Ars Combinatoria, 158: 117–128
DOI:10.61091/ars158-12
http://www.combinatorialpress.com/ars
Received 12 February 2018, Accepted 10 November 2018, Published 31 March 2024

Combinatorial Press

Article

q-Analogue of the Generalized Fibonacci and Lucas Polynomials

Adem ŞAHİN[1,*]

[1] Faculty of Education, Tokat Gaziosmanpaşa University, 60250 Tokat, Turkey

* **Correspondence:** adem.sahin@gop.edu.tr, hessenberg.sahin@gmail.com

Abstract: In this article, we define q-generalized Fibonacci polynomials and q-generalized Lucas polynomials using q-binomial coefficient and obtain their recursive properties. In addition, we introduce generalized q-Fibonacci matrix and generalized q-Lucas matrix, then we derive their basic identities. We define (k, q, t)-symmetric generalized Fibonacci matrix and (k, q, t)-symmetric generalized Lucas matrix, then we give the Cholesky factorization of these matrices. Finally, we give determinantal and permanental representations of these new polynomial sequences.

Keywords: q-binomial coefficient, q-generalized Fibonacci polynomials, q-generalized Lucas polynomials, q-generalized Fibonacci matrix and q-generalized Lucas matrices
2010 Mathematics Subject Classification: 11B39, 11C20, 15B36

---

1. Introduction

MacHenry [1] defined generalized Fibonacci polynomials and generalized Lucas polynomials. The generalized Fibonacci polynomials and the generalized Lucas polynomials already have comprehensive representation properties. These polynomials are a general form of generalized bivariate Fibonacci and Lucas p-polynomials, ordinary Fibonacci, Lucas, Pell, Pell-Lucas and Perrin sequences, Chebyshev polynomials of the second kind, and the Tribonacci numbers, etc.

The generalized Fibonacci polynomials, $F_{k,n}(t)$, and the generalized Lucas polynomials, $G_{k,n}(t)$, are defined inductively by as follows:

$$F_{k,0}(t) = 1,\ F_{k,n+1}(t) = t_1 F_{k,n}(t) + \cdots + t_k F_{k,n-k+1}(t)(n > 1),$$

and

$$\begin{aligned} G_{k,0}(t) &= k,\ G_{k,1}(t) = t_1,\ G_{k,n}(t) = G_{k-1,n}(t)(1 \le n \le k), \\ G_{k,n}(t) &= t_1 G_{k,n-1}(t) + \cdots + t_k G_{k,n-k}(t)(n > k), \end{aligned}$$

where the vector $t = (t_1, t_2, \ldots, t_k)$ and t_i $(1 \le i \le k)$ are constant coefficients of the core polynomial

$$P(x; t_1, t_2, \ldots, t_k) = x^k - t_1 x^{k-1} - \cdots - t_k.$$

In [2], authors gave explicit formula for the $F_{k,n}(t)$ and $G_{k,n}(t)$ as follows:

$$F_{k,n}(t) = \sum_{a \vdash n} \binom{|a|}{a_{1,\ldots,}a_k} t_1^{a_1} \ldots t_k^{a_k}, \tag{1}$$

$$G_{k,n}(t) = \sum_{a \vdash n} \frac{n}{|a|} \binom{|a|}{a_{1,\dots,a_k}} t_1^{a_1} \dots t_k^{a_k}.$$

The notations $a \vdash n$ and $|a|$ are used instead of $\sum_{j=1}^{k} ja_j = n$ and $\sum_{j=1}^{k} a_j$, respectively.

In addition, in [2–6], the authors studied algebraic properties of these polynomials.

