem-type sequence or a hooked Skolem-type sequence of order t, construct the pairs (a_i, b_i) such that $b_i - a_i = i$ for $1 \leq i \leq t$. From these pairs, along with an arbitrary positive integer c, form one of the following sets of triples*

1. *$(0, a_i + c, b_i + c)$, $1 \leq i \leq t$, or*

2. *$(0, i, b_i + c)$, $1 \leq i \leq t$.*

The triples give a labelling of a C_3^t, with the central vertex labelled 0.

Lemma 2. *Let $c \geq t$ be an arbitrary positive integer.*

1. *Using any Skolem sequence of order t, with Construction 6(1) gives the edge labels $[1, t] \cup [c + 1, c + 2t]$, and the vertex labels from $\{0\} \cup [c + 1, c + 2t]$, such that each nonzero label occurs exactly once, and*

2. *using any hooked Skolem sequence of order t, with Construction 6(1) gives the edge labels $[1, t] \cup [c + 1, c + 2t - 1] \cup \{c + 2t + 1\}$, and the vertex labels $\{0\} \cup [c + 1, c + 2t - 1] \cup \{c + 2t + 1\}$, such that each nonzero label occurring exactly once, and*

3. *using any Skolem sequence with Construction 6(2) gives the edge labels $[1, t] \cup [c + 1, c + 2t]$ and the vertex labels from $\{0\} \cup [1, t] \cup [c + 2, c + 2t]$, each nonzero label occurring exactly once, and*

4. *using any hooked Skolem sequence with Construction 6(2) gives the edge labels $[1, t] \cup [c + 1, c + 2t - 1] \cup \{c + 2t + 1\}$ and the vertex labels from $\{0\} \cup [1, t] \cup [c + 2, c + 2t - 1,] \cup \{c + 2t + 1\}$, each nonzero label occurring exactly once.*

Proof. Start with a Skolem sequence S_t of order t, and construct the triples $(0, a_i + c, b_i + c)_{i=1}^t$ by using Construction 6(1). Since $1 \leq a_i \leq 2t - 1$ and $2 \leq b_i \leq 2t$, and all a_i and b_i are distinct, the vertex

labels are $\{0\} \cup [c + 1, c + 2t]$, where no vertex label other than 0 is repeated. Considering the edge labels, we see that they are $b_i - a_i = i$, $a_i + c$, and $b_i + c$. Since by construction, $1 \le i \le t$ and all the a_i and b_i are distinct, we obtain edge labels $[1, t] \cup [c + 1, c + 2t]$, a union of disjoint sets since $c \ge t$, all of which are distinct.

A similar argument holds for hooked Skolem sequences, and for each case of Construction 6(2).

$\square$

We make note of one more special case of a Skolem-type sequence used with Construction 6(1).

Lemma 3. *Using a Langford sequence with Construction 6(1) and $c \ge d + l - 1$ gives edge labels $[d, d + l - 1] \cup [c + 1, c + 2l]$ and vertex labels from $\{0\} \cup [c + 1, c + 2l]$, each nonzero label occurring exactly once.*

To illustrate Constructions 6(1) and 6(2), consider for example, $(3, 1, 1, 3, 2, 0, 2)$, a hooked Skolem sequence of order 3 which yields the pairs $(2, 3)$, $(5, 7)$, $(1, 4)$. Letting $c = 3$, these pairs yield the triples to near gracefully label a C_3^3: $(0, 5, 6)$, $(0, 8, 10)$, and $(0, 4, 7)$ by Construction 6(1), and $(0, 1, 6)$, $(0, 2, 10)$, and $(0, 3, 7)$ by Construction 6(2). Taking $c = t$ and considering only Skolem and hooked Skolem sequences in this construction, we can use the resulting triples to (near) gracefully label Dutch windmills (C_3^t) for any $t \ge 1$, as well as some related graphs. See [11] and [19].

Koh, Rogers, Lee, and Toh [3] conjectured that C_n^t is graceful if and only if $nt \equiv 0, 3 \pmod 4$. In 2005, Yang et al. in [6] have shown the conjecture true for $n = 5$. In Theorem 8, we prove that $G = C_5^p$ is near graceful when $p \equiv 1, 2 \pmod 4$, and verify that G is graceful when $p \equiv 0, 3 \pmod 4$ through the use of (hooked) Skolem and (hooked) near-Skolem sequences. Then, in Section 5, we use this construction to prove (near) graceful labellings exist for families of $C_3^t C_5^p$. In Construction 7 we will use (hooked) Skolem and (hooked) near-Skolem sequences together to form the 5-tuples to label a C_5^p.

Construction 7. *Let $p > 0$. Given a (hooked) Skolem sequence S_1 of order $p = 4m + k$ where $0 \le k \le 3$, construct the pairs (a_i, b_i) such that $b_i - a_i = i$ for $1 \le i \le p$. Select Skolem sequence S_2 and construct the pairs (c_j, d_j) as follows.*

1. *For $k = 0$, select the Skolem sequence of order $2p$ using Table 12, in Appendix A.*

2. *For $k = 1$, select the hooked Skolem sequence of order $2p$ using Table 14, in Appendix A.*

3. *For $k = 2$, select the hooked near-Skolem sequence of order $2p + 1$ using Table 2.*

4. *For $k = 3$, select the near-Skolem sequence of order $2p + 1$ using Table 3.*

Then form the 5-tuple $(0, d_{b_i} + p, b_i, a_i, d_{a_i} + p)$ for each $1 \le i \le p$.

Note that by construction, for $k = 0$ and $k = 1$, no right endpoints occur in positions 1 to p of S_2. Similarly, S_2 has no right endpoints in positions $1, 2, \ldots, p - 1$, nor in $p + 1$ when $k = 2$ and $k = 3$.

Theorem 8. *If $G = C_5^p$, then G is near graceful when $p \equiv 1, 2 \pmod 4$ and G is graceful when $p \equiv 0, 3 \pmod 4$.*

Proof. **Case 1.** Let $p \equiv 1 \pmod 4$. Form the 5-tuples $(0, d_{b_i} + p, b_i, a_i, d_{a_i} + p)$ for each $1 \le i \le p$ as indicated in Construction 7. We begin by considering the vertex labels used by the 5-tuples.

Note that from the Skolem sequence S_1 the entries a_i and b_i in the third and fourth entries of the 5-tuples give the distinct numbers $[1, 2p]$. As we know, there are no right endpoints (d_j) in the first p positions in the hooked Skolem sequence. The first right endpoint is in position $p + 1$, so the set of possible positions for right endpoints are $[p + 1, 4p - 1] \cup \{4p + 1\}$. Thus, the second and fifth entries of the 5-tuples are all elements of $[2p + 1, 5p - 1] \cup \{5p + 1\}$. In Skolem sequences, a_i and b_i are distinct, so in the hooked Skolem sequence, d_{a_i} and d_{b_i} are distinct. Further, we know that

the entries a_i and b_i on the third and fourth entries of the 5-tuples give the distinct numbers $[1, 2p]$, and $d_{a_i}, d_{b_i} \geq p + 1$ so the minimum value of the second and fifth entries of the 5-tuples is $2p + 1$. Therefore, all the nonzero entries of the 5-tuples are distinct.

From the above discussion, it is clear that the only vertex label repeated is 0 (p times), and that all vertices are distinct and come from the set $[0, 5p - 1] \cup \{5p + 1\}$.

We now consider the edge labels defined by the difference between subsequent entries (taken cyclically) in the 5-tuple $(0, d_{b_i} + p, b_i, a_i, d_{a_i} + p)$.

Since $b_i - a_i = i$, these differences are all distinct and comprise the set $[1, p]$. Based on our previous discussion, the differences between $d_{b_i} + p$ and 0 and the differences between $d_{a_i} + p$ and 0 give distinct numbers from the set $[2p + 1, 5p - 1] \cup \{5p + 1\}$. Considering the remaining differences, we see that $(d_{b_i} + p) - b_i = (d_{b_i} - b_i) + p = c_{b_i} + p$ and $(d_{a_i} + p) - a_i = (d_{a_i} - a_i) + p = c_{a_i} + p$. These differences are all distinct numbers in the set $[p + 1, 5p - 1]$. Since $\bigcup_{i=1}^{p} \{a_i, b_i\} = [1, p]$, then

$\bigcup_{i=1}^{p} \{c_{a_i}, c_{b_i}, d_{a_i}, d_{b_i}\} = [1, 4p - 1] \cup \{4p + 1\}$, and hence $\bigcup_{i=1}^{p} \{c_{a_i} + p, c_{b_i} + p, d_{a_i} + p, d_{b_i} + p\}$ is exactly $[p + 1, 5p - 1] \cup \{5p + 1\}$, since all the c_j and d_j are distinct.

From the above discussion, it is clear that all edges are distinct and are exactly the set $[1, 5p - 1] \cup \{5p + 1\}$. We can conclude that since the vertex labels are a subset of $[0, 5p - 1] \cup \{5p + 1\}$, each nonzero label occurring exactly once, and the edge labels are exactly $[1, 5p - 1] \cup \{5p + 1\}$, C_5^p can be near gracefully labelled when $p \equiv 1 \pmod 4$.

Case 2. The case $p \equiv 0 \pmod 4$, is proved similarly to Case 1. Use a Skolem sequence to construct the pairs a_i and b_i. Instead of a hooked Skolem sequence, use a Skolem sequence to construct the pairs d_{a_i} and d_{b_i}. The resulting C_5^p can be gracefully labelled when $p \equiv 0 \pmod 4$.

Case 3. If $p \equiv 2 \pmod 4$ and $p > 2$, the statement is proved similarly to Case 1. Instead of a Skolem sequence, use a hooked Skolem sequence to construct the pairs a_i and b_i. Instead of a hooked Skolem sequence, use a hooked near-Skolem sequence to construct the pairs d_{a_i} and d_{b_i}. We know such a sequence exists by Lemma 1. For $p = 2$, use the 5-tuples $(0, 11, 2, 9, 1)$ and $(0, 6, 3, 7, 5)$. Therefore, C_5^p can be near gracefully labelled when $p \equiv 2 \pmod 4$.

Case 4. If $p \equiv 3 \pmod 4$ and $p > 3$, the statement is proved similarly to Case 1. Instead of a Skolem sequence, use a hooked Skolem sequence to construct the pairs a_i and b_i. Instead of a hooked Skolem sequence, use a near-Skolem sequence to construct the pairs d_{a_i} and d_{b_i}. We know such a sequence exists by Lemma 1. For $p = 3$, use the 5-tuples $(0, 15, 1, 14, 12)$, $(0, 5, 6, 3, 10)$ and $(0, 9, 13, 2, 8)$. Therefore, C_5^p can be gracefully labelled when $p \equiv 3 \pmod 4$.   □

3.2. Two-fold Skolem-type Sequence Constructions

In this section we discuss how to label C_4^s using two-fold Skolem-type sequences.

