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Article

On the Shelter of a Poset & Its Algebraic Applications

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Abstract: For a poset $P = C_a \times C_b$, a subset $A \subseteq P$ is called a chain blocker for P if A is inclusion wise minimal with the property that every maximal chain in P contains at least one element of A, where C_i is the chain $1 < \cdots < i$. In this article, we define shelter of the poset P to give complete description of all chain blockers of $C_5 \times C_b$ for $b \geq 1$.

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1. Introduction

Let $P = C_a \times C_b$ be a poset where C_i is the chain $1 < \cdots < i$. A chain blocker of P is defined as a subset $B \subseteq P$ such that every maximal chain in P contains at least one element of B and B is inclusion wise minimal with this property. Initial concept of a chain blocker came from its algebraic properties. Let $S = k[x_1, \ldots, x_n]$ be the polynomial ring over a field k and in n variables. A monomial ideal in S is an ideal generated by only monomials. A monomial ideal $I \subset S$ is called a squarefree monomial ideal if it is generated by squarefree monomials. An ideal J is called primary if $u_1u_2 \in J$ implies either $u_1 \in J$ or $u_2^l \in J$ for some $l > 0$. A primary decomposition of an ideal I is an expression of I as a finite intersection of primary ideals [1–3]. To each poset P of cardinality n we associate an ideal $I_P \subset S = k[x_1, \ldots, x_n]$ in the following way.

We take the Hasse diagram of P as a directed graph $G = (V, E)$. In [4], Conca and Negri defined path ideals I_G of G as a monomial ideal generated by all paths of length t in G, i.e. $I_G = \langle x_{i_1} \cdots x_{i_t} : x_{i_1}, \ldots, x_{i_t} \rangle$ is a path of length t in $G >\subseteq k[x_1, \ldots, x_n]$. The path ideals associated to Hasse diagram of a poset have also been studied in [5]. In our case $n = ab$ and we denote this ideal by I_P such that the generators of I_P correspond to the maximal chains in P. The chain blockers of P have one to one correspondence with the irreducible primary components of I_P (Proposition 9).

In [6] chain blockers of $C_a \times C_b$ are being discussed for $a \leq 4$ and $b \geq 1$. In [7] a new combinatorial interpretation of the convoluted Catalan numbers were studied when chain blockers of P were studied for some special cases. The Catalan numbers $C(m) = C(m, 1)$ were introduced in 1887 by Catalan [8–10].

In Section 2 we define an *i-shelter* of $S_i(j)$ as a subset C of $P \setminus \{\mathcal{R}(P), \mathcal{L}(P)\}$ with the property that each maximal chain of P containing $v \in S_i(j)$ contains at least one element of C and C is inclusion wise minimum with this property. Here $\mathcal{R}(P)$ and $\mathcal{L}(P)$ denote right maximum and left maximum chains of P, respectively. We use this definition to describle chain blockers of $C_5 \times C_b$ for $b \geq 1$. In Section 3 some algebraic consequences of chain blockers are given.

2. Main Result

Let $P = C_a \times C_b$ be a poset where C_i is the chain $1 < \cdots < i$. Let

$$\mathcal{L}(P) := (1,1) < \cdots < (1,b) < \cdots < (a,b)$$

be the left maximal chain of P and

$$\mathcal{R}(P) := (1,1) < \cdots < (a,1) < \cdots < (a,b)$$

be the right maximal chain of P. We quote the following Corollary of [7].

Corollary 1. *Let $B \subseteq P = C_a \times C_b$ be a chain blocker. Then B contains exactly one element from $\mathcal{R}(P)$ and exactly one element from $\mathcal{L}(P)$.*

For a particular subset of P, we define its shelter in the following way.

Definition 1. *Let $S_i(j) = \{(2,j),(3,j),\ldots,(i+1,j)\} \subset P$. An* i-shelter *of $S_i(j)$ is a subset C of $P \setminus \{\mathcal{R}(P), \mathcal{L}(P)\}$ with the property that each maximal chain of P containing $v \in S_i(j)$ contains at least one element of C and C is inclusion wise minimum with this property.*

For example in Figure 1 (a), $i = 1$, $j = 2$ and an 1-shelter is $\{(2,5),(3,6),(4,5),(4,4),(4,3),(4,2)\}$. Also, in Figure 1(b), $i = 2$, $j = 4$ and a 2-shelter is $\{(2,7),(3,7),(4,6),(4,5),(4,4)\}$. Note that i-shelter is not unique. We denote set of all i-shelters of $S_i(j)$ as $\widehat{S}_i(j)$ with $|\widehat{S}_i(j)| = \xi_i(j)$.