On the other hand, there exists several different q-analogues of the Fibonacci-type sequences, see [7–17]. For example, Cigler [14] defined q-Fibonacci polynomials ($F_{n,q}(x, s)$) and q-Lucas polynomials ($L_{n,q}(x, s)$) as follows:

$$\begin{aligned} F_n(x, s) &= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{\binom{k+1}{2}} x^{n-1-2k} s^k, \\ L_n(x, s) &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{[n]}{[n-i]} \begin{bmatrix} n-i \\ i \end{bmatrix} q^{\binom{i}{2}} x^{n-2i} s^i. \end{aligned}$$

The q-binomial coefficient is defined as:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q; q)_n}{(q; q)_k (q; q_{n-k})} = \frac{[n]_q!}{[k]_q! [n-k]_q!},$$

with $(a; q)_n := \prod_{j=0}^{n-1} (1 - aq^j)$, $[n]_q = 1 + q + \cdots + q^{n-1}$ and $[n]_q! = [1]_q [2]_q \cdots [n]_q$.

Many researchers have studied matrices whose elements are binomial coefficients, Fibonacci-type sequences and q-binomial coefficients, see [18–25]. Lee et al. [21] defined $n \times n$ k-Fibonacci matrix $\mathcal{F}(k)_n = [f(k)_{i,j}]_{i,j=1,2,\dots,n}$ as:

$$f(k)_{i,j} = \begin{cases} f_{k,i-j+k-1}, & \text{if } i - j + 1 \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $f_{k,n}$ nth k-Fibonacci numbers defined by Miles in [26]. The $\mathcal{F}(k)_n^{-1} = [f'_{i,j}]$ was given as follows:

$$f'_{i,j} = \begin{cases} 1, & \text{if } i = j \\ -1, & \text{if } i - k \leq j \leq i - 1 \\ 0, & \text{otherwise.} \end{cases}$$

Lee et al. [21] also defined $n \times n$ k-symmetric Fibonacci matrix $Q(k)_n = [q(k)_{i,j}]$ as follows:

$$q(k)_{i,j} = q(k)_{j,i} = \begin{cases} \sum_{l=1}^{k} q(k)_{i,i-l} + f_{k,k-1}, & \text{if } i = j, \\ \sum_{l=1}^{k} q(k)_{i,j-l}, & \text{if } i + 1 \leq j, \end{cases}$$

and obtained the Cholesky factorization of $Q(k)_n$ as follows:

$$Q(k)_n = \mathcal{F}(k)_n (\mathcal{F}(k)_n)^T.$$

In addition, many researchers have studied determinantal and permanental representations of Fibonacci-type sequences and polynomials. More examples can be found in [3, 27–37].

2. q-analogue of the Generalized Fibonacci Polynomials and Generalized Lucas Polynomials

In this section, we define two families of polynomials, the q-generalized Fibonacci polynomial and the q-generalized Lucas polynomial using q-binomials and obtain properties of these polynomials. In the following two definitions, the summation takes place over all integers $c_1, c_2, ..., c_k$ such that $\sum_{j=1}^{k} jc_j = n$, and $c = \sum_{j=1}^{k} c_j$.

Definition 1. *For any integers $n \geq 0$, the q-generalized Fibonacci polynomial, $F_{k,n}(t;q)$, is defined by*

$$F_{k,n}(t;q) := \sum \frac{[c]_q!}{[c_1]_q![c_2]_q!...[c_k]_q!} t_1^{c_1} \dots t_k^{c_k}.$$

In particular, if we take $q = 1$, we obtain the $F_{k,n}(t;1) := F_{k,n}(t)$.

The first few $F_{k,n}(t;q)$ are

$$1,\ t_1,\ t_2 + t_1^2,\ t_3 + (1+q)t_1t_2 + t_1^3,\ t_4 + (1+q)t_1t_3 + t_2^2 + t_1^4 + (1+q+q^2)t_1^2t_2, \dots .$$

Definition 2. *For any integers $n \geq 0$, the q-generalized Lucas polynomial, $L_{k,n}(t;q)$, is defined by*

$$L_{k,n}(t;q) := \sum \frac{[n]_q([c]_q!)}{[c]_q([c_1]_q![c_2]_q!...[c_k]_q!)} t_1^{c_1} \dots t_k^{c_k}.$$

In particular, if we take $q = 1$, we obtain the $L_{k,n}(t;1) := G_{k,n}(t)$.