Construction 9. *From a two-fold Skolem-type sequence of order s, construct the pairs of the form $\left(c_j, d_j\right)$ and $\left(e_j, f_j\right)$ where $c_j, d_j, e_j,$ and f_j are the entries where j occurs in the two-fold Skolem-type sequence, $c_j < d_j, e_j < f_j$, with $j \in H$ and $d_j - c_j = f_j - e_j = j$. From these pairs we can obtain s quadruples of the form $\left(0, d_j + c, j, f_j + c\right)$, where c is a fixed positive integer. These quadruples admit a labelling of C_4^s.*

In this paper, we will use Construction 9 with a variety of two-fold Skolem-type sequences which will produce edge and vertex labels as given in Table 9. The symbols t and c are constants, s and l are the order of the sequences, and d is the defect of the Langford sequence.

Lemma 4. *The results in Table 9 are correct, with each nonzero label occurring exactly once.*

Proof. We will prove that the result in the first row of Table 9 is correct. The other rows follow similarly.

Sequence	Edge labels are:	Vertex labels from:
Two-fold Skolem sequences of order $s \le c + 1$	$[c + 1, c + 4s]$	$[0, s] \cup [c + 2, c + 4s]$
Two-fold Skolem sequences from Table 5 of order $s \le 2c + 1$ and s is odd	$[c + 1, c + 4s]$	$[0, s] \cup \left[c + \frac{s+3}{2}, c + 4s\right]$
Two-fold Skolem sequences from Table 6 of order $s \le 2c$ and s is even	$[c + 1, c + 4s]$	$[0, s] \cup \left[c + \frac{s+2}{2}, c + 4s\right]$
Double Langford sequence from Table 1 with defect $d \le c + 1$ and order l	$[c + 1, c + 4l]$	$\{0\} \cup [d, 3d - 2] \cup [c + 2d, c + 4l]$
Double Skolem sequences of order $s \le 2c + 2$ from Tables 12, 14	$[c + 1, c + 4s]$	$[0, s] \cup \left[c + \frac{s+4}{2}, c + 4s\right]$
Double Skolem sequences of order $s \le 2c + 1$ from Tables 13, 15	$[c + 1, c + 4s]$	$[0, s] \cup \left[c + \frac{s+3}{2}, c + 4s\right]$
Two-fold Skolem-type sequence of order $s \le \frac{(c+4)}{2}$ from Table 4	$[c + 1, c + 4s]$	$\{0\} \cup [c + 2s + 1, c + 4s] \cup H$
Two-fold Skolem-type sequence C^1, with $c \ge 0$	$[c + 1, c + 4]$	$\{0, 2\} \cup [c + 3, c + 4]$
Two-fold Skolem-type sequence C^2, with $c \ge 1$	$[c + 1, c + 8]$	$\{0, 2, 3\} \cup \{c + 3, c + 5, c + 7, c + 8\}$
Two-fold Skolem-type sequence C^3, with $c \ge 3$	$[c + 1, c + 12]$	$\{0, 2, 3, 5\} \cup [c + 3, c + 4]$ $\cup [c + 9, c + 12]$
Two-fold Skolem-type sequence C^4, with $c \ge 2$	$[c + 1, c + 16]$	$\{0, 2, 3, 5, 6\} \cup [c + 5, c + 8]$ $\cup [c + 13, c + 16]$
Two-fold Langford sequences from Table 7 with $d = 6k - 1$ and $l = 4k - 1$, and $c \ge 2k - 1$	$[c + 1, c + 16k - 4]$	$\{0\} \cup [6k - 1, 10k - 3]$ $\cup [c + 8k - 1, c + 16k - 4]$

Table 9. Summary of Results of Construction 9 with a Variety of two-fold Skolem-type Sequences

From a two-fold Skolem sequence S_s^2 made up of pairs (c_j, d_j) and (e_j, f_j), construct the quadruples $\left(0, d_j + c, j, f_j + c\right)_{j=1}^{s}$ as given by Construction 9 with $c \ge s - 1$. As we know $2 \le d_j \le 4s - 1$ and $4 \le f_j \le 4s$ and $1 \le j \le s$, all the vertex labels will be distinct and from the set $[1, s] \cup [c + 2, c + 4s]$, except 0, which will be repeated s times. Considering the differences of these quadruples, we see that $\left(d_j + c\right) - j = c_j + c, \left(f_j + c\right) - j = e_j + c, d_j + c - 0 = d_j + c$, and $f_j + c - 0 = f_j + c$. Since, by construction, c_j, d_j, e_j, f_j are all distinct, we obtain $[c + 1, c + 4s]$ as the set of distinct edge labels. □

Note that for the result in the first row of Table 9 there are no restrictions on the sequence so the set of vertex labels is large. If we restrict the sequence, as in the second and third rows of Table 9, we will refine the set of vertex labels.

In Constructions 10–13, we will give labellings of $C_3^t C_4^s$ when $2t < s \le \frac{(13t+37)}{2}$, with the central vertex labelled 0.

Let P_x be the two-fold Skolem-type sequence of order x given by Construction 4. Let P_x' be the the sequence obtained from P_x by removing the pair $(1, 1)$ from the end of the sequence. Define P_0' to be the sequence $(1, 1)$.

Construction 10. *From a double Langford sequence with defect $d = t + 1$ and order $l = 2t + 1$, construct the quadruples of the form $\left(0, d_j + c, j, f_j + c\right)$ as indicated in Construction 9 with $c = t$. From a Skolem sequence of order t, construct the triples $(0, a_i + c, b_i + c)$ as indicated in Construction 6 with $c = 4l + t$. These triples and quadraples give a labelling for a $C_3^t C_4^s$ where $s = 2t + 1$, with the central vertex labelled 0.*

Construction 11. *By concatenating a double Langford sequence with defect $d = t + 1$ and order*

$l = 2t + 1$, *with a two-fold Skolem sequence of order k ($k \leq t$), construct the quadruples of the form* $\left(0, d_j + c, j, f_j + c\right)$ *with $c = t$ as indicated in Construction 9. From a Skolem sequence of order t, construct the triples $(0, a_i + c, b_i + c)$ as indicated in Construction 6 with $c = 4k + 4l + t$. These triples and quadruples give a labelling for a $C_3^t C_4^s$ where $2t + 2 \leq s \leq 3t + 1$, with the central vertex labelled 0.*

Construction 12. *By concatenating a two-fold Skolem-type sequence P'_{x-1} of order $x - 1$ and $1 \leq x \leq \frac{(t+3)}{2}$, with a double Langford sequence with defect $d = t + 4x - 1$ and order $l = 2t + 8x - 3$, with the sequence P'_0, and a two-fold Skolem-type sequence from Proposition 1 of order y, for $0 \leq y \leq 4$, construct the quadruples of the form $\left(0, d_j + c, j, f_j + c\right)$ with $c = t$ as indicated in Construction 9. From a Skolem sequence of order t construct the triples $(0, a_i + c, b_i + c)$ as indicated in Construction 6 with $c = 4l + t + 4x + 4y + 4$. These triples and quadruples give a labelling for a $C_3^t C_4^s$ where $s = 2t + 9x + y - 3$ and $3t + 2 \leq s \leq \frac{(13t+37)}{2}$, with the central vertex labelled 0.*

Construction 13. *By concatenating a two-fold Skolem-type sequence P_x of order x with $1 \leq x \leq \frac{(t+3)}{2}$, with a double Langford sequence with defect $d = t + 4x + 1$ and order $l = 2t + 8x + 1$, with a two-fold Skolem-type sequence from Proposition 1 of order y, with $0 \leq y \leq 4$, construct the quadruples of the form $\left(0, d_j + c, j, f_j + c\right)$ with $c = t$ as indicated in Construction 9. From a Skolem sequence of order t construct the triples $(0, a_i + c, b_i + c)$ as indicated in Construction 6 with $c = 4l + t + 4x + 4y$. These triples and quadruples give a labelling for a $C_3^t C_4^s$ where $s = 2t + 9x + y + 1$ and $3t + 2 \leq s \leq \frac{(13t+37)}{2}$, with the central vertex labelled 0.*

Note that Constructions 12 and 13 will cover all the values of $s \in \left[3t + 2, \frac{(13t+37)}{2}\right]$.

The bound $s \leq \frac{(13t+37)}{2}$ comes from Constructions 12 and 13 as we know $s = 2t + 9x + z$ or $x = \frac{(s-2t-z)}{9}$. Since $t \geq 2x - 3$ then $s \leq \frac{(13t+2z+27)}{2}$ and since $z \in [-3, 5]$ we obtain that $s \leq \frac{(13t+37)}{2}$.

Lemma 5. *In the labellings of $C_3^t C_4^s$ given by Constructions 10 - 13 for $4 \leq t \leq s \leq \frac{(13t+37)}{2}$, and $t \geq 2x - 3$ when $s \geq 3t + 2$,*

1. *if $t \equiv 0, 1 \pmod 4$, then the edge labels used are $[1, 4s + 3t]$ and the vertex labels used are from $\{0\} \cup [1, 4s + 3t]$, where each nonzero label occurs exactly once,*

2. *if $t \equiv 2, 3 \pmod 4$ then the edge labels used are $[1, 4s + 3t - 1] \cup \{4s + 3t + 1\}$ and the vertex labels used are from $\{0\} \cup [1, 4s + 3t - 1] \cup \{4s + 3t + 1\}$, where each nonzero label occurs exactly once.*

Proof. We will prove these results for Construction 13. The proof for Construction 12 follows in the same fashion as Construction 13. For Constructions 10 and 11 the proofs also follow in the same fashion as Construction 13, but with no Skolem-type sequences.

Let $t \equiv 0, 1 \pmod 4$. We begin by considering the edge labels. Consider the quadruples formed by the concatenated sequence in Construction 13, with $c = t$. Those quadruples corresponding to P_x yield edge labels $[t + 1, t + 4x]$, by Table 9 (row 7). For the quadruples corresponding to the double Langford sequence, we obtain the edge labels $[t + 4x + 1, 4l + t + 4x]$, by Table 9 (row 4), by considering $c = t + 4x$ (the length of P_x). For the quadruples corresponding to the two-fold Skolem-type sequence from Proposition 1, we obtain the edge labels $[4l + t + 4x + 1, 4l + t + 4x +4y]$, by Table 9 (rows 9-12), by considering $c = 4l + t + 4x$ (the length of P_x and the double Langford sequence). Note that if $y = 0$, then we are considering C^0, the empty sequence, and hence produce no edge labels.

Consider the triples formed by the Skolem sequence of order t with $c = 4l + t + 4x + 4y$ (the length of P_x, the double Langford sequence, and the two-fold Skolem-type sequence). By Lemma 2(1), this construction yields edge labels $[1, t] \cup [4l + t + 4x + 4y + 1, 4l + 3t + 4x + 4y]$.