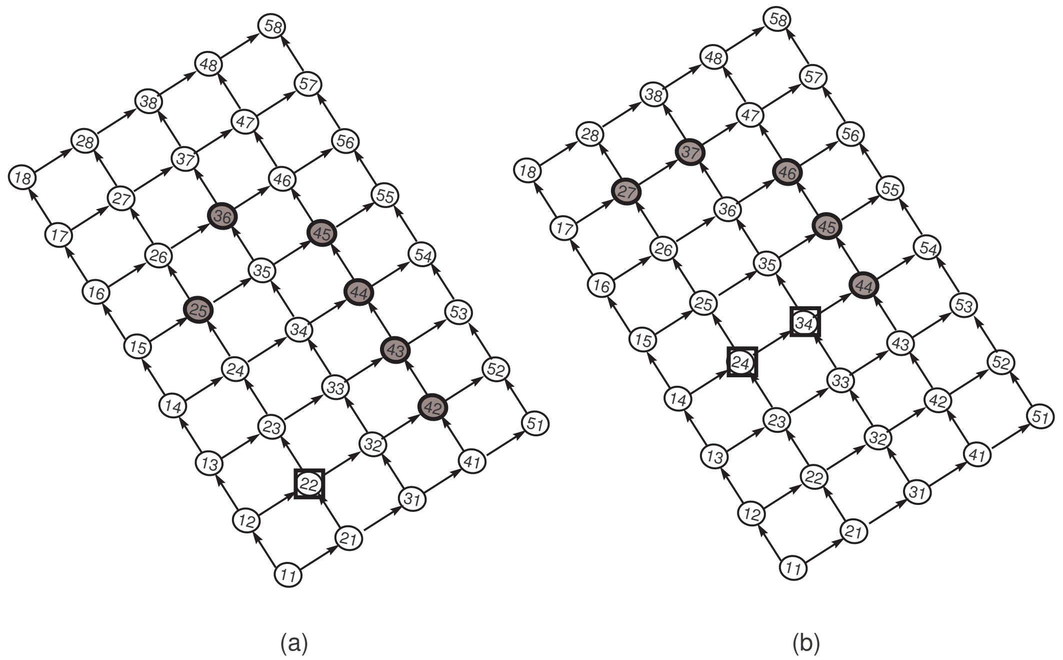

Figure 1. Two Shelters $S_1(2)$ and $S_2(4)$ for the Poset $C_5 \times C_8$

Proposition 1. *Let $P = C_5 \times C_b$ be a poset and $\widehat{S}_1(j)$ be the set of all 1-shelter of $S_1(j) \subset P$. Then*

$$|\widehat{S}_1(j)| = \xi_1(j) = 3(2^{b-j-1}) - b + j - 1.$$

Proof. Let C be a 1-shelter of $\{(2, j)\}$. Then C must contains exactly one element from the set $\Gamma = \{(2, j), \ldots, (2, b-1)\}$. Because if C does not contain any element from Γ then C does not fulfil the definition of 1-shelter and if there are two elements $\alpha, \beta \in \Gamma \cap C$ with $\alpha < \beta$ then by minimality of C $\beta \notin C$, a contradiction. Now if $(2, j) \in C$ then $C = \{(2, j)\}$ is itself a 1-shelter. Now let $(2, i) \in \Gamma \cap C$ for $i \in \{j+1, \ldots, b-1\}$, then for C we have two independent choices: (a) either $(3, l) \in C$ or $(4, l) \in C$ for all $l \in \{j, \ldots, i-2\}$, thus 2^{i-j-1} such possibilities. (b) $\{(3, k), (4, k-1), \ldots, (4, j)\} \subset C$ for each $k \in \{i-1, \ldots, b-1\}$, thus $b - i + 1$ such options. Hence over all we have $2^{i-j-1}(b - i + 1)$ number of 1-shelters C for this choice of i. Hence total number of i-shelter C is given by

$$1 + \sum_{i=j+1}^{b-1} 2^{i-j-1}(b - i + 1) = 3(2^{b-j-1}) - b - 1 + j.$$

□

Corollary 2. *Let $P = C_5 \times C_b$ be a poset and $\widehat{S}_2(j)$ be the set of all 2-shelter of $S_2(j) \subset P$. Then*

$$|\widehat{S}_2(j)| = \xi_2(j) = 3(2^{b-j-1}) - 2.$$

Proof. Let C be a 2-shelter of $\{(2, j), (3, j)\}$, then again as in the proof of previous proposition, C must contain exactly one element from the set $\Gamma_1 = \{(2, j), \ldots, (2, b-1)\}$. If $(2, j) \in C$, then $\{(2, j), (3, k), (4, k-1) \ldots, (4, j)\}$ is the 2-shelter for each $k \in \{(j+1, \ldots, b-1)\}$. Thus there are $b - j - 1$ number of 2-shelters containing $(2, j)$. Also if $\{(2, j), (3, j)\} \in C$, then $C = \{(2, j), (3, j)\}$ is itself a 2-shelter. Now if $(2, i) \in C$, where $i \in \{j+1, \ldots, b-1\}$ then a 1-shelter of $\{(2, j)\}$ is also a 2-shelter of

$$\{(2, j), (3, j)\}$$

and vice versa. Hence by previous proposition total number of 2-shelters of $\{(2, j), (3, j)\}$ is given by

$$b - j + \sum_{i=j+1}^{b-1} 2^{i-j-1}(b - i + 1) = 3(2^{b-j-1}) - 2.$$

□

Remark 1. *Let $P = C_5 \times C_b$ be a poset. Note that by above corollary if we fix $(2, 1) \in \mathcal{R}(P)$ and $(1, 3) \in \mathcal{L}(P)$, then clearly $\{(1, 3)\} \cup C \cup \{(2, 1)\}$ is chain blocker of P, where C is a 1-shelter of $\{(2, 2)\}$. Thus number of chain blockers of P containing $(2, 1)$ and $(1, 3)$ is given by $|\widehat{S}_1(2)|$. Similarly the number of chain blockers of P containing $(3, 1)$ and $(1, 3)$ is given by $|\widehat{S}_1(2)|$. Also in the same way the number of chain blockers of P containing $(4, 1)$ and $(1, 3)$ is given by $|\widehat{S}_2(2)|$.*

Let $P = C_5 \times C_b$ be a poset and $P^n_m \subset P \setminus \{\mathcal{L}(P), \mathcal{R}(P)\}$ be a subset of P defined by $(p, q) \in P^n_m \Leftrightarrow p \in \{2, 3, 4\}$ and $q \in \{m, m+1, \ldots, m+n-1\}$. We extend the idea of i-shelter in the following way. Let δ^n_m be a subset of P^n_m such that for any maximal chain B of P if $B \cap P^n_m \cap \mathcal{R}(P^n_m) \neq \emptyset$ then we must have $B \cap \delta^n_m \neq \emptyset$ and δ^n_m is minimum with this property, where $\mathcal{R}(P^n_m) = \{(5, i) \mid i \in \{m, \ldots, m+n-1\}\}$.

Remark 2. *A set δ^n_m blocks each maximal chain of P which passes through both P^n_m and $\mathcal{R}(P^n_m)$.*

Clearly we must have exactly one element from each row of P^n_m thus we have $|\delta^n_m| = n$. Obviously δ^n_m is not unique. Let $\beta(n, \lambda_1, \lambda_2)$ be the cardinality of set of all δ^n_m, where $\lambda_1 = \beta(0, \lambda_1, \lambda_2)$ and $\lambda_2 = \beta(1, \lambda_1, \lambda_2)$.