The first few $L_{k,n}(t;q)$ are

$$[k]_q, t_1, (1+q)t_2 + t_1^2, (1+q+q^2)t_3 + (1+q+q^2)t_1t_2 + t_1^3, \dots .$$

We need the following definitions and lemmas in our proofs.

Definition 3. *$\mathcal{S}_{k,n}$ is the sequence defined by $\mathcal{S}_{k,0} = 1$, $\mathcal{S}_{k,1} = t_1$ and for $n \geq 2$*

$$\mathcal{S}_{k,n} = t_1\mathcal{S}_{k,n-1} + \sum_{j=1}^{n-1}(-1)^{n-j}F_{k,n-j+1}(t;q)\mathcal{S}_{k,j-1}. \tag{2}$$

The first few terms of $\mathcal{S}_{k,n}$ are

$$1, t_1,\ -t_2,\ t_3 - t_1t_2 + qt_1t_2,\ -t_4 + t_1t_3 - qt_1t_3 + qt_1^2t_2 - q^2t_1^2t_2, \dots .$$

Lemma 1. *Let $n \geq 1$ be an integer. Then*

$$F_{k,n}(t;q) = \sum_{j=1}^{k}(-1)^{j+1}\mathcal{S}_{k,j}F_{k,n-j}(t;q). \tag{3}$$

Proof. This is obvious from Eq. (2). □

Definition 4. *$\mathcal{T}_{k,n}$ is the sequence defined by $\mathcal{T}_{k,0} = 1$, $\mathcal{T}_{k,1} = t_1$ and $n \geq 2$*

$$\mathcal{T}_{k,n} = t_1\mathcal{T}_{k,n-1} + \sum_{j=1}^{n-1}(-1)^{n-j}L_{k,n-j+1}(t;q)\mathcal{T}_{k,j-1}. \tag{4}$$

The first few terms of $\mathcal{T}_{k,n}$ are

$$1, t_1, -t_2 - qt_2, t_3 + qt_3 - t_1t_2 - qt_1t_2 + q^2t_3 + q^2t_1t_2, \dots .$$

Lemma 2. *Let $n \geq 1$ be an integer. Then*

$$L_{k,n}(t;q) = \sum_{j=1}^{k} (-1)^{j+1} \mathcal{T}_{k,j} L_{k,n-j}(t;q).$$

Proof. This is obvious from Eq. (4). □

Theorem 1. *Let $F_{k,n}(t_1, t_2, \dots, t_k)$ be the generalized Fibonacci polynomial and $F_{k,n}(t;q)$ be the q-generalized Fibonacci polynomial, then*

$$F_{k,n}(t_1, t_2, \dots, t_k; q) = F_{k,n}(\mathcal{S}_{k,1}, -\mathcal{S}_{k,2}, \dots, (-1)^{k+1}\mathcal{S}_{k,k}).$$

Proof. We proceed by induction on n. The result clearly holds for $n = 1$, since $F_{k,1}(t;q) = t_1 = \mathcal{S}_{k,1} = F_{k,1}(\mathcal{S}_{k,1}, -\mathcal{S}_{k,2}, \dots, (-1)^{k+1}\mathcal{S}_{k,k})$. Now suppose that the result is true for all positive integers less than or equal to n. We prove it for $(n+1)$. In fact, by the definition of generalized Fibonacci polynomials for the vector $\mathcal{S} = (\mathcal{S}_{k,1}, -\mathcal{S}_{k,2}, \dots, (-1)^{k+1}\mathcal{S}_{k,k})$, we have

$$F_{k,n+1}(\mathcal{S}) = \mathcal{S}_{k,1} F_{k,n}(\mathcal{S}) + \cdots + (-1)^{k+1}\mathcal{S}_{k,k} F_{k,n-k+1}(\mathcal{S}).$$