From the above discussion, it is clear that all edges are distinct and are exactly the set $[1, 4s + 3t]$, where $s = l + x + y$.

We now consider the vertex labels. Consider the quadruples formed by the concatenated sequence in Construction 13, with $c = t$. Those quadruples corresponding to P_x yield vertex labels that are a subset of $\{0, 1\} \cup [t + 2x, t + 4x] \cup \{4i | 1 \leq i \leq x - 1\}$, by Table 9 (row 7) and since $t \geq 2x - 3$ there are no vertices repeated. For the quadruples corresponding to the double Langford sequence, we obtain vertex labels from $\{0\} \cup [t + 4x + 1, 9t + 36x + 4]$, by Table 9 (row 4), by considering $c = t + 4x$ (the length of P_x). For the quadruples corresponding to the two-fold Skolem-type sequence from Proposition 1, we obtain vertex labels from $\{0, 2, 3, 5, 6\} \cup [9t + 36x + 7, 9t + 36x + 4y + 4]$, by Table 9 (rows 9-12), by considering $c = 4l + t + 4x$ (the length of P_x and double Langford sequence). The labels $\{0, 2, 3, 5, 6\}$ are only used once so they do not conflict with any other vertex labels, since $t + 2x \geq 6$ and none of these labels is a multiple of four. Note that if $y = 0$, then we are considering C^0, the empty sequence, and hence produce no vertex labels.

Consider the triples formed by the Skolem sequence of order t with $c = 4l + t + 4x + 4y$ (the length of P_x, double Langford sequence, and the two-fold Skolem-type sequence). By Lemma 2(1), this construction yields vertex labels from $\{0\} \cup [9t + 36x + 4y + 5, 11t + 36x + 4y + 4]$.

From the above discussion, it is clear that all vertex labels are distinct and are from the set $[0, 4s + 3t]$ where $s = l + x + y$.

If $t \equiv 2, 3 \pmod 4$, we proceed similarly to the proof for the case $t \equiv 0, 1 \pmod 4$, but use a hooked Skolem sequence instead of Skolem sequence with Construction 6 and Lemma 2(2). $\square$

4. $C_3^t C_4^s$

In this section, we prove (near) graceful labellings exist for $C_3^t C_4^s$.

Theorem 14. *If $G = C_3^t C_4^s$, where $t \geq s \geq 1$, then G is graceful when $t \equiv 0, 1 \pmod 4$ and near graceful when $t \equiv 2, 3 \pmod 4$.*

Proof. **Case 1**. $t \equiv 0, 1 \pmod 4$.

Use a two-fold Skolem sequence of order s, with Construction 9 and with $c = t$, to get $\left(0, d_j + c, j, f_j + c\right)$, $1 \leq j \leq s$. Using a Skolem sequence of order t in Construction 6 with $c = 4s + t$ and $t \geq s$, gives $(0, a_i + c, b_i + c)$, $1 \leq i \leq t$. These sequences are known to exist; see Table 8.

These two constructions give the vertex labels and the induced edge labels of G. By Table 9 (row 1) the quadruples use vertex labels in $[0, s] \cup [t + 2, 4s + t]$ and edge labels $[t + 1, 4s + t]$ and by Lemma 2(1) the triples use vertex labels in $\{0\} \cup [4s + t + 1, 4s + 3t]$ and edge labels $[1, t] \cup [4s + t + 1, 4s + 3t]$. Thus, this is a graceful labelling.

Case 2. $t \equiv 2, 3 \pmod 4$.

This is similar to Case 1 using a hooked Skolem sequence and Lemma 2(2) instead of a Skolem sequence and Lemma 2(1). $\square$

As an example, consider the two-fold Skolem sequence $(3, 1, 1, 3, 2, 2, 2, 2, 3, 1, 1, 3)$. This sequence gives quadruples and triples that together gracefully label $G = C_3^4 C_4^3$ (see Figure 1).

Theorem 15. *If $G = C_3^t C_4^s$, where $4 \leq t \leq s \leq \frac{(13t+37)}{2}$, then G is graceful when $t \equiv 0, 1 \pmod 4$ and is near graceful when $t \equiv 2, 3 \pmod 4$.*

Proof. **Case 1**. $t \equiv 0, 1 \pmod 4$.

For $4 \leq t < s \leq 2t$ and s odd, use a two-fold Skolem sequence of order s as given in Table 5, with Construction 9 and with $c = t$, to get $\left(0, d_j + c, j, f_j + c\right)$, $1 \leq j \leq s$. Using a Skolem sequence of order t in Construction 6 with $c = 4s + t$ and $t < s$, gives $(0, a_i + c, b_i + c)$, $1 \leq i \leq t$. These sequences exist, as detailed in Table 8.

These two constructions give the vertex labels and the induced edge labels of G. By Table 9 (row 2) the quadruples use vertex labels in $[0, s] \cup \left[\frac{(s+3)}{2} + t, 4s + t\right]$ and edge labels $[t + 1, 4s + t]$

and by Lemma 2(1) the triples use vertex labels in $\{0\} \cup [4s + t + 1, 4s + 3t]$ and produce edge labels $[1, t] \cup [4s + t + 1, 4s + 3t]$. Thus, this is a graceful labelling.

For $4 \leq t < s \leq 2t$ and s even, use a two-fold Skolem sequence of order s as given in Table 6, with Construction 9 and $c = t$, to get $(0, d_j + c, j, f_j + c)$, $1 \leq j \leq s$. Using a Skolem sequence of order t in Construction 6 with $c = 4s + t$ and $t < s$, gives $(0, a_i + c, b_i + c)$, $1 \leq i \leq t$. These sequences exist, as detailed in Table 8.

These two constructions give the vertex labels and the induced edge labels of G. By Table 9 (row 2) the quadruples use vertex labels in $[0, s] \cup \left[\frac{(s+2)}{2} + t, 4s + t \right]$ and edge labels $[t + 1, 4s + t]$ and by Lemma 2(1) the triples use vertex labels in $\{0\} \cup [4s + t + 1, 4s + 3t]$ and edge labels $[1, t] \cup [4s + t + 1, 4s + 3t]$. Thus, this is a graceful labelling.

For $4 \leq t < s = 2t + 1$ use Construction 10, and for $2t + 2 \leq s \leq 3t + 1$ use Construction 11. These constructions give the vertex labels and the induced edge labels of G. By Lemma 5(1) the quadruples and triples use vertex labels in $[0, 4s + 3t]$ and edge labels $[1, 4s + 3t]$. Thus, this is a graceful labelling.

We now consider the case $3t + 2 \leq s \leq \frac{(13t+37)}{2}$. Define $I_x = [2t + 9x - 3, 2t + 9x + 1]$, and $J_x = [2t + 9x + 1, 2t + 9x + 5]$ for $x \geq 1$ and fixed t. Define $K_x = [2t + 9x - 3, 2t + 9x + 5] = I_x \cup J_x$. Note that $K_x \cap K_{x+1} = \emptyset$, but for any interval K_x the largest element is $2t + 9x + 5$ and the smallest element in K_{x+1} is $2t + 9x + 6$. Note $\bigcup_{x \geq 1} K_x = [2t + 6, \infty)$, and so for all integers $s \in [2t + 6, \infty)$, there exists x such that $s \in K_x$, and hence $s \in I_x$ or $s \in J_x$. That is, s can be written either in the form $s = 2t + 9x + y - 3$ or the form $s = 2t + 9x + y + 1$, where $0 \leq y \leq 4$, for some x.

For $s = 2t + 9x + y - 3$, and $3t + 2 \leq s \leq \frac{(13t+37)}{2}$ use Construction 12, and for $s = 2t + 9x + y + 1$, and $3t + 2 \leq s \leq \frac{(13t+37)}{2}$ use Construction 13. These constructions give the vertex labels and the induced edge labels of G. By Lemma 5(1) the quadruples and triples use vertex labels in $[0, 4s + 3t]$ and edge labels $[1, 4s + 3t]$. Thus, this is a graceful labelling.

Case 2. $t \equiv 2, 3 \pmod 4$.

Similar to Case 1, but use a hooked Skolem sequence, Lemma 2(2), and Lemma 5(2) instead of a Skolem sequence, Lemma 2(1) and Lemma 5(1). Then, this is a near graceful labelling.   □

In Theorem 14 and Theorem 15 we proved (near) graceful labellings exist for $C_3^t C_4^s$ if $1 \leq s \leq t$ or $4 \leq t \leq s \leq \frac{(13t+37)}{2}$, omitting the cases for $t = 1, 2, 3$. So, in the following lemma we consider those cases.

Lemma 6. *For $k \geq 1$ and $x \geq 1$, if*

   1. $x \equiv 0 \pmod 4$ when $w \geq 1$ and $\frac{(2k-12w+2)}{4} \leq s \leq \frac{(6k-12w-5)}{4}$, or

   2. $x \equiv 1 \pmod 4$ when $w \geq 0$ and $\frac{(2k-12w-1)}{4} \leq s \leq \frac{(6k-12w-8)}{4}$, and

a graceful labelling of $C_3^{4w+x} C_4^s$ exists, then a graceful labelling of $C_3^{4w+x} C_4^{s+4k-1}$ exists. Alternatively, if

   3. $x \equiv 2 \pmod 4$ when $w \geq 0$ and $\frac{(2k-12w-2)}{4} \leq s \leq \frac{(6k-12w-9)}{4}$, or

   4. $x \equiv 3 \pmod 4$ when $w \geq 0$ and $\frac{(2k-12w-3)}{4} \leq s \leq \frac{(6k-12w-10)}{4}$, and

a near graceful labelling of $C_3^{4w+x} C_4^s$ exists which contains a triangle $(0, 4s + 12w + 5, 4s + 12 + 7)$ if $x \equiv 2 \pmod 4$ or contains triangles $(0, 4s + 12w + 7, 4s + 12w + 8)$ and $(0, 4s + 12w + 6, 4s + 12w + 10)$, then a near graceful labelling of $C_3^{4w+x} C_4^{s+4k-1}$ exists.

Proof. First, note that if $C_3^{4w+x} C_4^s$ is graceful, it uses edge labels $[1, 4s + 12w + 3x]$ and vertex labels from $[0, 4s + 12w + 3x]$. If if $C_3^{4w+x} C_4^s$ is near graceful, it uses edge labels $[1, 4s + 12w + 3x - 1] \cup \{4s + 12w + 3x + 1\}$ and vertex labels from $[0, 4s + 12w + 3x - 1] \cup \{4s + 12w + 3x + 1\}$. We consider four cases corresponding to the four possibilities of the lemma.