Proposition 2. *With the same notations as defined above we have:*

$$\beta(n, \lambda_1, \lambda_2) = 3\beta(n-1, \lambda_1, \lambda_2) - \beta(n-2, \lambda_1, \lambda_2). \quad (1)$$

Proof. To choose one element from each row, let $(\alpha,\beta) \in \delta_m^n$. If $\alpha = 2$ or 3 then we can choose any element from $(\beta+1)$th row of P_m^n. If $\alpha = 4$ then we can not choose $(2,\beta+1)$ since in this case a maximal chain containing $\{(2,\beta),(3,\beta),(3,\beta+1),(4,\beta+1)\} \subset P_m^n$ is not blocked. Thus for each additional row, 3 choices each for $\alpha = 2$ or 3 and 2 choices for $\alpha = 4$. Hence to get $\beta(n,\lambda_1,\lambda_2)$ we have $3\beta(n-1,\lambda_1,\lambda_2)$ minus one choice $\alpha = 4$ which is exactly equal to $\beta(n-2,\lambda_1,\lambda_2)$. □

Using elementary methods from Combinatorics, we have the following corollary.

Corollary 3. *The generating function for the recursive relation 1 is given by*

$$\beta\Big(n,\lambda_1,\lambda_2\Big) = \Big(\frac{\lambda_2-\lambda_1\alpha_2}{\alpha_1-\alpha_2}\Big)\alpha_1^n + \Big(\frac{\lambda_2-\lambda_1\alpha_1}{\alpha_2-\alpha_1}\Big)\alpha_2^n,$$

where $\alpha_1 = \frac{3+\sqrt{5}}{2}$ *and* $\alpha_2 = \frac{3-\sqrt{5}}{2}$.

We use Corollary 1 to calculate number of all chain blockers of $P = C_5 \times C_b$. We fix one element from $\mathcal{R}(P)$ and run over all elements of $\mathcal{L}(P)$ one by one to calculate all chain blockers which contain these two elements. Here is the first proposition regarding this technique.

Proposition 3. *Let* $P = C_5 \times C_b$ *be a poset. Then the number of chain blockers of* P *containing* $(2,1) \in \mathcal{R}(P)$ *is given by*

$$C_{2,1} = 1 + \sum_{j=0}^{b-3}\{\beta(j,1,2)\xi_1(b-j-2) + \beta(j,0,1)\xi_2(b-j-2)\} + 3\beta(b-2,1,2) + 2\beta(b-2,0,1).$$

Proof. If we take $(1,2) \in \mathcal{L}(P)$, then $\{(1,2),(2,1)\}$ is itself a chain blocker. If we choose $(1,j) \in \mathcal{L}(P)$ for $j \in \{3,\dots,b\}$ then we need to calculate all possibilities of chain blockers containing $\{(2,1),(1,j)\}$ and elements from P_2^{b-1}, where P_m^n is defined above. Now we divide the rows of P_2^{b-1} into two disjoint parts. One part is from row 2 to row $j-2$ and second part is from row $j-1$ to row $b-1$. Now a chain blocker of P must contains elements from both of these parts. In particular, each chain blocker must contain exaclty one element from row $j-2$ of P_2^{b-1}.

If it contains $(4,j-2)$ then the number of such chain blockers having elements from first part i.e. from row 2 to row $j-2$ of P_2^{b-1} is exactly equal to $\beta(j-3,\lambda_1,\lambda_2)$ where in this case we have $\lambda_1 = 0$ and $\lambda_2 = 1$. For this case, number of chain blockers from the second part i.e. from row $j-1$ to row $b-1$ is given by 2-shelter $\xi_2(b-j+1)$. Since choices from these two parts are disjoint so total number of chain blockers in this case are given by $\beta(j-3,0,1)\xi_2(b-j+1)$.

On the other hand if a chain blocker contains $(2,j-2)$ or $(3,j-2)$, then number of such chain blockers having elements from first part i.e. from row 2 to row $j-2$ is given by $\beta(j-3,\lambda_1,\lambda_2)$ where in this case we have $\lambda_1 = 1$ and $\lambda_2 = 2$. Also in this case number of chain blockers containing elements from part 2 of P_2^{b-1} is given by the 1-shelter $\xi_1(b-j+1)$. Again since both of these choices are independent so have total number chain blockers in this case equals to $\beta(j-3,1,2)\xi_1(b-j+1)$. Summing over j from 3 to b we have partial proof to our formula.

Now we are left to count the number of chain blockers of P contain $(2,1)$ and (j,b), where $j \in \{2,3,4\}$. A chain blocker containing $(2,b)$ must contain either $(2,b-1)$ or $(3,b-1)$, thus the number of such chain blockers equals to $\beta(b-2,1,2)$. Also a chain blockers containing $(3,b)$ must contain $(2,b-1)$, $(3,b-1)$ or $(4,b-1)$ and hence the number of such chain blockers equal to $\beta(b-2,1,2)+\beta(b-2,0,1)$. Finally the number of chain blockers containing $(4,b)$ equals to $\beta(b-2,1,2)+\beta(b-2,0,1)$. So we obatined

$$C_{2,1} = 1 + \sum_{j=0}^{b-3}\{\beta(j,1,2)\xi_1(b-j-2) + \beta(j,0,1)\xi_2(b-j-2)\} + 3\beta(b-2,1,2) + 2\beta(b-2,0,1),$$

which complete the proof. □

Corollary 4. *Let $P = C_5 \times C_b$ be a poset. Then the number of chain blockers of P containing* $(3,1)$ *is given by*

$$C_{3,1} = C_{2,1} + \xi_1(b-2) - 1.$$

Proof. Since any chain blocker of P containing $(2,1)$ and any $p \in \mathcal{L}(P)\backslash\{1,2\}$ remains a chain blocker if we replace $(2,1)$ by $(3,1)$. Thus we left with those chain blockers which contains $(3,1)$ and $(1,2)$. But number of such chain blockers are given by $\xi_1(b-2)$. □

