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Article

On Fractional ID-(g, f)-factor-critical Covered Graphs

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Abstract: A graph G is called a fractional ID-(g, f)-factor-critical covered graph if for any independent set I of G and for every edge $e \in E(G - I)$, $G - I$ has a fractional (g, f)-factor h such that $h(e) = 1$. We give a sufficient condition using degree condition for a graph to be a fractional ID-(g, f)-factor-critical covered graph. Our main result is an extension of Zhou, Bian and Wu's previous result [S. Zhou, Q. Bian, J. Wu, A result on fractional ID-k-factor-critical graphs, Journal of Combinatorial Mathematics and Combinatorial Computing 87(2013)229–236] and Yashima's previous result [T. Yashima, A degree condition for graphs to be fractional ID-$[a, b]$-factor-critical, Australasian Journal of Combinatorics 65(2016)191–199].

Keywords: Graph, Degree condition, Fractional (g, f)-factor, Fractional (g, f)-covered graph, Fractional ID-(g, f)-factor-critical covered graph
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1. Introduction

The graphs discussed here are finite, undirected and simple. For a graph G, its vertex set is denoted by $V(G)$ and its edge set is denoted by $E(G)$. For $x \in V(G)$, we use $N_G(x)$ to denote the set of vertices adjacent to x in G, $N_G[x] = N_G(x) \cup \{x\}$ and $d_G(x) = |N_G(x)|$ is the degree of x in G. Setting $\delta(G) = \min\{d_G(x) : x \in V(G)\}$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S. We write $G - S = G[V(G) \setminus S]$ and $N_G(S) = \bigcup_{x \in S} N_G(x)$. If $N_G(S) \cap S = \emptyset$, then we call S independent. For $X \subseteq E(G)$, $G[X]$ denotes the subgraph of G induced by X.

Let g and f be two integer-valued functions defined on $V(G)$ satisfying $0 \le g(x) \le f(x)$ for any $x \in V(G)$ and let $h : E(G) \to [0, 1]$ be a function with $g(x) \le \sum_{e \ni x} h(e) \le f(x)$ for any $x \in V(G)$. Define $F_h = \{e : h(e) > 0, e \in E(G)\}$. Then we call $G[F_h]$ a fractional (g, f)-factor of G with indicator function h. Naturally, a fractional (g, f)-factor is a fractional $[a, b]$-factor if $g(x) = a$ and $f(x) = b$ for all $x \in V(G)$, and a fractional $[k, k]$-factor is called a fractional k-factor. If $h(e) \in \{0, 1\}$ for any $e \in E(G)$, then a fractional (g, f)-factor, a fractional $[a, b]$-factor and a fractional k-factor are called a (g, f)-factor, an $[a, b]$-factor and a k-factor, respectively.

A graph G is defined as a fractional ID-(g, f)-factor-critical graph if $G - I$ possesses a fractional (g, f)-factor for any independent set I of G. A graph G is defined as a fractional (g, f)-covered graph if for any $e \in E(G)$, G admits a fractional (g, f)-factor with indicator function h satisfying $h(e) = 1$. Similarly, we may define a fractional ID-$[a, b]$-factor-critical graph, a fractional ID-k-factor-critical graph, a fractional $[a, b]$-covered graph and a fractional k-covered graph.

There are a rich literature on the existence of factors and fractional factors in graphs. More specifically, a great deal of results on the existence of factors in graphs with given properties can be discovered in [1–7], and a lot of results can be discovered in [8–12] related to the existence of fractional factors in graphs with prescribed properties. Zhou, Bian and Wu [13] demonstrated a degree condition for a graph being fractional ID-k-factor-critical. Yashima [14] posed a degree condition for a graph to be fractional ID-$[a, b]$-factor-critical.

Theorem 1. ([13]) *Let k be an integer with $k \geq 1$, and let G be a graph of order n with $n \geq 6k - 2$ and $\delta(G) \geq \frac{n}{3} + k$. If G satisfies*

$$\max\{d_G(x), d_G(y)\} \geq \frac{2n}{3}$$

for each pair of nonadjacent vertices x, y of G, then G is fractional ID-k-factor-critical.