Case 1. By using a two-fold Langford sequence with defect $d = 6k - 1$ and order $l = 4k - 1$ from Table 7 with Construction 9, $c = 4s + 12w$, we can label a C_4^{4k-1} with edge labels $[4s + 12w + 1, 16k + 4s + 12w - 4]$ and vertex labels from $\{0\} \cup [6k - 1, 10k - 3] \cup [8k + 4s + 12w - 1, 16k + 4s + 12w - 4]$.

Note that to make sure any vertex label is only used at most once, $6k - 1$ must be greater than $4s + 12w$ and since $\frac{(6k-12w-5)}{4} \geq s$, we have no vertex labels used twice. Also, to avoid a conflict in vertex labelling, $8k + 4s + 12w - 1 > 10k - 3$ and since $s \geq \frac{(2k-12w+2)}{4}$, we have no vertex labels used twice.

By identifying the vertices with label zero in $C_3^{4w}C_4^s$ and C_4^{4k-1}, the labelling we obtain is a graceful labelling of $C_3^{4w}C_4^{s+4k-1}$ with edge labels $[1, 16k + 4s + 12w - 4]$ and vertex labels from $[0, 16k + 4s + 12w - 4]$.

Case 2. Again, by using a two-fold Langford sequence with defect $d = 6k - 1$ and order $l = 4k - 1$ from Table 7 with Construction 9, $c = 4s + 12w + 3$, we can label a C_4^{4k-1} with edge labels $[4s + 12w + 4, 16k + 4s + 12w - 1]$ and vertex labels from $\{0\} \cup [6k - 1, 10k - 3] \cup [8k + 4s + 12w + 2, 16k + 4s + 12w - 1]$. As in the previous case, by choice of s, we avoid re-using labels.

Then by identifying the vertices with label zero in $C_3^{4w+1}C_4^s$ and C_4^{4k-1}, we obtain is a graceful labelling of $C_3^{4w+1}C_4^{s+4k-1}$.

Case 3. By using a two-fold Langford sequence with defect $d = 6k - 1$ and order $l = 4k - 1$ from Table 7 with Construction 9, $c = 4s + 12w + 4$, we can label a C_4^{4k-1} with edge labels $[4s + 12w + 5, 16k + 12w + 4s]$ and vertex labels from $\{0\} \cup [6k - 1, 10k - 3] \cup [8k + 4s + 12w + 3, 16k + 4s + 12w]$. As in case 1, choice of s allows us to avoid re-using vertex and edge labels.

If we replace the triangle containing edge length 2, $(0, 4s + 12w + 5, 4s + 12w + 7)$, by $(0, 16k + 4s + 12w + 1, 16k + 4s + 12w + 3)$ and identify the vertices with label zero in the $C_3^{4w+2}C_4^s$ and the C_4^{4k-1} labelling, then we obtain a near graceful labelled $C_3^{4w+2}C_4^{s+4k-1}$.

Case 4. By using a two-fold Langford sequence with defect $d = 6k - 1$ and order $l = 4k - 1$ from Table 7 with Construction 9, $c = 4s + 12w + 5$, we can label a C_4^{4k-1} with edge labels $[4s + 12w + 6, 16k + 4s + 12w + 1]$ and vertex labels from $\{0\} \cup [6k - 1, 10k - 3] \cup [8k + 4s + 12w + 4, 16k + 4s + 12w + 1]$. As in case 1, by choice of s we eliminate the possibility of re-using labels.

Replace the triangles containing edge lengths 1 and 4, $(0, 4s + 12w + 7, 4s + 12w + 8)$ and $(0, 4s + 12w + 6, 4s + 12w + 10)$, by $(0, 16k + 4s + 12w + 3, 16k + 4s + 12w + 4)$ and $(0, 16k + 4s + 12w + 2, 16k + 4s + 12w + 6)$. If we then identify the vertices with label zero in the $C_3^{4w+3}C_4^s$ and the C_4^{4k-1} labelling, we obtain a near graceful labelled $C_3^{4w+3}C_4^{s+4k-1}$. □

Note that in the near graceful cases, the resulting near graceful labellings contain triangles of the appropriate form to satisfy the criteria of the lemma, allowing an iterative use.

One of the difficulties of Lemma 6 is that it gives what is essentially an existential result. Consider the graph $C_3^4 C_4^{100}$. Then certainly, if we are going to apply Lemma 6, we must consider its first case. But for what combination of s and k? We must find a pair (s, k) such that $s + 4k - 1 = 100$ and such that $\frac{(2k-10)}{4} \leq s \leq \frac{(6k-117)}{4}$. One such pair is $(21, 20)$. In that case, if $C_3^4 C_4^{21}$ is graceful, we are done. Happily, we can conclude that $C_3^4 C_4^{21}$ is graceful, by Theorem 15. Of course, this isn't the only pair that satisfies these two constraints, and different pairs will produce different labellings. (The pair $(17, 21)$ is another such solution, since Theorem 15 tells us that $C_3^4 C_4^{17}$ is graceful.)

Thus, to prove that an arbitrary graph $C_3^t C_4^s$ is (near) graceful, we will show that an appropriate k can be found to satisfy the appropriate condition in Lemma 6, and that a smaller (near) graceful labelling exists by one of Theorems 14 or 15.

Theorem 16. *If $s \in \mathbb{N}$, and $t = 4w + z$ with $z = 0, 1, 2, 3$ then $C_3^t C_4^s$ can be gracefully labelled if $t = 4w, 4w + 1$ and near gracefully labelled if $t = 4w + 2, 4w + 3$.*

Proof. We consider four cases based on the modularity of t. We will repeatedly use induction on s, the number of 4-cycles in $C_3^{4w+z}C_4^s$. For $w = 0$ and $z = 1, 2,$ or 3, see Appendix B for (near) graceful labellings when s is small. For $w \geq 1$ and any z note that the base cases needed for the induction (below the indicated value of s in each case) can be given by Theorems 14 and 15.

Case 1. $t = 4w$.

Define I_k to be the real interval $\left[\frac{(18k-12w-2)}{4}, \frac{(22k-12w-9)}{4}\right]$ for fixed k. If $k \geq \frac{25}{4}$ then $\frac{(18(k+1)-12w-2)}{4} \leq \frac{(22k-12w-9)}{4}$. That is, $I_k \cap I_{k+1} \neq \emptyset$. This implies $\bigcup_{k \geq 7} I_k = \left[\frac{124-12w}{4}, \infty\right)$, for fixed arbitrary w, and so for all $s \in \left[\frac{121-12w}{4}, \infty\right)$, there exists k such that $s \in I_k$.

We proceed by induction on s. Let $s \geq 28$ be an integer. There exists some k such that $s \in I_k$. Therefore, letting $s = 4k + s' - 1$, then $\frac{(2k-12w+2)}{4} \leq s' \leq \frac{(6k-12w-5)}{4}$. By induction, a graceful labelling of $C_3^{4w}C_4^{s'}$ exists, and by Lemma 6 a graceful labelling of $C_3^{4w}C_4^s$ exists.

Case 2. $t = 4w + 1$.

As in case 1, if $I_k = \left[\frac{(18k-12w-5)}{4}, \frac{(22k-12w-12)}{4}\right]$, then for all $s \in \bigcup_{k \geq 7} I_k = \left[\frac{121-12w}{4}, \infty\right)$, there exists k such that $s \in I_k$. Again, following by induction on s, a graceful labelling of $C_3^{4w+1}C_4^{s'}$ exists, and by Lemma 6 a graceful labelling of $C_3^{4w+1}C_4^s$ exists.

Case 3. $t = 4w + 2$.

Again, define $I_k = \left[\frac{(18k-12w-6)}{4}, \frac{(22k-12w-13)}{4}\right]$. Then for all $s \in \bigcup_{k \geq 7} I_k = \left[\frac{120-12w}{4}, \infty\right)$, there exists k such that $s \in I_k$. With induction on s, a graceful labelling of $C_3^{4w+2}C_4^{s'}$ exists, and by Lemma 6 a graceful labelling of $C_3^{4w+2}C_4^s$ exists.

Case 4. $t = 4w + 3$.

As previously, let $I_k = \left[\frac{(18k-12w-7)}{4}, \frac{(22k-12w-14)}{4}\right]$, so that for all $s \in \bigcup_{k \geq 7} I_k = \left[\frac{119-12w}{4}, \infty\right)$, there exists k such that $s \in I_k$. Again using induction on s, a graceful labelling of $C_3^{4w+3}C_4^{s'}$ exists, and by Lemma 6 a graceful labelling of $C_3^{4w+3}C_4^s$ exists. $\square$

5. $C_3^t C_5^p$

Using Theorem 8 with Langford sequences, we can obtain (near) graceful labellings of $C_3^t C_5^p$.

In Constructions 17, we will give labellings of $C_3^t C_5^p$, with the central vertex labelled 0.

Construction 17. *Given a C_5^p gracefully labelled by the 5-tuples $(0, d_{b_i} + p, b_i, a_i, d_{a_i} + p)$ for each $1 \leq i \leq p$ formed by*

1. *Construction 7(1), if $p \equiv 0 \pmod 4$, or*

2. *Construction 7(2), if $p \equiv 1 \pmod 4$, or*

3. *Construction 7(3), if $p \equiv 2 \pmod 4$, or*

4. *Construction 7(4), if $p \equiv 3 \pmod 4$, then*

replace the 5-tuples with $(0, d_{b_i} + p + 3t, b_i, a_i, d_{a_i} + p + 3t)$ for each $1 \leq i \leq p$ and $t \geq 2p + 1$. From a Langford sequence with defect $p + 1$ and order t, form triples via Construction 6 with $c = p + t$. These 5-tuples and triples give a labelling of a $C_3^t C_5^p$, with the central vertex labelled 0.

Recall the result of Theorem 8: if $G = C_5^p$, then G is near graceful when $p \equiv 1, 2 \pmod 4$ and G is graceful when $p \equiv 0, 3 \pmod 4$. We consider both of these cases in the following theorem.

Lemma 7. *The labellings of $C_3^t C_5^p$ given by Construction 17(1) and 17(4) use edge labels $[1, 5p + 3t]$ exactly once and vertex labels from $[0, 5p + 3t]$, each nonzero label occurring exactly once, whereas the labellings given by Construction 17(2) and 17(3) use edge labels $[1, 5p + 3t - 1] \cup \{5p + 3t + 1\}$ exactly once and vertex labels from $[0, 5p + 3t - 1] \cup \{5p + 3t + 1\}$, each nonzero label occurring exactly once.*

Proof. We have four cases based on congruence modulo 4. We begin with $p \equiv 0 \pmod 4$ and $p \equiv 3 \pmod 4$ where we are using gracefully labelled C_5^p.

Case 1. If $p \equiv 0 \pmod 4$, form the 5-tuples $(0, d_{b_i} + p + 3t, b_i, a_i, d_{a_i} + p + 3t)$ for each $1 \leq i \leq p$. We begin by considering vertex labels.