Proposition 4. *Let $P = C_5 \times C_b$ be a poset. Then the number of chain blockers of P containing* $(4,1)$ *is given by*

$$\begin{aligned} C_{4,1} &= 2\xi_2(b-2) + \sum_{j=1}^{b-3}\{\beta(j,0,1)\xi_1(b-j-2) + \beta(j,1,1)\xi_2(b-j-2)\} \\ &\quad +3\beta(b-2,0,1) + 2\beta(b-2,1,1). \end{aligned}$$

Proof. If we fix $(1,2) \in \mathcal{L}(P)$, then clearly number of chain blockers containing $(4,1)$ and $(1,2)$ is given by the 2-shelter $\xi_2(b-2)$. Same is true if we fix $(1,3) \in \mathcal{L}(P)$. To calculate remaining case of chain blockers we will use the same technique as in the proof of Proposition 3. By the similar arguments the number of chain blockers for the remaining cases are given by

$$\sum_{k=1}^{b-3}\{\beta(k,\lambda_1,\lambda_2)\xi_1(b-k-2) + \beta(k,\lambda_1',\lambda_2')\xi_2(b-k-2)\}$$

$$+3\beta(b-2,\lambda_1,\lambda_2) + 2\beta(b-2,\lambda_1',\lambda_2').$$

Now in this case $\lambda_1 = 0$ and $\lambda_2 = 1$. Also we have $\lambda_1' = 1$ and $\lambda_2' = 1$. So we conclude

$$\begin{aligned} C_{4,1} &= 2\xi_2(b-2) + \sum_{j=1}^{b-3}\{\beta(j,0,1)\xi_1(b-j-2) + \beta(j,1,1)\xi_2(b-j-2)\} \\ &\quad +3\beta(b-2,0,1) + 2\beta(b-2,1,1). \end{aligned}$$

which complete the proof. □

Proposition 5. *Let $P = C_5 \times C_b$ be a poset. Then the number of chain blockers of P containing* $(5,1)$ *is given by*

$$\begin{aligned} C_{5,1} &= 2b - 1 + 3\xi_2(b-3) + \sum_{j=2}^{b-3}\{\beta(j-2,2,5)\xi_1(b-j-2) \\ &\quad +\beta(j-2,1,3)\xi_2(b-j-2)\} + 3\beta(b-4,2,5) + 2\beta(b-4,1,3). \end{aligned}$$

Proof. Let B be a chain blocker of P. Fix $(1,2) \in \mathcal{L}(P) \cap B$. Then B must contains $(4,2)$. Now if $(3,2) \in B$ then either $B = \{(1,2),(2,2),(3,2),(4,2),(5,1)\}$ or $B = \{(1,2),(2,3),(3,2),(4,2),(5,1)\}$. On the other hand if $(3,2) \notin B$ and $(2,2) \in B$. Then $B = \{(1,2),(2,2),(3,j),(4,j-1),\ldots,(4,2),(5,1)\}$, where $j = 3,\ldots,b-1$. Lastly if both $(2,2) \notin B$ and $(3,2) \notin B$ holds then number of such chain blockers are given by 2-shelter $\xi_2(b-3)$. Similarly if we fix $(1,3) \in \mathcal{L}(P) \cap B$ then we have the same number of chain blockers. Now we fix $(1,4) \in \mathcal{L}(P) \cap B$, then the only difference from the previous two cases is while $(3,2) \in B$, in this we have only one chain blocker given by $B = \{(1,4),(2,3),(3,2),(4,2),(5,1)\}$ as shown in figure 1. Thus total number of chain blockers containing $(5,1)$ and a element from the set $\{(1,2),(1,3),(1,4)\}$ is given by $2b - 1 + 3\xi_2(b-3)$. Now for the remaining cases we will use the same technique as we use in the proof of Proposition 3. Here for the multiplier of ξ_1, we have

$\beta(k-2,\lambda_1,\lambda_2)$, where λ_1 is equal to 2 since both $\{(4,2),(3,2),(2,3)\}$ and $\{(4,2),(3,3)\}$ ends up for 1-shelter. Now if we extend to next row, then $\{(4,2),(3,2),(2,3),(2,4)\}$, $\{(4,2),(3,2),(2,3),(3,4)\}$, $\{(4,2),(3,3),(2,4)\}$, $\{(4,2),(3,3),(3,4)\}$ and $(4,2),(4,3),(3,4)$ lead to 1-shelter. Hence $\lambda_2 = 5$. On the other hand, for the multiplier of ξ_2, we have $\beta(k-2,\lambda'_1,\lambda'_2)$. Now $\lambda'_1 = 1$ due to $\{(4,2),(4,3)\}$ and $\lambda'_2 = 3$ due to $\{(4,2),(3,2),(2,3),(4,4)\}$, $\{(4,2),(3,3),(4,4)\}$ and $\{(4,2),(4,3),(4,4)\}$. Thus

$$v\sum_{k=2}^{b-3}\{\beta(k-2,2,5)\xi_1(b-k-2)$$

counts number of all chain blockers contain $(5,1)$ and $(1,j)$, where $j = 5,\ldots,b$. The remaining case are thus given by $3\beta(b-4,2,5)+2\beta(b-4,1,3)$. As a result we conclude

$$\begin{aligned} C_{5,1} &= 2b-1+3\xi_2(b-3)+\sum_{j=2}^{b-3}\{\beta(j-2,2,5)\xi_1(b-j-2) \\ &\quad +\beta(j-2,1,3)\xi_2(b-j-2)\}+3\beta(b-4,2,5)+2\beta(b-4,1,3), \end{aligned}$$

which complete the proof. □

Proposition 6. *Let $P = C_5 \times C_b$ be a poset. Then the number of chain blockers of P containing $(5,2)$ is given by*

$$\begin{aligned} C_{5,2} &= 5\xi_1(b-4)+5b+5+4\xi_2(b-4)+\sum_{j=5}^{b-1}\{\beta(j-5,7,17)\xi_1(b-j) \\ &\quad +\beta(j-5,3,10)\xi_2(b-j)\}+3\beta(b-5,7,17)+2\beta(b-5,3,10). \end{aligned}$$

Proof. Let B be an arbitrary chain blocker of P containing $(5,2)$. If we fix $(1,2) \in B\cap\mathcal{L}(P)$ then obviously B must contain at least one element from each middle columns. Let $\{(4,2),(3,2)\} \in B$ then either $(2,2)$ or $(2,3)$ belongs to B. Similarly if $\{(4,2),(3,3)\} \in B$ the B must contain at least one element from the set $\{(2,2),(2,3)\}$ or one element from 1-shelter starting at $(2,4)$. Thus number of chain blockers containing the set $\{(5,2),(4,2),(1,2)\}$ is given by $4+\xi_1(b-4)$.