Theorem 2. ([14]) *Let $b \geq a \geq 1$ be integers, and let G be a graph of order n with $n \geq \frac{(a+2b)(2a+b+1)}{b}$ and $\delta(G) \geq \frac{bn}{a+2b} + a$. If G satisfies*

$$\max\{d_G(x), d_G(y)\} \geq \frac{(a+b)n}{a+2b}$$

for each pair of nonadjacent vertices x, y of G, then G is fractional ID-$[a, b]$-factor-critical.

Combining the definition of a fractional ID-(g, f)-factor-critical graph with that of a fractional (g, f)-covered graph, we present the definition of a fractional ID-(g, f)-factor-critical covered graph, that is, a graph G is called fractional ID-(g, f)-factor-critical covered if $G - I$ is fractional (g, f)-covered for any independent set I of G. A fractional ID-(k, k)-factor-critical covered graph is simply called a fractional ID-k-factor-critical covered graph. In this article, we prove the following theorem for a graph being fractional ID-(g, f)-factor-critical covered, which is a generalization of Theorems 1 and 2.

Theorem 3. *Let a, b, r be integers with $r \geq 0$ and $b - r \geq a \geq 1$, let G be a graph of order n with $n \geq \frac{(a+b)(2a+b+r)+2}{a+r}$ and $\delta(G) \geq \frac{(a+r)n}{2a+b+r} + \frac{(b-r+1)^2-(a+r)(b-a-2r+1)}{a+r}$, and let $g, f : V(G) \to \mathbb{Z}$ be two functions with $a \leq g(x) \leq f(x) - r \leq b - r$ for each $x \in V(G)$. If G satisfies*

$$\max\{d_G(x), d_G(y)\} \geq \frac{(a+b)n+2}{2a+b+r}$$

for each pair of nonadjacent vertices x, y of G, then G is fractional ID-(g, f)-factor-critical covered.

Using Theorem 3, the following two results hold.

Corollary 1. *Let a, b be integers with $b \geq a \geq 1$, let G be a graph of order n with $n \geq \frac{(a+b)(2a+b)+2}{a}$ and $\delta(G) \geq \frac{an}{2a+b} + \frac{(b+1)^2-a(b-a+1)}{a}$, and let $g, f : V(G) \to \mathbb{Z}$ be two functions with $a \leq g(x) \leq f(x) \leq b$ for each $x \in V(G)$. If G satisfies*

$$\max\{d_G(x), d_G(y)\} \geq \frac{(a+b)n+2}{2a+b}$$

for each pair of nonadjacent vertices x, y of G, then G is fractional ID-(g, f)-factor-critical covered.

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Corollary 2. *Let k be an integer with $k \geq 1$, let G be a graph of order n with $n \geq 6k + \frac{2}{k}$ and $\delta(G) \geq \frac{n}{3} + k + 1 + \frac{1}{k}$. If G satisfies*

$$\max\{d_G(x), d_G(y)\} \geq \frac{2kn+2}{3k}$$

for each pair of nonadjacent vertices x, y of G, then G is fractional ID-k-factor-critical covered.

2. The proof of Theorem 3

The following result acquired by Li, Yan and Zhang [15] will be used to prove Theorem 3.

Theorem 4. ([15]) *Let G be a graph, and let $g, f : V(G) \to \mathbb{Z}$ be two functions with $0 \le g(x) \le f(x)$ for every $x \in V(G)$. Then G is fractional (g, f)-covered if and only if*

$$\delta_G(S,T) = f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) \ge \varepsilon(S)$$

for any $S \subseteq V(G)$, where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \le g(x)\}$ and $\varepsilon(S)$ is defined by

$$\varepsilon(S) = \begin{cases} 2, & \text{if } S \text{ is not independent,} \\ 1, & \text{if } S \text{ is independent and there is an edge joining} \\ & S \text{ and } V(G) \setminus (S \cup T), \text{ or there is an edge } e = uv \\ & \text{joining } S \text{ and } T \text{ such that } d_{G-S}(v) = g(v) \text{ for } v \in T, \\ 0, & \text{otherwise.} \end{cases}$$