In a Skolem-type sequence, the a_i, b_i, c_i and d_i are all unique. From the Skolem sequence S_1 of order p indicated in Construction 7, we notice that the entries a_i and b_i in the third and fourth entries of the 5-tuples give the distinct numbers $[1, 2p]$. Recall that the Skolem sequence S_2 of order $2p$ used in Construction 7 contains no right endpoints in the first p positions, with the first right endpoint occurring in position $p + 1$. Note $\bigcup_{i=1}^{p} \{d_{a_i}, d_{b_i}\}$ is the set of all right endpoints, and so a subset of $[p + 1, 4p]$. From the 5-tuples, we use the right endpoints as indicated in the second and fifth entries. We add $p + 3t$ to each element based on our construction so we obtain a subset of $[2p + 3t + 1, 5p + 3t]$.

Consider the triples of the form $(0, a_i + p + t, b_i + p + t)$ with $1 \leq i \leq t$ given by using Construction 6 with $c = p + t$. These triples give the vertex labels for t triangles C_3^t, where 0 is repeated t times as a common vertex. Thus by Lemma 3 these triples use vertex labels from $\{0\} \cup [p + t + 1, p + 3t]$.

Thus, in this labelling, the common vertex 0 is repeated $p + t$ times. The remaining vertices are all distinct and from the union of disjoint sets $[1, 2p] \cup [p + t + 1, p + 3t] \cup [2p + 3t + 1, 5p + 3t]$.

We now examine the edge labels from this construction, by considering the differences between subsequent entries (taken cyclically) in $(0, d_{b_i} + p + 3t, b_i, a_i, d_{a_i} + p + 3t)$. The differences between b_i and a_i produce the distinct numbers $[1, p]$.

The differences $(d_{b_i} + p + 3t) - 0$ and the differences $(d_{a_i} + p + 3t) - 0$ produce distinct numbers. Call this set of distinct numbers A and observe that $A \subseteq [2p + 3t + 1, 5p + 3t]$. Also, from $(d_{b_i} + p + 3t) - b_i = (d_{b_i} - b_i) + p + 3t = c_{b_i} + p + 3t$ and $(d_{a_i} + p + 3t) - a_i = (d_{a_i} - a_i) + p + 3t = c_{a_i} + p + 3t$, we get distinct numbers. Call this set of distinct numbers B and observe that $B \subseteq [p + 3t + 1, 5p + 3t - 1]$. We know in a Skolem sequence, the c_i and d_i are all unique so A and B are disjoint. Note that $|A \cup B| = 4p$. Now, we can conclude that all the previous differences give exactly the edge labels $[p + 3t + 1, 5p + 3t]$ exactly once.

Consider the differences from the triples of the form $(0, a_i + p + t, b_i + p + t)$ with $1 \leq i \leq t$ given by using Construction 6 with $c = p + t$. By Lemma 3 these triples use edge labels $[p + 1, p + 3t]$. If we take the union of the sets of edge labels, we obtain $[1, 5p + 3t]$.

Case 2. The case $p \equiv 3 \pmod 4$, is proved similarly to Case 1. Instead of a Skolem sequence for S_1, use a hooked Skolem sequence of order p to construct the entries a_i and b_i. Instead of a Skolem sequence for S_2, use a near-Skolem sequence of order $2p + 1$ to construct the entries d_{a_i} and d_{b_i}. The result follows in a similar fashion.

For the two remaining cases, we consider $p \equiv 1 \pmod 4$ and $p \equiv 2 \pmod 4$ where we are using near gracefully labelled C_5^p.

Case 3. If $p \equiv 1 \pmod 4$, use a hooked Skolem sequence of order p for S_1 to obtain the entries a_i and b_i and use a hooked Skolem sequence of order $2p$ for S_2 to obtain the entries d_{a_i} and d_{b_i}.

Case 4. If $p \equiv 2 \pmod 4$, use a hooked Skolem sequence of order p for S_1 to obtain the entries a_i and b_i and use a hooked near-Skolem sequence of order $2p + 1$ for S_2 to obtain the entries d_{a_i} and d_{b_i}. □

We give an example of a labelling of $G = C_3^9 C_5^4$. Consider the graceful labelling of C_5^4 obtained using Construction 7 with $t = 9$ and Construction 17 to obtain $(0, 43, 7, 8, 40)$, $(0, 38, 4, 2, 37)$, $(0, 44, 3, 6, 39)$, $(0, 46, 1, 5, 47)$ as the vertex labels of the C_5 vanes. Apply Construction 6 with the Langford sequence $(13, 11, 9, 7, 5, 12, 10, 8, 6, 5, 7, 9, 11, 13, 6, 8, 10, 12)$ and $c = 13$ to construct the triples $(0, 18, 23)$, $(0, 22, 28)$, $(0, 17, 24)$, $(0, 21, 29)$, $(0, 16, 25)$, $(0, 20, 30)$, $(0, 15, 26)$, $(0, 19, 31)$, $(0, 14, 27)$. Together, these give a graceful labelling the graph G.

Theorem 18. *If $G = C_3^t C_5^p$ and $t \geq 2p + 1$ then*

1. G is graceful when $p \equiv 0 \pmod 4$ and $t \equiv 0, 1 \pmod 4$,

   2. G is graceful when $p \equiv 3 \pmod 4$ and $t \equiv 0, 3 \pmod 4$,

   3. G is near graceful when $p \equiv 1 \pmod 4$ and $t \equiv 0, 3 \pmod 4$,

   4. G is near graceful when $p \equiv 2 \pmod 4$ and $t \equiv 0, 1 \pmod 4$.

Proof. The (near) graceful C_5^p exists by Theorem 8. Construction 17 gives the edge and vertex labels of G. Then G is graceful when $p \equiv 0, 3 \pmod 4$ and near graceful when $p \equiv 1, 2 \pmod 4$ by Lemma 7. $\square$

Theorem 18 only contains results for half of the possible combinations of p and t and only for large t. That is not to say the other cases are not (near) graceful. For example, the case $p = t = 1$ does not lend itself to our construction. However the labelling of $C_3^1 C_5^1$ with the vertices of the vanes labelled by $(0, 5, 7)$ and $(0, 8, 4, 3, 6)$ is graceful.

6. $C_3^t C_6^h$

In this section, we extend the technique of [2] to obtain labellings for $C_3^t C_6^h$.

Lemma 8. *Suppose there exists a labelling of a windmill with two triangles labelled $(0, i, b_i + n)$ and $(0, j, b_j + n)$, where (a_i, b_i) and (a_j, b_j) are the positions of i and j in a Skolem sequence. If these two triangles are removed and replaced by a C_6 with vertex labels $(0, b_i + n, i, i + j, j, b_j + n)$, for $1 \le i, j \le n$ and $i \ne j$, then the edge labels are preserved.*

Proof. The edge labels induced by the triples $(0, i, b_i + n)$ and $(0, j, b_j + n)$ are $\{i, j, a_j + n, b_j + n, a_i + n, b_i + n\}$. The edge labels induced by $(0, b_i + n, i, i + j, j, b_j + n)$ are the same. $\square$

Construction 19. *From a (near) gracefully labelled C_3^n labelled by $(0, i, b_i + n)$ with $1 \le i \le n$ formed by one of the (hooked) Skolem sequences of order n in Table 12 - 15, in Appendix A as in Construction 6(2), replace $2h$ triples $\left(h \le \left\lfloor \frac{(2n+1)}{5} \right\rfloor\right)$ with h 6-tuples by replacing the pair of triples $(0, i, b_i + n)$ and $(0, j, b_j + n)$ with $(0, b_i + n, i, i + j, j, b_j + n)$, with pairs indicated as in Table 10.*

Table 10 gives a family of possible pairs (i, j) corresponding to triples $(0, i, b_i + n)$ and $(0, j, b_j + n)$ with $1 \le i, j \le n$ that may be paired to form hexagons of the form $(0, b_i + n, i, i + j, j, b_j + n)$ using the vertex label $i + j$. If the value of $i + j$ does not conflict with any other vertex labels, we obtain a (near) graceful labelling of $C_3^t C_6^h$, where $n = t + 2h$ and $1 \le t, h \le n$.

We notice from Table 10, when $n \in \{5k, 5k + 1\}$ with $k \ge 1$, we can obtain up to $2k$ distinct values of $i + j$. It is straightforward to check that this is the best possible result using this method of generating hexagons.

The bound $h \le \left\lfloor \frac{(2n+1)}{5} \right\rfloor$ comes from the maximum number of possible pairs of triples formed by using the method of Table 10. Combining this with the fact $n = t + 2h$, it is straightforward to show these restrictions are equivalent to $h \le 2t + 1$.

Recall from the example following Lemma 3 the hooked Skolem sequence yielded the triples $(0, 1, 6), (0, 2, 10)$, and $(0, 3, 7)$. Pair the triples $(0, 1, 6)$ and $(0, 3, 7)$ to form the 6-tuple $(0, 6, 1, 4, 3, 7)$ using the label 4 which was not already used, so we obtain a near graceful labelling of $C_3^1 C_6^1$. We cannot pair $(0, 1, 6)$ with $(0, 2, 10)$ because $1 + 2 = 3$ which duplicates another vertex label.

We now present how to use Construction 19 with Table 10 to get a family of possible pairs of triples corresponding to i and j that may be paired to form hexagons with $i + j$. Start with the Skolem sequence $(8, 6, 4, 2, 7, 2, 4, 6, 8, 3, 5, 7, 3, 1, 1, 5)$ constructed from Table 12 in Appendix A. Take the triples of the form $(0, i, b_i + 8)$. From Table 10, we can get up to three pairs $(2, 7), (3, 8)$, and $(4, 6)$ which give three different gracefully labelled graphs. We have three possible replacements: replace

n	(i, j)	
$5k$	$(k+z, 4k+z+1), (2k+z'+1, 3k+z'+1)$	$0 \le z, z' \le k-1$
$5k+1$	$(k+z+1, 4k+z+1), (2k+z'+2, 3k+z'+1)$	$0 \le z \le k, 0 \le z' \le k-2$
$5k+2$	$(k+z+1, 4k+z+2), (2k+z'+2, 3k+z'+2)$	$0 \le z \le k, 0 \le z' \le k-1$
$5k+3$	$(k+z+1, 4k+z+3), (2k+z'+2, 3k+z'+3)$	$0 \le z \le k, 0 \le z' \le k-1$
$5k+4$	$(k+z+1, 4k+z+4), (2k+z'+3, 3k+z'+3)$	$0 \le z \le k, 0 \le z' \le k-1$

Table 10. Possible Pairs of Triples Corresponding to i and j that may be Paired to form Hexagons with $i + j$

$(0, 2, 14)$ and $(0, 7, 20)$ by $(0, 14, 2, 9, 7, 20)$; replace $(0, 3, 21)$ and $(0, 8, 17)$ by $(0, 21, 3, 11, 8, 17)$; and replace $(0, 4, 15)$ and $(0, 6, 16)$ by $(0, 15, 4, 10, 6, 16)$. We can gracefully label $C_3^6 C_6^1$ by using any one of these replacements, $C_3^4 C_6^2$ by using any two, and $C_3^2 C_6^3$ by simultaneously using all three.