Now let $\{(4,3),(2,2)\} \in B$. Then B must contain either one element from the set $\{(3,2),(3,3),(3,4)\}$ or contain $(3,j),(4,j-1),\ldots,(4,4)$ for $j = 5,\ldots,b-1$. The same is true if $\{(4,3),(2,3)\} \in B$. Now if we take $B = \{(5,2),(4,3),(3,3)(2,4),(1,2)$, then the only case remaining for B containing $(5,2)$ and $(1,2)$ is given by the 2-shelter starting at $\{(2,4),(3,4)\}$. Thus number of chain blockers containing $(5,2)$ and $(1,2)$ is given

$$2b+1+\xi_1(b-4)+\xi_2(b-4). \tag{2}$$

Lets move to second element of $\mathcal{L}(P)$ that is $(1,3) \in B\cap\mathcal{L}(P)$. Note that each chain blocker of P containing $\{(1,2),(5,2)\}$ will remain a chain blocker if we replace $(1,2)$ with $(1,3)$ and vice versa. Thus in this case we have the same number of chain blockers as given in Equation 2. Now let $(1,4) \in B\cap\mathcal{L}(P)$. If $(4,2) \in B$ then we have three possibilities, namely

$$B = \{(5,2),(4,2),(3,2),(2,3),(1,4)\},$$

or

$$B = \{(5,2),(4,2),(3,3),(2,3),(1,4)\},$$

or

$$B = \{(5,2),(4,2),(3,3),(1,4)\}\cup X_1,$$

where X_1 is an element of 1-shelter having base element $(2,4)$. Secondly if $(4,3) \in B$ again we have three possibilities. Either

$$B = \{(5,2),(4,3),(4,4),\ldots,(4,j-1),(3,j),(2,3),(1,4)\}$$

for $j = 2,\ldots,b-1$, or $B = \{(5,2),(4,3),(3,3),(2,4),(1,4)\}$, or

$$B = \{(5,2),(4,3),(1,4)\} \cup X_2$$

where X_2 is an element of 2-shelter having base elements $\{(2,4),(3,4)\}$. Thus number of chain blockers containing $(5,2)$ and $(1,4)$ is given by

$$b + 1 + \xi_1(b-4) + \xi_2(b-4).$$

Now let $(1,5) \in B \cap \mathcal{L}(P)$. If $(4,2) \in B$ then either

$$B = \{(5,2),(4,2),(3,2),(2,3),(1,5)\} \cup X_1,$$

or

$$B = \{(5,2),(4,2),(3,3),(1,5)\} \cup X_1,$$

where X_1 is an element of 1-shelter starting from $(2,4)$. Secondly if $(4,3) \in B$ then following three cases occurs.

$$B = \{(5,2),(4,3),(3,2),(2,3),(2,4),(1,5)\},$$

or

$$B = \{(5,2),(4,3),(3,3),(2,4),(1,5)\},$$

or

$$B = \{(5,2),(4,3),(1,5)\} \cup X_2,$$

where X_2 is an element of 2-shelter having base elements $\{(2,4),(3,4)\}$. Now summing over all above cases i.e. number of chain blockers containing $(5,2)$ and exactly one element from the set $\{(1,2),\ldots,(1,5)\}$ is given by

$$5b + 5 + 5\xi_1(b-4) + 4\xi_2(b-4).$$

For the remaining part of the proof we will use the same technique as used in the proves of previous results. Namely, for $k = 5,\ldots,b-1$ the multiplier of $\xi_1(b-k)$, we have $\beta(k-5,\lambda_1,\lambda_2)$, where λ_1 is equal to 7 due to $\{(4,2),(3,2),(2,3),(2,4)\}$, $\{(4,2),(3,2),(2,3),(3,4)\}$, $\{(4,2),(3,3),(2,4)\}$, $\{(4,2),(3,3),(3,4)\}$, $\{(4,3),(3,2),(2,3),(2,4)\}$, $\{(4,3),(3,3),(2,4)\}$ and $\{(4,3),(3,4)\}$. Similarly if we extend to next level we have $\lambda_2 = 17$.

On the other hand, for the multiplier of $\xi_2(b-k)$, we have $\beta(k-5,\lambda'_1,\lambda'_2)$. Now $\lambda'_1 = 3$ due to $\{(4,2),(3,2),(2,3),(4,4)\}$, $\{(4,2),(3,3),(4,4)\}$ and $\{(4,3),(4,4)\}$. Similarly if we extend to next level we have $\lambda'_2 = 10$. Thus summing over all k and adding the similar cases for $(2,b)$, $(3,b)$ and $(4,b) \in B \cap \mathcal{L}(P)$. So we obtained our required result as:

$$\begin{aligned} C_{5,2} &= 5\xi_1(b-4) + 5b + 5 + 4\xi_2(b-4) + \sum_{j=5}^{b-1}\{\beta(j-5,7,17)\xi_1(b-j) \\ &\quad +\beta(j-5,3,10)\xi_2(b-j)\} + 3\beta(b-5,7,17) + 2\beta(b-5,3,10). \end{aligned}$$

□

Remark 3. *So far we have calculated all chain blockers containing v where $v \in \{(1,1),\ldots,(5,1),(5,2)\} \subset \mathcal{R}(P)$. Since each chain blocker of P contains at least one element from $\mathcal{R}(P)$ thus we are left with the case of calculating number of chain blockers containing $\mathcal{R}(P) \setminus \{(1,1),\ldots,(5,1),(5,2)\} \equiv \{(5,3),\ldots,(5,b-1)\}$. We first calculate number of chain blockers B containing $(5,k)$ where $k \in \{3,\ldots,b-2\}$. Then B must contains exactly one element w from $\mathcal{L}(R)$. Then we have following choices of w.*

1. $w = (1, i)$ *where* $2 \leq i \leq k+1$

2. $w = (1, i)$ *where* $k+2 \leq i \leq b$

3. $w = (o, b)$ *where* $2 \leq o \leq 4$

In the following results we will use notations from this remark.