From the hypothesis of induction, we obtain

$$F_{k,n+1}(\mathcal{S}) = \mathcal{S}_{k,1} F_{k,n}(t;q) + \cdots + (-1)^{k+1}\mathcal{S}_{k,k} F_{k,n-k+1}(t;q).$$

Thus, we obtain

$$F_{k,n+1}(\mathcal{S}) = F_{k,n+1}(t;q),$$

using Eq. (3). □

Theorem 2. *Let $F_{k,n}(t_1, t_2, \dots, t_k)$ be the generalized Fibonacci polynomial and $L_{k,n}(t;q)$ be the q-generalized Lucas polynomial, then*

$$L_{k,n}(t_1, t_2, \dots, t_k; q) = F_{k,n}(\mathcal{T}_{k,1}, -\mathcal{T}_{k,2}, \dots, (-1)^{k+1}\mathcal{T}_{k,k}).$$

Proof. The proof runs like in Theorem 1. □

Corollary 1.

$$F_{k,n}(t;q) := \sum \frac{c!}{c_1! \dots c_k!} \mathcal{S}_{k,1}^{c_1} (-\mathcal{S}_{k,2}^{c_2}) \dots ((-1)^k \mathcal{S}_{k,k-1}^{c_{k-1}})((-1)^{k+1}\mathcal{S}_{k,k}^{c_k}).$$

Proof. This is obvious from Eq. (1) and Theorem 1. □

Corollary 2.

$$L_{k,n}(t;q) := \sum \frac{c!}{c_1! \dots c_k!} \mathcal{T}_{k,1}^{c_1} (-\mathcal{T}_{k,2}^{c_2}) \dots ((-1)^k \mathcal{T}_{k,k-1}^{c_{k-1}})((-1)^{k+1}\mathcal{T}_{k,k}^{c_k}).$$

Proof. This is obvious from Eq. (1) and Theorem 2. □

Corollary 3. *Let $y_{k,n} = (-1)^{n+1}\mathcal{T}_{k,n}$ for $q = 1$, $F_{k,n}(t_1, t_2, \dots, t_k)$ be the generalized Fibonacci polynomial and $G_{k,n}(t_1, t_2, \dots, t_k)$ be the generalized Lucas polynomial, then*

$$G_{k,n}(t_1, t_2, \dots, t_k) = F_{k,n}(y_{k,1}, y_{k,2}, \dots, y_{k,k}).$$

Proof. If we rewrite Theorem 2. for $q = 1$, we obtain

$$L_{k,n}(t;1) = F_{k,n}(y_{k,1}, \dots, y_{k,k}).$$

Further, this is obvious from the definitions of $L_{k,n}(t;q)$ and $G_{k,n}(t)$ that $L_{k,n}(t;1) = G_{k,n}(t)$. Therefore, we obtain the desired result. □

There have been several studies on the generalized Fibonacci polynomials and the generalized Lucas polynomials and the relationship between them. Corollary 3. gives a new relation between these polynomials. Using this corollary, different results can be obtained. From Propositions 1, 3, 4 and Lemma 3 in [5], we can give the following corollaries.

Corollary 4. *Let* $y_{k,n} = (-1)^{n+1}\mathcal{T}_{k,n}$ *for* $q = 1$ *and* $F_{k,n}(t_1, t_2, \ldots, t_k)$ *be the generalized Fibonacci polynomial, then*

$$F_{k,n}(y_{k,1}, y_{k,2}, \ldots, y_{k,k}) = -t_{k-1}F_{k,n-k+1}(t) - \cdots - (k-1)t_1F_{k,n-1}(t) + kF_{k,n}(t).$$

Corollary 5. *Let* $y_{k,n} = (-1)^{n+1}\mathcal{T}_{k,n}$ *for* $q = 1$ *and* $F_{k,n}(t_1, t_2, \ldots, t_k)$ *be the generalized Fibonacci polynomial, then*

$$F_{k,n}(y_{k,1}, y_{k,2}, \ldots, y_{k,k}) = \sum_{a \vdash n} n(-1)^{a+1}\binom{|a|-1}{a_{1,\ldots,}a_k}F_{k,1}^{a_1}\ldots F_{k,k}^{a_k}.$$