Note that the first right endpoint in the Skolem sequences of order n described in Tables 12–15, in Appendix A, is always at $\left\lceil \frac{(n+3)}{2} \right\rceil$. This fact will be useful later in the proof of Theorem 20.

In Theorem 20, we use Construction 19 with the appropriate Skolem sequence of order n which will give (near) graceful labellings for $C_3^t C_6^h$ and $h \le 2t + 1$ as given in Table 11.

$n = t + 2h$	Graceful	Near graceful
$5k$	$k \equiv 0, 1 \pmod 4$	$k \equiv 2, 3 \pmod 4$
$5k + 1$	$k \equiv 0, 3 \pmod 4$	$k \equiv 1, 2 \pmod 4$
$5k + 2$	$k \equiv 2, 3 \pmod 4$	$k \equiv 0, 1 \pmod 4$
$5k + 3$	$k \equiv 1, 2 \pmod 4$	$k \equiv 0, 3 \pmod 4$
$5k + 4$	$k \equiv 0, 1 \pmod 4$	$k \equiv 2, 3 \pmod 4$

Table 11. Summary of Results of (near) Graceful Labelling $C_3^t C_6^h$ and $h \le 2t + 1$

Theorem 20. *If $G = C_3^t C_6^h$ with $h \le 2t + 1$, then G is graceful or near graceful.*

Proof. Let $n = 2h + t$. We begin by considering $n = 5k$. Proofs for other n follow in the same fashion.

Start with a (near) gracefully labelled C_3^n with $n = 5k$ with $k \equiv 0, 1 \pmod 4$ labelled by $(0, i, b_i + n)$ with $1 \le i \le n$ formed by a Skolem sequence as indicated in Construction 19, where $2h + t = n$ and $h \le 2t + 1$. By Construction 19, we replace $2h$ triples with h 6-tuples by $(0, b_i + n, i, i + j, j, b_j + n)$. By Lemma 8 the edge labels will not be changed. Therefore, in this new labelling we use the edge labels $[1, 3n]$. The new labelling uses the same vertex labels, as well as some new labels of the form $i + j$.

We constructed $(0, i, b_i + n)$ with $1 \le i \le n$ by using Table 12 or Table 13, in Appendix A. These triples use the elements i in the interval $[1, n]$ and use the elements $b_i + n$ in the interval $\left[\left\lceil \frac{(3n+3)}{2} \right\rceil, 3n \right]$. Thus any subset of these triples do not use any vertex labels in the interval $\left[n + 1, \left\lceil \frac{(3n+3)}{2} \right\rceil - 1 \right]$.

By construction, the sums $i + j$ are all distinct and in the interval $\left[n + 1, \left\lceil \frac{(3n+3)}{2} \right\rceil - 1 \right]$. The new vertex labels do not duplicate any previously used labels; this new vertex labelling is injective and uses labels from the set $[1, 3n]$.

Finally, we can conclude that since the vertex labels are a subset of $[0, 3n]$ and the edge labels are exactly $[1, 3n]$, that $G = C_3^t C_6^h$ with $2h + t = n$ and $h \le 2t + 1$ can be gracefully labelled when $k \equiv 0, 1 \pmod 4$.

The case $n = 5k$ with $k \equiv 2, 3 \pmod 4$, works in the same way except the construction of $(0, i, b_i + n)$ with $1 \le i \le n$ is by using Table 14 or Table 15, in Appendix A. This construction then uses the edge labels $[1, 3n - 1] \cup \{3n + 1\}$ and vertex labels from $[0, 3n - 1] \cup \{3n + 1\}$. We conclude that $G = C_3^t C_6^h$ with $2h + t = 5k$ and $h \le 2t + 1$ can be near gracefully labelled when $k \equiv 2, 3 \pmod 4$. $\square$

Theorem 20 only contains results for $h \leq 2t + 1$. We cannot (near) gracefully label $G = C_3^t C_6^h$ with $h > 2t + 1$ by our construction, so these problems remain open.

7. Discussion

In this paper, we have completely characterized the situation where $C_3^t C_4^s$ is graceful or near graceful, and given partial solutions for $C_3^t C_5^p$ and $C_3^t C_6^h$.

Many of the techniques of this paper can be combined, but are difficult to reduce to theorems. From the example in Section 5 we gracefully labelled $C_3^9 C_5^4$. If we replace the triples $(0, 16, 25)$, $(0, 20, 30)$, $(0, 19, 31)$ by $(0, 9, 25)$, $(0, 10, 30)$, $(0, 12, 31)$, we notice that the edge labels are the same and the new vertex labels do not appear elsewhere in the labelling of $C_3^9 C_5^4$. Thus we have obtained another graceful labelling of $C_3^9 C_5^4$. Now by the same technique we used in Section 6, we can obtain a graceful labelling for $C_3^7 C_5^4 C_6^1$ by replacing $(0, 9, 25)$, $(0, 10, 30)$ with $(0, 25, 9, 19, 10, 30)$.

Since Skolem-type sequences have proven useful in (near) gracefully labelling C_3 windmills, as well as enabling us to label $C_3 C_4$ windmills, it is natural to ask can we use Skolem-type sequences to gracefully label all C_4 windmills? While the result is known, this would be an extension of the Skolem techniques used in this paper.

In the process of writing this paper, many Skolem-type sequences were considered. We pose two open problems whose solutions would be of interest not only to these constructions, but in their own right. First, can we find a family of m-fold Skolem-type sequences with first right endpoint as large as possible? Second, what are necessary and sufficient conditions for the existence of m-fold Langford sequences with $m \geq 2$?

Acknowledgement

References

1. Rosa, A., 1966, July. On certain valuations of the vertices of a graph. In *Theory of Graphs* (Internat. Symposium, Rome (pp. 349-355).

2. Bermond, J.C., 1979. Graceful graphs, radio antennae and French windmills. In *Proceedings One day Combinatorics Conference, Research Notes in Mathematics*, vol 34, Pitman (pp. 18-37). Pitman.

3. Koh, K.M., Rogers, D.G., Lee, P.Y. and Toh, C.W., 1979. On graceful graphs V: unions of graphs with one vertex in common. *Nanta Math, 12*(2), pp.133-136.

4. Bermond, J.C., Brouwer, A.E. and Germa, A., 1978. Systemes de triplets et differences associees. In *Probèmes Combinatoires Et Théorie Des Graphes, Colloque International Cnrs 260* (Vol. 260, pp. 35-38). CNRS.

5. Kejie, M., 1989. Gracefulness of $P(n_1, n_2, \ldots, n_m)$ and $D_{m,4}$, *Applied Math, 4*, pp.95-97.

6. Yang, Y., Lin, X. and Yu, C., 2005. The graphs C_5^t are graceful for $t \equiv 0, 3 \pmod 4$, *Ars Combinatoria, 74*, pp.239–244.

7. Yang, Y., Xu, X., Xi, Y., Li, H. and Haque, K., 2006. The graphs C_7^t are graceful for $t \equiv 0, 1 \pmod 4$, *Ars Combinatoria, 79*, pp.295–301.

8. Xu, X., Yang, Y., Li, H. and Xi, Y., 2008. The graphs C_{11}^t are graceful for $t \equiv 0, 1 \pmod 4$, *Ars Combinatoria, 88*, pp.429–435.

9. Xu, X., Yang, Y., Han, L. and Li, H., 2009. The graphs C_{13}^t are graceful for $t \equiv 0, 3 \pmod 4$, *Ars Combinatoria, 90*, pp.25–32.

10. Yang, Y., Xu, X., Xi, Y. and Huijun, H., 2007. The graphs C_9^t are graceful for $t \equiv 0, 3 \pmod 4$, *Ars Combinatoria, 85*, pp.361–368.

11. Dyer, D., Payne, I., Shalaby, N. and Wicks, B., 2012. On the graceful conjecture for triangular cacti, *Australasian Journal of Combinatorics, 53*, pp.151–170.

12. Gallian, J.A., 1998. A Dynamic Survey of Graph Labelling, *Electronic Journal of Combinatorics, 5*, pp.6–34.

13. Baker, C. and Manzer, J.D.A., 2008. Skolem-labeling of generalized three-vane windmills, *Australasian Journal of Combinatorics, 41*, pp.175–204.

14. Skolem, T., 1957. On Certain Distributions of Integers in Pairs with Given Differences, *Mathematica Scandinavica, 5*, pp.57–68.

15. Shalaby, N., 1994. The existence of near-Skolem and hooked near-Skolem sequences, *Discrete Mathematics, 135*, pp.303–319.

16. Simpson, J.E., 1983. Langford Sequences; Perfect and Hooked, *Discrete Mathematics, 44*, pp.97–104.

17. O'Keefe, E.S., 1961. Verification of a conjecture of Th. Skolem, *Mathematica Scandinavica, 9*, pp.80–82.

18. Baker, C., Nowakowski, R., Shalaby, N. and Sharary, A., 1994. *M*-fold and extended *M*-fold Skolem sequences, *Utilitas Mathematica, 45*, pp.153–167.