Proposition 7. *Let* $C(k, b)$ *be number of chain blockers containing* $(5, k)$ *and* $(1, i)$ *for* $3 \leq k \leq b-2$ *and* $2 \leq i \leq k+1$. *Then*

$$\begin{aligned} C(k,b) &= f_a(k) + \sum_{m=6}^{k+3} f_b(k,m) + 2k(b-2) - k^2 + 3k - 2 \\ &+ \sum_{j=4}^{k+2} f_c(k,j) + (4k-3)\xi_1(b-k-2) \\ &+ \sum_{p=6}^{k+2} f_d(k,p) + (k+1)(1+\xi_2(b-k-2)) + k + \xi_2(b-k-2), \end{aligned}$$

where

$$\begin{aligned} f_a(k) &= \frac{4k^3 + 27k^2 - 37k - 6}{6}, \\ f_b(k,i) &= \sum_{j=2}^{i-3} 2^{i-j-1}(i-2)(k-i+3+\xi_1(b-k-2)), \\ f_c(k,i) &= (k-i+3)(b-2) - (\frac{k^2-3k+2}{2} - \frac{i^2-7i+12}{2}), \\ f_d(k,i) &= \sum_{j=1}^{k-i+3}(k-1+\frac{j^2+j}{2}) + (k-i+3)\xi_1(b-k-2). \end{aligned}$$

Proof. To calculate number of chain blockers containing $(5, k) \in \mathcal{R}(P)$ and $w = (1, i)$ where $3 \leq k \leq b-2$ and $2 \leq i \leq k+1$, we divide into two main cases.

Case 1. $2 \leq i \leq 5$

Since a chain blocker B contains only these two elements from $\mathcal{R}(P)$ and $\mathcal{L}(P)$ so B must contains at least one element $(4, l)$ from 4^{th} column. If $l \leq k$, then $(4, l)$ is the only element which B contains from column 4. Also B must contains at least one element from the column 2 and 3. Now if $(2, m) \in B$ then either $2 \leq m \leq k+1$ or $m > k+1$. For the first case the number of chain blockers are same for $m = 2$ and $m = 3$. Now if $k+1 \geq m \geq l+2$, then the set $\{(3, l+1), \ldots, (3, m-1)\}$ must belong to B and hence we have $k-l-1$ number of choices. But if $l+1 \geq m \geq 3$, then

$$B = \{(1,2), (2,m), (3,n), (4,l), (5,k)\},$$

where $l+1 \geq n \geq m-1$ and we have $l+1-(m-1)+1$ number of choices for this case. Thus subtotal for number of chain blockers in this case is given by

$$\sum_{l=2}^{k}(\sum_{m=3}^{l+1}(l-m+3) + l + \sum_{m=l+2}^{k+1}) = \frac{1}{6}k^3 + \frac{3}{2}k^2 - \frac{5}{3}k. \tag{3}$$

Note that in the above case, if we replace $(1, 2)$ with $(1, 3) \in \mathcal{R}(P)$ the number of chain blockers remains the same. On the other hand if we replace $(1, 2)$ with $(1, 4) \in \mathcal{R}(P)$ then there is no chain blocker containing $(1,4)$, $(5,k)$ and $(2,2)$, where as the remaining calculations will be the same as above thus subtotal of number of chain blockers in this case is given by

$$\sum_{l=2}^{k}(\sum_{m=3}^{l+1}(l-m+3)+\sum_{m=l+2}^{k+1}) = \frac{1}{6}k^3+k^2-\frac{13}{6}k+1. \tag{4}$$

With the similar arguments and considering the same limits as above, the number of chain blockers containing $(1,5)$ is given by

$$\frac{1}{6}k^3+\frac{1}{2}k^2-\frac{2}{3}k-2. \tag{5}$$

Let $f_a(k)$ be function obtained by summing over the equation 3 two times(for both $(1,2)$ with $(1,3)$), equations 4 and equation 5. Thus

$$f_a(k) = \frac{4k^3+27k^2-37k-6}{6}. \tag{6}$$

Now for $(1,i)$ where $6 \leq i \leq k+3$, and fixing $(5,k) \in \mathcal{R}(P)$ for $3 \leq k \leq b-2$. We take $(4,j) \in B$ with $j = 2,\ldots,i-3$. Then there are four different calculations are involved, namely:

- For each $j = 2,\ldots,i-3$, we must have either $\{(4,j),(3,n),(2,n+1),\ldots,(2,i-1)\} \in B$ for $n = 2,\ldots,j$ or $\{(4,j),(3,j+1)\} \in B$. Thus it counts $i-2$ possibilities as whole.
- Either $(2,o)$ or $(3,o)$ belongs to B for $o = j+1,\ldots,i-1$. Thus it counts 2^{i-j-1}.v
- Either $(2,i-1) \in B$ or $\{(2,o),(3,o-1),\ldots,(3,i-1)\}$ for $o = i,\ldots,k+1$. Thus as a whole we have $k-i+3$ possibilities.
- i-shelter having $(2,k+2)$ as the base point so it counts $\xi_1(b-k-2)$.