Proof of Theorem 3. Let I be an independent set of G, and let $H = G - I$. In order to prove Theorem 3, it suffices to show that H is fractional (g, f)-covered. Assume that H is not fractional (g, f)-covered. Then by Theorem 4, we have

$$\delta_H(S,T) = f(S) - g(T) + \sum_{x \in T} d_{H-S}(x) \le \varepsilon(S) - 1 \tag{1}$$

for some $S \subseteq V(H)$, where $T = \{x : x \in V(H) \setminus S, d_{H-S}(x) \le g(x)\}$.
Claim 1. $|I| \le \frac{(a+r)n-2}{2a+b+r}$.

Proof. Note that $n \ge \frac{(a+b)(2a+b+r)+2}{a+r}$. Thus, the inequality holds for $0 \le |I| \le 1$. Next, we assume that $|I| \ge 2$. It follows from the hypothesis of Theorem 3 and I being an independent set of G that

$$\max\{d_G(x), d_G(y)\} \ge \frac{(a+b)n+2}{2a+b+r}$$

for any two distinct vertices $x, y \in I$. Thus, we acquire

$$|I| + \frac{(a+b)n+2}{2a+b+r} \le |I| + \max\{d_G(x), d_G(y)\} \le n,$$

namely,

$$|I| \le \frac{(a+r)n-2}{2a+b+r}.$$

Claim 1 is demonstrated. □

Note that $\varepsilon(S) \le |S|$. If $T = \emptyset$, then using (1), we gain $\varepsilon(S) - 1 \ge \delta_H(S,T) = f(S) \ge (a+r)|S| \ge |S| \ge \varepsilon(S)$, a contradiction. Hence, $T \ne \emptyset$. Let

$$h_1 = \min\{d_{H-S}(x) : x \in T\},$$

and select $x_1 \in T$ with $d_{H-S}(x_1) = h_1$. If $T \setminus N_T[x_1] \ne \emptyset$, then we write

$$h_2 = \min\{d_{H-S}(x) : x \in T \setminus N_T[x_1]\},$$

and select $x_2 \in T \setminus N_T[x_1]$ with $d_{H-S}(x_2) = h_2$. Apparently, $0 \le h_1 \le h_2 \le b - r$.

In what follows, the proof is divided into two cases.
Case 1. $T = N_T[x_1]$.
Claim 2. $|S| + d_{H-S}(x) > b - r + 1$ for every $x \in T$.

Proof. According to Claim 1, $\delta(G) \geq \frac{(a+r)n}{2a+b+r} + \frac{(b-r+1)^2-(a+r)(b-a-2r+1)}{a+r}$ and $H = G - I$, we have

$$\begin{aligned}
|S| + d_{H-S}(x) &\geq d_H(x) = d_{G-I}(x) \geq d_G(x) - |I| \geq \delta(G) - |I| \\
&\geq \frac{(a+r)n}{2a+b+r} + \frac{(b-r+1)^2-(a+r)(b-a-2r+1)}{a+r} \\
&\quad -\frac{(a+r)n-2}{2a+b+r} \\
&= \frac{(b-r+1)^2-(a+r)(b-a-2r+1)}{a+r} + \frac{2}{2a+b+r} \\
&> \frac{(b-r+1)^2-(a+r)(b-a-2r+1)}{a+r} \\
&= \frac{(b-a-2r+1)^2+(a+r)(b-r+1)}{a+r} \\
&\geq b-r+1
\end{aligned}$$

for every $x \in T$. Claim 2 is verified. □

Claim 3. $|T| \geq a + r + 1$.