Corollary 6. *Let* $y_{k,n} = (-1)^{n+1}\mathcal{T}_{k,n}$ *for* $q = 1$ *and* $F_{k,n}(t_1, t_2, \ldots, t_k)$ *be the generalized Fibonacci polynomial, then*

$$\frac{\partial F_{k,n}(y_{k,1}, y_{k,2}, \ldots, y_{k,k})}{\partial t_i} = nF_{k,n-i}(t_1, t_2, \ldots, t_k).$$

Example 1. *We give a companion matrix* $C_{(k)}$ *and obtain* $C_{(k)}^{\infty}$ *using the method in [5] as follows:*

$$C_{(k)} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ (-1)^{k+1}\mathcal{S}_k & (-1)^k\mathcal{S}_{k-1} & (-1)^{k-1}\mathcal{S}_{k-2} & \ldots & \mathcal{S}_1 \end{bmatrix}.$$

In particular, if we take $q = 1$*, we obtain the companion matrix A in [5]. We let the companion matrix operate on its last row vector on the right and append the image vector to the companion matrix as a new last row. We repeat this process, obtaining a matrix with infinitely many rows. Note that* $C_{(k)}$ *is invertible if and only if* $\mathcal{S}_k \neq 0$*. Assuming that* $\mathcal{S}_k \neq 0$*, we can also extend the matrix from the top row upward by operating on the top row with* $C_{(k)}^{-1}$*, obtaining a doubly infinite matrix, that is, one with infinitely many rows in either direction and k-columns. We call this the infinite companion matrix* $C_{(k)}^{\infty}$*. If we take* $q = 1$*, we obtain the companion matrix* A^{∞} *in [5]. It is obvious that, the right hand column of* $C_{(k)}^{\infty}$ *in the positive direction is* $F_{k,n}(t; q)$*.*

3. q-analogue of the Generalized Fibonacci and Lucas Matrices

In this section, we introduce the generalized q-Fibonacci matrix and generalized q-Lucas matrix, then we find their inverse matrices and give the Cholesky factorization of (k, q, t)-symmetric generalized Fibonacci and Lucas matrices. These results generalize the k-Fibonacci matrix and its inverse and k-symmetric Fibonacci matrix in [21]($q = t_i = 1$).

Definition 5. *The generalized q-Fibonacci matrix* $\mathcal{F}_{k,n}^{(t;q)} := [f_{i,j}]_{0 \le i,j \le n}$ *is defined by*

$$f_{i,j} = \begin{cases} F_{k,i-j}(t; q), & \text{if } i \geq j, \\ 0, & \text{otherwise.} \end{cases}$$

Example 2. *The generalized q-Fibonacci matrix for $n = k = 4$ is*

$$\mathcal{F}_{4,4}^{(t;q)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ t_1 & 1 & 0 & 0 & 0 \\ t_2 + t_1^2 & t_1 & 1 & 0 & 0 \\ t_3 + (1+q)t_1t_2 + t_1^3 & t_2 + t_1^2 & t_1 & 1 & 0 \\ \begin{matrix}(t_4 + (1+q)t_1t_3 + t_2^2 + t_1^4 \\ +(1+q+q^2)t_1^2t_2)\end{matrix} & t_3 + (1+q)t_1t_2 + t_1^3 & t_2 + t_1^2 & t_1 & 1 \end{pmatrix}.$$

In particular, if we take $q = t_i = 1$, we obtain the equation $\mathcal{F}_{k,n}^{(1;1)} = \mathcal{F}(k)_n$.