Keeping in view the nature of above four choices we have the following description for this case.

$$f_b(k,i) = \sum_{j=2}^{i-3} 2^{i-j-1}(i-2)(k-i+3+\xi_1(b-k-2)). \tag{7}$$

Now again if $\{(1,i),(5,k)\} \in B$ where $i = 3,\ldots,k+2$. Then if $(4,k+1)$ then for each $(2,m) \in B$ with $m = 3,\ldots,k+1$ we have

$$B = \{(1,i),(2,m),(3,n),(4,k+1),(5,k)\}$$

where $n = m-1,\ldots,k+2$. Also for $k+2 < n \leq b-2$, the element $(3,n)$ is replaced by $(3,n),(4,n-1),\ldots,(4,k)$. Thus for each m we have $b-m$ number of chain blockers thus total number of chain blockers are given by

$$f_c(k,i) = (k-i+3)(b-2)-(\frac{k^2-3k+2}{2}-\frac{i^2-7i+12}{2}).$$

Note that the number of chain blockers containing $(1,3)$ and $(1,2)$ is same so we add one additional $f_c(k,3)$ in the final calculations. Now for $(5,k)$ and $(1,i)$ in B with $4 \leq k \leq b-2$ we restrict $6 \leq i \leq k+2$ and $i-2 \leq l \leq k$ for $(4,l) \in B$. We have following possibilities of calculations. For each $(4,l) \in B$ there must be at least one element from the 2nd and the 3rd column. Now there are three types of possibilities. Namely:

- $\{(2,i-1),\ldots,(2,j),(3,j-1)\}$ belongs to B for $j = 3,\ldots,i-3$.
- $\{(2,m),(3,n)\} \in B$ for $i-1 \leq m \leq k+2$ and $2 \leq n \leq m$.
- for $\{(4,k+1),(3,k+2)\}$ the number of chain blockers is given by $\xi_1(b-k-2)$.

Thus total number for this subcase is given by:

$$f_d(k,i) = \sum_{j=1}^{k-i+3}(k-1+\frac{j^2+j}{2})+(k-i+3)\xi_1(b-k-2).$$

Now finally the number of chain blockers containing $(1,i),(4,k+1),(5,k)$ and $m,n \geq k+2$ is given by $\xi_2(b-k-2)$ with $\{(2,k+2),(3,k+2)\}$ as the base elements. It completes the proof. □

Proposition 8. *Let* $P = C_5 \times C_b$ *be a poset. Let* B *be a chain blocker containing* $(5, k)$ *and* $(1, i)$ *such that* $3 \le k \le b-2$ *and* $i \ge k+2$. *Let* $\beta(b, \lambda_1, \lambda_2)$ *be the coefficient of* ξ_1 *and* $\beta(b, \mu_1, \mu_2)$ *be the coefficient of* ξ_2. *Then*

$$\begin{aligned} \lambda_1 &= g(k), \\ \lambda_2 &= 2g(k) + h(k), \\ \mu_1 &= h(k), \\ \mu_2 &= g(k) + h(k), \end{aligned}$$

where

$$\begin{aligned} g(k) &= \sum_{j=0}^{k-1} 2^{k-1-i}(2+i), \\ h(k) &= \sum_{i=0}^{k-2} 2^{k-i-2}(2+i) + 1. \end{aligned}$$

Proof. It follows from the proofs of Propositions 3-6 and from the definitions of $g(k)$ and $h(k)$. □

Theorem 1. *Let* $\mathcal{D}(k, b)$ *be the number of chain blockers containing* $(5, k)$ *and* $(1, i)$ *for* $3 \le k \le b-2$ *and* $i \ge k+2$. *Then*

$$\begin{aligned} \mathcal{D}(k, b)) &= C(k, b) + \sum_{q=k+3}^{b-1} (\beta(q-k-3, g(k), 2g(k)+h(k))\xi_1(b-q) \\ &\quad +\beta(q-k-3, h(k), g(k)+h(k))\xi_2(b-q)) \\ &\quad +2\beta(b-k-4, g(k), 2g(k)+h(k)) \\ &\quad +\beta(b-k-4, h(k), g(k)+h(k)) \\ &\quad +2\beta(b-k-3, g(k), 2g(k)+h(k)) \\ &\quad +2\beta(b-k-3, h(k), g(k)+h(k)). \end{aligned}$$

Proof. By using the above calculations and selecting the elements one by one from both left end maximal chain and right end maximal chain, we have the result. □

On combining all the results and after simplification, we have the following theorem.

Theorem 2. *Let* $\mathcal{T}(k, b)$ *be the total number of chain blockers of* $C_5 \times C_b$ *for* $b \ge 1$. *Then*

$$\begin{aligned} \mathcal{T}(k, b) &= C_{2,1} + C_{3,1} + C_{4,1} + C_{5,1} + C_{5,2} + \sum_{k=3}^{b-3} \mathcal{D}(k, b)) + 2\sum_{m=6}^{b+1} f_{bb}(b-2, m) \\ &\quad +2\sum_{n=4}^{b} f_c(b-2, n) + 2\sum_{o=6}^{b} f_{dd}(b-2, o) \\ &\quad +5\sum_{p=0}^{b-4} (2+p)2^{b-4-p} + \frac{4}{3}b^3 + 3b^2 - \frac{97}{3}b + 40, \end{aligned} \tag{8}$$

where

$$\begin{aligned} f_{bb}(k, i) &= \sum_{j=4}^{i-1} 2^{i-j-1}(i-2)(k-i+3), \\ f_c(k, i) &= (k-i+3)(b-2) - (\frac{k^2-3k+2}{2} - \frac{i^2-7i+12}{2}), \\ f_{dd}(k, i) &= \sum_{j=1}^{k-i+3} (k-1+\frac{j^2+j}{2}). \end{aligned}$$

3. Applications

In this section we provide algebraic consequences associated to a chain blocker B of P. A simplicial complex Δ on the vertex set $V = [n]$ is a collection of subsets of $2^{[n]}$ with the property that if $A \in \Delta$ then Δ contains all subsets of A. The inclusionwise maximum elements of Δ are called facets. Let $\{F_i, \ldots, F_r\}$ be the set of facets of Δ. A minimal vertex cover of Δ is a subset $A \subseteq V$ with the property that for every facet F_i of Δ there exist a vertex $v \in A$ such that $v \in F_i$ and A is minimal with this property.