Proof. Assume $|T| \leq a + r$. Then it follows from (1), $\varepsilon(S) \leq 2$, $T \neq \emptyset$ and Claim 2 that

$$\begin{aligned}
1 &\geq \varepsilon(S) - 1 \geq \delta_H(S,T) = f(S) - g(T) + \sum_{x\in T} d_{H-S}(x) \\
&\geq (a+r)|S| - (b-r)|T| + \sum_{x\in T} d_{H-S}(x) \\
&\geq |T||S| - (b-r)|T| + \sum_{x\in T} d_{H-S}(x) \\
&= \sum_{x\in T}(|S| + d_{H-S}(x) - (b-r)) \\
&> \sum_{x\in T}(b-r+1-(b-r)) \\
&= |T| \geq 1,
\end{aligned}$$

this is a confliction. The proof of Claim 3 is finished. □

Note that $|T| = |N_T[x_1]| \leq d_{H-S}(x_1) + 1 = h_1 + 1$. Combining this with $0 \leq h_1 \leq b - r$, we acquire

$$|T| \leq h_1 + 1 \leq b - r + 1. \tag{2}$$

Using (2) and Claim 3, we have

$$a + r + 1 \leq |T| \leq h_1 + 1 \leq b - r + 1,$$

which implies

$$h_1 \geq a + r \tag{3}$$

and

$$b \geq a + 2r. \tag{4}$$

By Claim 1, $H = G - I$ and $\delta(G) \geq \frac{(a+r)n}{2a+b+r} + \frac{(b-r+1)^2-(a+r)(b-a-2r+1)}{a+r}$, we get

$$\begin{aligned}
|S| + h_1 &= |S| + d_{H-S}(x_1) \geq d_H(x_1) = d_{G-I}(x_1) \\
&\geq d_G(x_1) - |I| \geq \delta(G) - |I| \\
&\geq \frac{(a+r)n}{2a+b+r} + \frac{(b-r+1)^2-(a+r)(b-a-2r+1)}{a+r}
\end{aligned}$$

$$
\begin{aligned}
& -\frac{(a+r)n-2}{2a+b+r} \\
= & \frac{(b-r+1)^2-(a+r)(b-a-2r+1)}{a+r}+\frac{2}{2a+b+r} \\
> & \frac{(b-r+1)^2-(a+r)(b-a-2r+1)}{a+r},
\end{aligned}
$$

namely,

$$|S| > \frac{(b-r+1)^2-(a+r)(b-a-2r+1)}{a+r} - h_1. \tag{5}$$

Using (1), (2), (3), (4), (5) and $\varepsilon(S) \le 2$, we obtain

$$
\begin{aligned}
1 \ \ge \ & \varepsilon(S)-1 \ge \delta_H(S,T) = f(S)-g(T)+\sum_{x\in T} d_{H-S}(x) \\
\ge \ & (a+r)|S|-(b-r)|T|+h_1|T| \\
= \ & (a+r)|S|-(b-r-h_1)|T| \\
> \ & (a+r)\Big(\frac{(b-r+1)^2-(a+r)(b-a-2r+1)}{a+r}-h_1\Big) \\
& -(b-r-h_1)(b-r+1) \\
= \ & (h_1-a-r)(b-a-2r+1)+b-r+1 \\
\ge \ & b-r+1 \ge a+1 \ge 2,
\end{aligned}
$$

which is a confliction.

Case 2. $T \neq N_T[x_1]$.

Note that $x_1 \in T$ and $x_2 \in T \setminus N_T[x_1]$. We easily see that $x_1x_2 \notin E(G)$. By the hypothesis of Theorem 3, $H = G - I$ and $0 \le h_1 \le h_2 \le b - r$, we have

$$
\begin{aligned}
\frac{(a+b)n+2}{2a+b+r} \ \le \ & \max\{d_G(x_1), d_G(x_2)\} \\
\le \ & \max\{d_{H-S}(x_1)+|S|+|I|, d_{H-S}(x_2)+|S|+|I|\} \\
= \ & \max\{h_1+|S|+|I|, h_2+|S|+|I|\} \\
= \ & h_2+|S|+|I|,
\end{aligned}
$$