Definition 6. *The $n \times n$ Hessenberg matrix $\mathcal{D}_{k,n} := [d_{i,j}]_{1 \le i,j \le n}$ is defined by*

$$d_{i,j} = \begin{cases} F_{k,i-j+1}(t;q), & \text{if } i \ge j, \\ 1, & \text{if } i+1 = j, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 3. [28] *Let A_n be an $n \times n$ lower Hessenberg matrix for all $n \ge 1$ and define* $\det(A_0) = 1$. *Then,* $\det(A_1) = a_{11}$ *and for $n \ge 2$*

$$\det(A_n) = a_{n,n}\det(A_{n-1}) + \sum_{r=1}^{n-1}[(-1)^{n-r}a_{n,r}(\prod_{j=r}^{n-1} a_{j,j+1})\det(A_{r-1})].$$

Lemma 4. *Let $n \ge 1$ be an integer. Then* $\det(\mathcal{D}_{k,n}) = \mathcal{S}_{k,n}$.

Proof. We proceed by induction on n. The result clearly holds for $n = 1$. Now, suppose that the result is true for all positive integers less than or equal to n. We prove it for $(n+1)$. In fact, using Lemma 3 we have

$$\begin{aligned} \det(\mathcal{D}_{k,n+1}) &= d_{n+1,n+1}\det(\mathcal{D}_{k,n}) + \sum_{i=1}^{n}\left[(-1)^{n+1-i}d_{n+1,i}\prod_{j=i}^{n} d_{j,j+1}\det(\mathcal{D}_{k,i-1})\right] \\ &= t_1\det(\mathcal{D}_{k,n}) + \sum_{i=1}^{n}\left[(-1)^{n+1-i}F_{k,n-i+2}(t;q)\det(\mathcal{D}_{k,i-1})\right] \\ &= t_1\mathcal{S}_{k,n} + \sum_{i=1}^{n}\left[(-1)^{n+1-i}F_{k,n-i+2}(t;q)\mathcal{S}_{k,i-1}\right] = \mathcal{S}_{k,n+1}. \end{aligned}$$

□

Theorem 3. *Let $\mathcal{F}_{k,n}^{(t;q)}$ be the $(n+1)\times(n+1)$ lower triangular generalized q-Fibonacci matrix. Then, we have*

$$(\mathcal{F}_{k,n}^{(t;q)})^{-1} = [b_{i,j}] = \begin{cases} (-1)^{i-j}\mathcal{S}_{k,i-j}, & \text{if } i-j \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Note that it suffices to prove that $\mathcal{F}_{k,n}^{(t;q)}(\mathcal{F}_{k,n}^{(t;q)})^{-1} = I_{n+1}$. For $i > j \ge 0$, we have

$$\begin{aligned} \sum_{k=0}^{n} f_{i,k}b_{k,j} &= \sum_{k=j}^{i} f_{i,k}b_{k,j} \\ &= F_{k,i-j}(t;q)\mathcal{S}_{k,0} - F_{k,i-j-1}(t;q)\mathcal{S}_{k,1} + \cdots + F_{k,0}(t;q)(-1)^{i-j}\mathcal{S}_{k,i-j} \end{aligned}$$

and we know

$$F_{k,i-j}(t;q) = \sum_{s=1}^{k}(-1)^{j+1}\mathcal{S}_{k,s}F_{k,i-j-s}(t;q)$$

from Eq. (2). Therefore, we obtain $\sum_{k=0}^{n} f_{i,k} b_{k,j} = 0$ for $i > j \geq 0$. It is obvious that $\sum_{k=0}^{n} f_{i,k} b_{k,j} = 0$ for $i - j < 0$ and $\sum_{k=0}^{n} f_{i,k} b_{k,j} = f_{i,i} b_{i,j} = \mathcal{S}_{k,0} F_{k,0}(t;q) = 1$ for $i = j$ which implies that $\mathcal{F}_{k,n}^{(t;q)} (\mathcal{F}_{k,n}^{(t;q)})^{-1} = I_{n+1}$, as desired. □

In particular, if we take $q = t_i = 1$, we obtain the inverse matrix of the k-Fibonacci matrix [21].