19. Alkasasbeh, A.H., Dyer, D. and Shalaby, N., 2020. Applying Skolem Sequences to Gracefully Label New Families of Triangular Windmills. *arXiv preprint arXiv:2006.16396.*

20. Marsh, D.C.B., 1967. Solution of Problem E1845, *American Mathematical Monthly, 74*, pp.591–592.

21. Hilton, A.J.W., 1969. Steiner and similar triple systems, *Mathematica Scandinavica, 24*, pp.208–216.

Appendix A: Skolem Constructions

i	a_i	b_i	
$2r + 2$	$2m - r$	$2m + 2 + r$	$0 \le r \le 2m - 1$
1	$7m$	$7m + 1$	–
$4m - 1$	$2m + 1$	$6m$	–
$2m + 2r + 1$	$5m + 1 - r$	$7m + r + 2$	$0 \le r \le m - 2$
$2m - 1$	$4m + 2$	$6m + 1$	–
$2m - 3 - 2r$	$5m + 2 + r$	$7m - 1 - r$	$0 \le r \le m - 3$

Table 12. Skolem Sequence Construction of Order $n = 4m$ and $m \ge 1$ from [14]

i	a_i	b_i	
$2r$	$2m + 1 - r$	$2m + 1 + r$	$1 \le r \le 2m$
$4m + 1$	$2m + 1$	$6m + 2$	–
$2m - 1 + 2r$	$5m + 2 - r$	$7m + 1 + r$	$1 \le r \le m$
$2m - 1$	$6m + 3$	$8m + 2$	–
1	$5m + 2$	$5m + 3$	–
$2r + 1$	$6m + 2 - r$	$6m + 3 + r$	$1 \le r \le m - 2$

Table 13. Skolem Sequence Construction of Order $n = 4m + 1$ and $m \ge 2$ from [20]

i	a_i	b_i	
$2r$	$2m+2-r$	$2m+2+r$	$1 \leq r \leq 2m+1$
1	$7m+4$	$7m+5$	$-$
$1+2r$	$6m+2-r$	$6m+3+r$	$1 \leq r \leq m$
$2m+3$	$6m+2$	$8m+5$	$-$
$2m+3+2r$	$5m+2-r$	$7m+5+r$	$1 \leq r \leq m-2$
$4m+1$	$2m+2$	$6m+3$	$-$

Table 14. Hooked Skolem Sequence Construction of Order $n = 4m+2$ and $m \geq 2$ from [21]

i	a_i	b_i	
$2r$	$2m+2-r$	$2m+2+r$	$1 \leq r \leq 2m+1$
1	$5m+4$	$5m+5$	$-$
$1+2r$	$6m+5-r$	$6m+6+r$	$1 \leq r \leq m-1$
$2m+1$	$6m+6$	$8m+7$	$-$
$2m+1+2r$	$5m+4-r$	$7m+5+r$	$1 \leq r \leq m$
$4m+3$	$2m+2$	$6m+5$	$-$

Table 15. Hooked Skolem Sequence Construction of Order $n = 4m+3$ and $m \geq 1$ from [21]

Appendix B

The following table includes graceful labellings of $C_3^1 C_4^s$, where $1 \leq s \leq 20$.

The vanes $(0,3,2,6)$ and $(0,5,7)$ are included with each of the sets in the following table. Note that these two vanes give a graceful labelling when $s = 1$.

s	Additional vanes
2	$(0,9,1,11)$
3	$(0,11,1,15), (0,12,4,13)$
4	$(0,13,1,19), (0,14,4,15), (0,16,8,17)$
5	$(0,15,1,19), (0,16,4,17), (0,20,11,22), (0,21,13,23)$
6	$(0,17,1,21), (0,18,4,19), (0,22,12,25), (0,23,14,26), (0,24,16,27)$
7	$(0,19,1,21), (0,22,10,27), (0,23,12,28), (0,24,14,29), (0,25,16,30), (0,26,18,31)$
8	$(0,23,1,39), (0,24,10,31), (0,25,12,32), (0,26,14,33), (0,27,16,34), (0,28,18,35),$ $(0,29,20,36), (0,30,22,37)$
9	$(0,23,1,39), (0,24,10,31), (0,25,12,32), (0,26,14,33), (0,27,16,34), (0,28,18,35),$ $(0,29,20,36), (0,30,22,37)$
10	$(0,23,1,41), (0,42,4,43), (0,24,10,31), (0,25,12,32), (0,26,14,33), (0,27,16,34),$ $(0,28,18,35), (0,29,20,36), (0,30,22,37), (0,3,2,6). (0,5,7)$
11	$(0,28,13,36), (0,29,15,37), (0,30,17,38), (0,44,18,45), (0,31,19,39), (0,32,21,40),$ $(0,46,22,47), (0,33,23,41), (0,34,25,42), (0,35,27,43)$
12	$(0,34,26,39), (0,33,24,38), (0,32,22,37), (0,31,20,36), (0,30,18,35), (0,45,27,51),$ $(0,44,25,50), (0,43,23,49), (0,42,21,48), (0,41,19,47), (0,40,17,46)$
13	$(0,38,26,43), (0,37,24,42), (0,36,22,41), (0,35,20,40), (0,34,18,39), (0,49,27,55),$ $(0,48,25,54), (0,47,23,53), (0,46,21,52), (0,45,19,51), (0,44,17,50)$
14	$(0,42,26,47), (0,41,24,46), (0,40,22,45), (0,39,20,44), (0,38,18,43), (0,53,27,59),$ $(0,52,25,58), (0,51,23,57), (0,50,21,56), (0,49,19,55), (0,48,17,54)$

15	(0, 55, 1, 59), (0, 56, 4, 57), (0, 60, 10, 61), (0, 28, 13, 36), (0, 62, 14, 63), (0, 29, 15, 37), (0, 30, 17, 38), (0, 44, 18, 45), (0, 31, 19, 39), (0, 32, 21, 40), (0, 46, 22, 47), (0, 33, 23, 41), (0, 34, 25, 42), (0, 35, 27, 43)
16	(0, 44, 36, 51), (0, 43, 34, 50), (0, 42, 32, 49), (0, 41, 30, 48), (0, 40, 28, 47), (0, 39, 26, 46), (0, 38, 24, 45), (0, 59, 37, 67), (0, 58, 35, 66), (0, 57, 33, 65), (0, 56, 31, 64), (0, 55, 29, 63), (0, 54, 27, 62), (0, 53, 25, 61), (0, 52, 23, 60)
17	(0, 48, 36, 55), (0, 47, 34, 54), (0, 46, 32, 53), (0, 45, 30, 52), (0, 44, 28, 51), (0, 43, 26, 50), (0, 42, 24, 49), (0, 63, 37, 71), (0, 62, 35, 70), (0, 61, 33, 69), (0, 60, 31, 68), (0, 59, 29, 67), (0, 58, 27, 66), (0, 57, 25, 65), (0, 56, 23, 64)
18	(0, 52, 36, 59), (0, 51, 34, 58), (0, 50, 32, 57), (0, 49, 30, 56), (0, 48, 28, 55), (0, 47, 26, 54), (0, 46, 24, 53), (0, 67, 37, 75), (0, 66, 35, 74), (0, 65, 33, 73), (0, 64, 31, 72), (0, 63, 29, 71), (0, 62, 27, 70), (0, 61, 25, 69), (0, 60, 23, 68)
19	(0, 56, 36, 63), (0, 55, 34, 62), (0, 54, 32, 61), (0, 53, 30, 60), (0, 52, 28, 59), (0, 51, 26, 58), (0, 50, 24, 57), (0, 71, 37, 79), (0, 70, 35, 78), (0, 69, 33, 77), (0, 68, 31, 76), (0, 67, 29, 75), (0, 66, 27, 74), (0, 65, 25, 73), (0, 64, 23, 72)
20	(0, 61, 1, 80), (0, 81, 4, 82), (0, 73, 9, 83), (0, 65, 10, 69), (0, 66, 12, 70), (0, 75, 13, 76), (0, 67, 14, 71), (0, 68, 16, 72), (0, 40, 17, 46), (0, 30, 18, 35), (0, 41, 19, 47), (0, 31, 20, 36), (0, 42, 21, 48), (0, 32, 22, 37), (0, 43, 23, 49), (0, 33, 24, 38), (0, 44, 25, 50), (0, 34, 26, 39), (0, 45, 27, 51)

The following table includes near graceful labellings of $C_3^2 C_4^s$, where $1 \le s \le 20$.
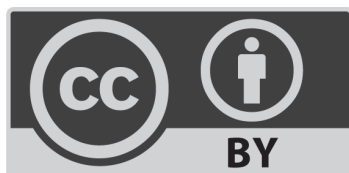
s	Vanes
1	(0, 5, 2, 6), (0, 7, 8), (0, 9, 11)
2	(0, 10, 1, 12), (0, 5, 2, 6), (0, 7, 8), (0, 15, 13)
3	(0, 12, 1, 16), (0, 13, 4, 14), (0, 5, 2, 6),(0, 7, 8), (0, 19, 17)
4	(0, 14, 1, 20), (0, 15, 4, 16), (0, 17, 8, 18), (0, 5, 2, 6), (0, 7, 8), (0, 23, 21)
5	(0, 16, 1, 20), (0, 17, 4, 18), (0, 21, 11, 23), (0, 22, 13, 24), (0, 5, 2, 6), (0, 7, 8), (0, 27, 25)
6	(0, 18, 1, 22), (0, 19, 4, 20), (0, 23, 12, 26), (0, 24, 14, 27),(0, 25, 16, 28), (0, 5, 2, 6), (0, 7, 8), (0, 31, 29)
7	(0, 20, 1, 22), (0, 23, 10, 28), (0, 24, 12, 29), (0, 25, 14, 30), (0, 26, 16, 31), (0, 27, 18, 32), (0, 5, 2, 6), (0, 7, 8), (0, 33, 35)
8	(0, 25, 16, 28), (0, 24, 14, 27), (0, 23, 12, 26), (0, 32, 17, 36), (0, 31, 15, 35), (0, 30, 13, 34), (0, 29, 11, 33), (0, 5, 2, 6), (0, 7, 8), (0, 39, 37)
9	(0, 24, 1, 40), (0, 25, 10, 32), (0, 26, 12, 33), (0, 27, 14, 34), (0, 28, 16, 35), (0, 29, 18, 36), (0, 30, 20, 37), (0, 31, 22, 38), (0, 5, 2, 6), (0, 7, 8), (0, 43, 41)
10	(0, 24, 1, 42), (0, 43, 4, 44), (0, 25, 10, 32), (0, 26, 12, 33), (0, 27, 14, 34), (0, 28, 16, 35), (0, 29, 18, 36), (0, 30, 20, 37), (0, 31, 22, 38), (0, 5, 2, 6), (0, 7, 8),(0, 47, 45)
11	(0, 29, 13, 37), (0, 30, 15, 38), (0, 31, 17, 39), (0, 45, 18, 46), (0, 32, 19, 40), (0, 33, 21, 41), (0, 47, 22, 48), (0, 34, 23, 42), (0, 35, 25, 43), (0, 36, 27, 44), (0, 5, 2, 6), (0, 7, 8), (0, 51, 49)
12	(0, 35, 26, 40), (0, 34, 24, 39), (0, 33, 22, 38), (0, 32, 20, 37), (0, 31, 18, 36), (0, 46, 27, 52), (0, 45, 25, 51), (0, 44, 23, 50), (0, 43, 21, 49), (0, 42, 19, 48), (0, 41, 17, 47), (0, 5, 2, 6), (0, 7, 8), (0, 55, 53)
13	(0, 39, 26, 44), (0, 38, 24, 43), (0, 37, 22, 42), (0, 36, 20, 41), (0, 35, 18, 40), (0, 50, 27, 56), (0, 49, 25, 55), (0, 48, 23, 54), (0, 47, 21, 53), (0, 46, 19, 52), (0, 45, 17, 51), (0, 10, 1, 12), (0, 5, 2, 6), (0, 7, 8), (0, 59, 57)