namely,

$$|S| \ge \frac{(a+b)n+2}{2a+b+r} - h_2 - |I|. \tag{6}$$

Note that $|S|+|T|+|I| \le n$, $h_2-h_1 \ge 0$, $b-r-h_2 \ge 0$ and $|N_T[x_1]| \le d_{H-S}(x_1)+1 = h_1+1$. Combining these with (1) and $\varepsilon(S) \le 2$, we derive

$$
\begin{aligned}
& (n-|S|-|T|-|I|)(b-r-h_2) \ge 0 \ge \delta_H(S,T)-\varepsilon(S)+1 \\
= \ & f(S)-g(T)+\sum_{x\in T} d_{H-S}(x)-\varepsilon(S)+1 \\
\ge \ & (a+r)|S|-(b-r)|T|+h_1|N_T[x_1]|+h_2(|T|-|N_T[x_1]|)-1 \\
= \ & (a+r)|S|-(b-r)|T|-(h_2-h_1)|N_T[x_1]|+h_2|T|-1 \\
= \ & (a+r)|S|-(h_2-h_1)|N_T[x_1]|-(b-r-h_2)|T|-1 \\
\ge \ & (a+r)|S|-(h_2-h_1)(h_1+1)-(b-r-h_2)|T|-1.
\end{aligned}
$$

Therefore,

$$(n-|S|-|I|)(b-r-h_2) \ge (a+r)|S|-(h_2-h_1)(h_1+1)-1.$$

The inequality above implies

$$(b-r-h_2)n-(a+b-h_2)|S|-(b-r-h_2)|I|+(h_2-h_1)(h_1+1)+1 \ge 0. \tag{7}$$

By (6), (7), Claim 1 and $n \geq \frac{(a+b)(2a+b+r)+2}{a+r}$, we have

$$\begin{aligned} 0 &\leq (b-r-h_2)n-(a+b-h_2)|S|-(b-r-h_2)|I| \\ &\quad +(h_2-h_1)(h_1+1)+1 \\ &\leq (b-r-h_2)n-(a+b-h_2)\Big(\frac{(a+b)n+2}{2a+b+r}-h_2-|I|\Big) \\ &\quad -(b-r-h_2)|I|+(h_2-h_1)(h_1+1)+1 \\ &= -\frac{((a+r)^2+(a+r)h_2)n}{2a+b+r}-\frac{2(a+b-h_2)}{2a+b+r}+(a+b-h_2)h_2 \\ &\quad +(a+r)|I|+(h_2-h_1)(h_1+1)+1 \\ &\leq -\frac{((a+r)^2+(a+r)h_2)n}{2a+b+r}-\frac{2(a+b-h_2)}{2a+b+r}+(a+b-h_2)h_2 \\ &\quad +\frac{(a+r)((a+r)n-2)}{2a+b+r}+(h_2-h_1)(h_1+1)+1 \\ &= -\frac{(a+r)h_2n}{2a+b+r}+(a+b-h_2)h_2+(h_2-h_1)(h_1+1) \\ &\quad +\frac{2h_2}{2a+b+r}-1. \end{aligned}$$

Hence,

$$-\frac{(a+r)h_2n}{2a+b+r}+(a+b-h_2)h_2+(h_2-h_1)(h_1+1)+\frac{2h_2}{2a+b+r}-1 \geq 0. \tag{8}$$

Note that $0 \leq h_1 \leq h_2 \leq b-r$. First assume $h_2 = 0$. Then $h_1 = 0$, and thus it follows from (8) that $-1 \geq 0$, a contradiction. We next discuss $1 \leq h_2 \leq b-r$.