Let Δ_P be a simplicial complex associated to a poset P in such a way that elements of Δ_P are exactly the chains in P. The set of facets of Δ_P are the maximal chains of P and hence each chain blockers of P is a minimal vertex cover of Δ_P. The simplicial complex Δ_P is called the order complex.

Now we are ready to relate a poset P to its algebraic counterpart. Let $S = k[x_1, \ldots, x_n]$ be the polynomial ring over the field k and in n variables. Recall that a monomial ideal in S is an ideal generated by monomials u_i. A monomial ideal is called a squarefree monomial ideal if it is generated by the squarefree monomials. Let $I = I_1 \cap \cdots \cap I_r$ be an irredundant primary decomposition of I, where ideals $I_1, \ldots, I_r$ are called irreducible primary components of I. For more details about primary decomposition see [1].

To each square free monomial ideal I one can associate a simplicial complex Δ. One way of this association is facet ideals and facet complex introduced by Sara Faridi [11]. A facet ideals $I^{\mathcal{F}}(\Delta)$ of Δ is an ideal generated by squarefree monomial $x_{i_1} \cdots x_{i_t}$ where $\{x_{i_1}, \ldots, x_{i_t}\}$ is a facet of Δ. Let $I = \langle u_1, \ldots, u_r\rangle$ be a squarefree monomial ideal. A facet complex $\Delta^{\mathcal{F}}(I)$ of I is a simplicial complex over the vertex set $\{v_1, \ldots, v_n\}$ and set of facets $\{F_1, \ldots, F_r\}$, where $F_i = \{v_j : x_j \mid u_i, \quad 1 \le j \le n\}$.

It is well known that minimal vertex covers of $\Delta^{\mathcal{F}}(I)$ correspond to the irreducible primary components of $I^{\mathcal{F}}(\Delta)$. Let $I_P = I^{\mathcal{F}}(\Delta_P)$. Note that I_P is also the path ideals of the directed graph of Hasse diagram of P. The path ideal was intorduced by Conca and De Negri in [4]. Some results of I_P were also studied in [5]. Since the facets of Δ_P are the maximal chains of P, hence by definition of a chain blocker we have the following proposition.

Proposition 9. *Let $P = C_a \times C_b$ be a poset and I_P is the path ideal of the directed graph of the Hasse diagram of P only if P is a pure poset, that is all the maximal chains have the same length.*

Following examples demonstrate the one to one correspondence given in above proposition. By using the Alexander duality, the minimal prime ideals of I_P are in one to one correspondence with the minimal generators of the Stanley-Reisner ideal of the Alexander dual of I_P, that is $I_P^\vee$.

Example 1. *Numbers of chain blockers in $C_5 \times C_5, C_5 \times C_6$, and $C_5 \times C_7$ are given in table 1 below. In this table, we have verified the corresponding number of irreducible primary components of I_P.*

Let $P = C_5 \times C_6$ be a poset. Consider the ideal containing x_{54} in $C_5 \times C_6$.

$$\begin{gathered}
I_P = (x_{21}x_{31}x_{41}x_{51}x_{52}x_{53}x_{54}x_{55}, x_{21}x_{31}x_{41}x_{42}x_{52}x_{53}x_{54}x_{55}, x_{21}x_{31}x_{41}x_{42}x_{43}x_{53}x_{54}x_{55},\\
x_{21}x_{31}x_{41}x_{42}x_{43}x_{44}x_{54}x_{55}, x_{21}x_{31}x_{32}x_{42}x_{52}x_{53}x_{54}x_{55}, x_{21}x_{31}x_{32}x_{42}x_{43}x_{53}x_{54}x_{55},\\
x_{21}x_{31}x_{32}x_{42}x_{43}x_{44}x_{54}x_{55}, x_{21}x_{31}x_{32}x_{33}x_{43}x_{53}x_{54}x_{55}, x_{21}x_{31}x_{32}x_{33}x_{43}x_{44}x_{54}x_{55}, x_{21}x_{31}x_{32}x_{33}x_{34}x_{44}x_{54}x_{55},\\
x_{21}x_{22}x_{32}x_{42}x_{52}x_{53}x_{54}x_{55}, x_{21}x_{22}x_{32}x_{42}x_{43}x_{53}x_{54}x_{55}, x_{21}x_{22}x_{32}x_{42}x_{43}x_{44}x_{54}x_{55}, x_{21}x_{22}x_{32}x_{33}x_{43}x_{53}x_{54}x_{55},\\
x_{21}x_{22}x_{32}x_{33}x_{43}x_{44}x_{54}x_{55}, x_{21}x_{22}x_{32}x_{33}x_{34}x_{44}x_{54}x_{55}, x_{21}x_{22}x_{23}x_{33}x_{43}x_{53}x_{54}x_{55}, x_{21}x_{22}x_{23}x_{33}x_{43}x_{44}x_{54}x_{55},\\
x_{21}x_{22}x_{23}x_{33}x_{34}x_{44}x_{54}x_{55}, x_{21}x_{22}x_{23}x_{24}x_{34}x_{44}x_{54}x_{55}).
\end{gathered}$$

Since each irreducible primary components of I_P correspond to a chain blocker of P. Thus by Theorem 1 number of irreducible primary components of I_P is given by 205. *The same number is verified by the Computer Algebra System CoCoA and Singular.*

poset	chain blockers containing ij	no. of chain blockers	no. of primary components of I_P containing x_{ij}
$C_5 \times C_5$	21	82	82
	31	89	89
	41	66	66
	51	45	45
	52	66	66
	53	89	89
	54	82	82
$C_5 \times C_6$	21	238	238
	31	256	256
	41	181	181
	51	114	114
	52	147	147
	53	170	170
	54	205	205
	55	187	187
$C_5 \times C_7$	21	676	676
	31	717	717
	41	488	488
	51	297	297
	52	366	366
	53	366	366
	54	374	374
	55	417	417
	56	376	376

Table 1. No. of Chain Blockers and Their Corresponding No. of Primary Components of I_P in $C_5 \times C_b$, $5 \leq b \leq 7$

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Conflict of Interest

The authors declare no conflict of interest.

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