14	(0, 43, 26, 48), (0, 42, 24, 47), (0, 41, 22, 46), (0, 40, 20, 45), (0, 39, 18, 44), (0, 54, 27, 60), (0, 53, 25, 59), (0, 52, 23, 58), (0, 51, 21, 57), (0, 50, 19, 56), (0, 49, 17, 55), (0, 12, 1, 16), (0, 13, 4, 14), (0, 5, 2, 6), (0, 7, 8), (0, 63, 61)
15	(0, 56, 1, 60), (0, 57, 4, 58), (0, 61, 10, 62), (0, 29, 13, 37), (0, 63, 14, 64), (0, 30, 15, 38), (0, 31, 17, 39), (0, 45, 18, 46), (0, 32, 19, 40), (0, 33, 21, 41), (0, 47, 22, 48), (0, 34, 23, 42), (0, 35, 25, 43), (0, 36, 27, 44), (0, 5, 2, 6), (0, 7, 8), (0, 67, 65)
16	(0, 45, 36, 52), (0, 44, 34, 51), (0, 43, 32, 50), (0, 42, 30, 49), (0, 41, 28, 48), (0, 40, 26, 47), (0, 39, 24, 46), (0, 60, 37, 68), (0, 59, 35, 67), (0, 58, 33, 66), (0, 57, 31, 65), (0, 56, 29, 64), (0, 55, 27, 63), (0, 54, 25, 62), (0, 53, 23, 61), (0, 5, 2, 6), (0, 7, 8), (0, 71, 69)
17	(0, 49, 36, 56), (0, 48, 34, 55), (0, 47, 32, 54), (0, 46, 30, 53), (0, 45, 28, 52), (0, 44, 26, 51), (0, 43, 24, 50), (0, 64, 37, 72), (0, 63, 35, 71), (0, 62, 33, 70), (0, 61, 31, 69), (0, 60, 29, 68), (0, 59, 27, 67), (0, 58, 25, 66), (0, 57, 23, 65), (0, 10, 1, 12), (0, 5, 2, 6), (0, 7, 8), (0, 75, 73)
18	(0, 53, 36, 60), (0, 52, 34, 59), (0, 51, 32, 58), (0, 50, 30, 57), (0, 49, 28, 56), (0, 48, 26, 55), (0, 47, 24, 54), (0, 68, 37, 76), (0, 67, 35, 75), (0, 66, 33, 74), (0, 65, 31, 73), (0, 64, 29, 72), (0, 63, 27, 71), (0, 62, 25, 70), (0, 61, 23, 69), (0, 12, 1, 16), (0, 13, 4, 14), (0, 5, 2, 6), (0, 7, 8), (0, 79, 77)
19	(0, 57, 36, 64), (0, 56, 34, 63), (0, 55, 32, 62), (0, 54, 30, 61), (0, 53, 28, 60), (0, 52, 26, 59), (0, 51, 24, 58), (0, 72, 37, 80), (0, 71, 35, 79), (0, 70, 33, 78), (0, 69, 31, 77), (0, 68, 29, 76), (0, 67, 27, 75), (0, 66, 25, 74), (0, 65, 23, 73), (0, 14, 1, 20), (0, 15, 4, 16), (0, 17, 8, 18), (0, 5, 2, 6), (0, 7, 8), (0, 83, 81)
20	(0, 62, 1, 81), (0, 82, 4, 83), (0, 74, 9, 84), (0, 66, 10, 70), (0, 67, 12, 71), (0, 76, 13, 77), (0, 68, 14, 72), (0, 69, 16, 73), (0, 41, 17, 47), (0, 31, 18, 36), (0, 42, 19, 48), (0, 32, 20, 37), (0, 43, 21, 49), (0, 33, 22, 38), (0, 44, 23, 50), (0, 34, 24, 39), (0, 45, 25, 51), (0, 35, 26, 40), (0, 46, 27, 52), (0, 5, 2, 6), (0, 7, 8), (0, 87, 85)

In the following table we include near graceful labellings of $C_3^3 C_4^s$, where $1 \le s \le 19$.

s	Vanes
1	(0, 8, 2, 9), (0, 3, 5), (0, 11, 12), (0, 10, 14)
2	(0, 11, 1, 13), (0, 8, 2, 9), (0, 3, 5), (0, 14, 18), (0, 15, 16)
3	(0, 13, 1, 17), (0, 14, 4, 15), (0, 8, 2, 9), (0, 3, 5), (0, 22, 18), (0, 20, 19)
4	(0, 15, 1, 19), (0, 16, 4, 17), (0, 20, 10, 21), (0, 8, 2, 9), (0, 3, 5), (0, 26, 22), (0, 24, 23)
5	(0, 17, 1, 21), (0, 18, 4, 19), (0, 22, 11, 24), (0, 23, 13, 25), (0, 8, 2, 9), (0, 3, 5), (0, 30, 26), (0, 28, 27)
6	(0, 19, 1, 23), (0, 20, 4, 21), (0, 24, 12, 27), (0, 25, 14, 28), (0, 26, 16, 29), (0, 8, 2, 9), (0, 3, 5), (0, 34, 30), (0, 32, 31)
7	(0, 21, 1, 23), (0, 24, 10, 29), (0, 25, 12, 30),(0, 26, 14, 31), (0, 27, 16, 32), (0, 28, 18, 33), (0, 8, 2, 9), (0, 3, 5), (0, 38, 34), (0, 36, 35)
8	(0, 26, 16, 29), (0, 25, 14, 28), (0, 24, 12, 27), (0, 33, 17, 37), (0, 32, 15, 36), (0, 31, 13, 35), (0, 30, 11, 34), (0, 8, 2, 9), (0, 3, 5), (0, 42, 38), (0, 40, 39)
9	(0, 25, 1, 41), (0, 26, 10, 33), (0, 27, 12, 34), (0, 28, 14, 35), (0, 29, 16, 36), (0, 30, 18, 37), (0, 31, 20, 38), (0, 32, 22, 39), (0, 8, 2, 9), (0, 3, 5), (0, 46, 42), (0, 44, 43)
10	(0, 25, 1, 43), (0, 44, 4, 45), (0, 26, 10, 33), (0, 27, 12, 34), (0, 28, 14, 35), (0, 29, 16, 36), (0, 30, 18, 37), (0, 31, 20, 38), (0, 32, 22, 39), (0, 8, 2, 9), (0, 3, 5), (0, 50, 46), (0, 48, 47)
11	(0, 30, 13, 38), (0, 31, 15, 39), (0, 32, 17, 40), (0, 46, 18, 47), (0, 33, 19, 41), (0, 34, 21, 42), (0, 48, 22, 49), (0, 35, 23, 43), (0, 36, 25, 44), (0, 37, 27, 45), (0, 8, 2, 9), (0, 3, 5),(0, 54, 50), (0, 52, 51)

12	(0, 36, 26, 41), (0, 35, 24, 40), (0, 34, 22, 39), (0, 33, 20, 38), (0, 32, 18, 37), (0, 47, 27, 53), (0, 46, 25, 52), (0, 45, 23, 51), (0, 44, 21, 50), (0, 43, 19, 49), (0, 42, 17, 48), (0, 8, 2, 9), (0, 3, 5), (0, 58, 54),(0, 56, 55)
13	(0, 40, 26, 45), (0, 39, 24, 44), (0, 38, 22, 43), (0, 37, 20, 42), (0, 36, 18, 41), (0, 51, 27, 57), (0, 50, 25, 56), (0, 49, 23, 55), (0, 48, 21, 54), (0, 47, 19, 53), (0, 46, 17, 52), (0, 11, 1, 13), (0, 8, 2, 9), (0, 3, 5), (0, 59, 60), (0, 58, 62)
14	(0, 55, 1, 59), (0, 56, 4, 57), (0, 60, 10, 61), (0, 30, 13, 38), (0, 31, 15, 39), (0, 32, 17, 40), (0, 46, 18, 47), (0, 33, 19, 41), (0, 34, 21, 42), (0, 48, 22, 49), (0, 35, 23, 43), (0, 36, 25, 44), (0, 37, 27, 45), (0, 8, 2, 9), (0, 3, 5), (0, 66, 62), (0, 64, 63)
15	(0, 57, 1, 61), (0, 58, 4, 59), (0, 62, 10, 63), (0, 30, 13, 38), (0, 64, 14, 65), (0, 31, 15, 39), (0, 32, 17, 40), (0, 46, 18, 47), (0, 33, 19, 41), (0, 34, 21, 42), (0, 48, 22, 49), (0, 35, 23, 43), (0, 36, 25, 44), (0, 37, 27, 45), (0, 8, 2, 9), (0, 3, 5), (0, 70, 66), (0, 68, 67)
16	(0, 46, 36, 53), (0, 45, 34, 52), (0, 44, 32, 51), (0, 43, 30, 50), (0, 42, 28, 49), (0, 41, 26, 48), (0, 40, 24, 47), (0, 61, 37, 69), (0, 60, 35, 68), (0, 59, 33, 67), (0, 58, 31, 66), (0, 57, 29, 65), (0, 56, 27, 64), (0, 55, 25, 63), (0, 54, 23, 62), (0, 8, 2, 9), (0, 3, 5), (0, 71, 72), (0, 70, 74)
17	(0, 50, 36, 57), (0, 49, 34, 56), (0, 48, 32, 55), (0, 47, 30, 54), (0, 46, 28, 53), (0, 45, 26, 52), (0, 44, 24, 51), (0, 65, 37, 73), (0, 64, 35, 72), (0, 63, 33, 71), (0, 62, 31, 70), (0, 61, 29, 69), (0, 60, 27, 68), (0, 59, 25, 67), (0, 58, 23, 66), (0, 11, 1, 13), (0, 8, 2, 9), (0, 3, 5), (0, 78, 74), (0, 76, 75)
18	(0, 54, 36, 61), (0, 53, 34, 60), (0, 52, 32, 59), (0, 51, 30, 58), (0, 50, 28, 57), (0, 49, 26, 56), (0, 48, 24, 55), (0, 69, 37, 77), (0, 68, 35, 76), (0, 67, 33, 75), (0, 66, 31, 74), (0, 65, 29, 73), (0, 64, 27, 72), (0, 63, 25, 71), (0, 62, 23, 70), (0, 13, 1, 17), (0, 14, 4, 15), (0, 8, 2, 9), (0, 3, 5), (0, 78, 82), (0, 79, 80)
19	(0, 58, 36, 65), (0, 57, 34, 64), (0, 56, 32, 63), (0, 55, 30, 62), (0, 54, 28, 61), (0, 53, 26, 60), (0, 52, 24, 59), (0, 73, 37, 81), (0, 72, 35, 80), (0, 71, 33, 79), (0, 70, 31, 78), (0, 69, 29, 77), (0, 68, 27, 76), (0, 67, 25, 75), (0, 66, 23, 74), (0, 15, 1, 19), (0, 16, 4, 17), (0, 20, 10, 21), (0, 8, 2, 9), (0, 3, 5), (0, 82, 86), (0, 84, 83).