Let

$$F(h_1,h_2) = -\frac{(a+r)h_2n}{2a+b+r}+(a+b-h_2)h_2+(h_2-h_1)(h_1+1)+\frac{2h_2}{2a+b+r}-1.$$

Using $0 \leq h_1 \leq h_2 \leq b-r$, $1 \leq h_2 \leq b-r$ and $n \geq \frac{(a+b)(2a+b+r)+2}{a+r}$, we obtain

$$\begin{aligned} \frac{\partial F(h_1,h_2)}{\partial h_2} &= -\frac{(a+r)n}{2a+b+r}+a+b-h_2-h_2+h_1+1+\frac{2}{2a+b+r} \\ &\leq -\frac{(a+r)n}{2a+b+r}+a+b-h_2+1+\frac{2}{2a+b+r} \\ &\leq -\frac{(a+b)(2a+b+r)+2}{2a+b+r}+a+b+\frac{2}{2a+b+r} \\ &= 0, \end{aligned}$$

which implies

$$F(h_1,h_2) \leq F(h_1,h_1). \tag{9}$$

It follows from (8), (9), $0 \leq h_1 \leq b-r$ and $n \geq \frac{(a+b)(2a+b+r)+2}{a+r}$ that

$$\begin{aligned} 0 &\leq F(h_1,h_2) \leq F(h_1,h_1) \\ &= -\frac{(a+r)h_1n}{2a+b+r}+(a+b-h_1)h_1+\frac{2h_1}{2a+b+r}-1 \\ &\leq -\frac{((a+b)(2a+b+r)+2)h_1}{2a+b+r}+(a+b-h_1)h_1 \\ &\quad +\frac{2h_1}{2a+b+r}-1 \\ &= -h_1^2-1 \leq -1, \end{aligned}$$

this is a confliction. We finish the proof of Theorem 3. □

3. Remark

Let $G = (b-r)tK_1 \vee (a+r)tK_1 \vee ((a+r)t+1)K_1$ be the complete 3-partite graph having three vertex sets of size $(b-r)t$, $(a+r)t$ and $(a+r)t+1$, respectively. So any two vertices contained in distinct vertex sets are adjacent and any two vertices contained in the same vertex set are not adjacent. Next, we show that the condition

$$\max\{d_G(x), d_G(y)\} \geq \frac{(a+b)n+2}{2a+b+r}$$

declared in Theorem 3 cannot be replaced by

$$\max\{d_G(x), d_G(y)\} \geq \frac{(a+b)n+2}{2a+b+r} - 1,$$

where a, b, r, t are nonnegative integers such tat $2 \leq a = b-r$ and t is enough large. Setting $|V(G)| = n$, we have $n = (2a+b+r)t+1$. For for any two vertices x and y of $((a+r)t+1)K_1$, we have

$$\begin{aligned}
\frac{(a+b)n+2}{2a+b+r} &> \max\{d_G(x), d_G(y)\} = (b-r)t + (a+r)t = (a+b)t \\
&= (a+b) \cdot \frac{n-1}{2a+b+r} = \frac{(a+b)n+2}{2a+b+r} - \frac{a+b+2}{2a+b+r} \\
&\geq \frac{(a+b)n+2}{2a+b+r} - 1
\end{aligned}$$

Thus, we easily see that

$$\max\{d_G(x), d_G(y)\} \geq \frac{(a+b)n+2}{2a+b+r} - 1$$

for any two nonadjacent vertices x, y of G. Let $I = V((a+r)tK_1)$. Then I is an independent set of G. Setting $H = G - I = (b-r)tK_1 \vee ((a+r)t+1)K_1$, $S = V((b-r)tK_1)$ and $T = V(((a+r)t+1)K_1)$. Define $g(x) = b-r$ and $f(x) = a+r$ for every $x \in V(G)$. Note that $\varepsilon(S) = 0$. Thus, we acquire

$$\begin{aligned}
\delta_H(S,T) &= f(S) - g(T) + \sum_{x \in T} d_{H-S}(x) \\
&= (a+r)|S| - (b-r)|T| \\
&= (a+r)(b-r)t - (b-r)((a+r)t+1) \\
&= -(b-r) = -a < 0 = \varepsilon(S).
\end{aligned}$$

In light of Theorem 4, H is not fractional (g, f)-covered, and so G is not fractional ID-(g, f)-factor-critical covered.

Data availability statement

My manuscript has no associated data.

Declaration of competing interest

The author declares no conflicts of interest to this work.

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