Definition 7. *The generalized q-Lucas matrix* $\mathcal{L}_{k,n}^{(t;q)} := [l_{i,j}]_{0 \leq i,j \leq n}$ *is defined by*

$$l_{i,j} = \begin{cases} L_{k,i-j}(t;q), & \text{if } i > j, \\ 1 & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

For example,

$$\mathcal{L}_{3,3}^{(t;q)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_1 & 1 & 0 & 0 \\ (1+q)t_2 + t_1^2 & t_1 & 1 & 0 \\ (1+q+q^2)t_3 + (1+q+q^2)t_1 t_2 + t_1^3 & (1+q)t_2 + t_1^2 & t_1 & 1 \end{pmatrix}.$$

Definition 8. *The* $n \times n$ *Hessenberg matrix* $\mathcal{E}_{k,n} := [e_{i,j}]_{1 \leq i,j \leq n}$ *is defined by*

$$e_{i,j} = \begin{cases} L_{k,i-j+1}(t;q), & \text{if } i \geq j, \\ 1, & \text{if } i+1 = j, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 5. *Let* $n \geq 1$ *be an integer. Then*

$$\det(\mathcal{E}_{k,n}) = \mathcal{T}_{k,n}.$$

Proof. The proof runs like in Lemma 4. □

Theorem 4. *Let* $\mathcal{L}_{k,n}^{(t;q)}$ *be the* $(n+1) \times (n+1)$ *lower triangular generalized q-Lucas matrix. Then, we have*

$$(\mathcal{L}_{k,n}^{(t;q)})^{-1} = [c_{i,j}] = \begin{cases} (-1)^{i-j} \mathcal{T}_{k,i-j}, & \text{if } i - j > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The proof runs like in Theorem 3. □

Definition 9. *Let* $n \geq 1$ *be an integer,* (k,q,t)*-symmetric generalized Fibonacci matrix* $Q_{k,n}^{(t,q)} := [m_{i,j}]_{0 \leq i,j \leq n}$ *is defined by*

$$m_{i,j} = \begin{cases} \sum_{l=0}^{i} F_{k,l}(t;q) F_{k,l}(t;q), & \text{if } i = j, \\ \sum_{l=0}^{i} F_{k,i-l}(t;q) F_{k,j-l}(t;q), & \text{if } i+1 \leq j. \end{cases}$$

Theorem 5. *The Cholesky factorization of* $Q_{k,n}^{(t,q)}$ *is given by*

$$Q_{k,n}^{(t,q)} = \mathcal{F}_{k,n}^{(t;q)} (\mathcal{F}_{k,n}^{(t;q)})^T$$

Proof. Note that it suffices to prove that $(\mathcal{F}_{k,n}^{(t;q)})^{-1}\mathcal{Q}_{k,n}^{(t,q)} = (\mathcal{F}_{k,n}^{(t;q)})^{T}$. We take $\mathcal{F}_{k,n}^{(t;q)})^{-1} = \left[b_{i,j}\right]$, $\mathcal{Q}_{k,n}^{(t,q)} = \left[m_{i,j}\right]$ and $(\mathcal{F}_{k,n}^{(t;q)})^{T} = \left[\overline{f}_{i,j}\right]$ and obtain $\sum_{s=1}^{k} b_{i,s}m_{s,j}$ for $i, j = 0, 1, 2, ..., k$. For $i = j = n$,

$$\begin{aligned}
\overline{f}_{i,i} &= \sum_{s=1}^{k} b_{i,s}m_{s,i} \\
&= (-1)^n\mathcal{S}_{k,n}F_{k,n}(t;q) + (-1)^{n-1}\mathcal{S}_{k,n-1}[F_{k,1}(t;q)F_{k,n}(t;q) + F_{k,0}(t;q)F_{k,n-1}(t;q)] + \cdots \\
&\quad -\mathcal{S}_{k,1}[F_{k,n-1}(t;q)F_{k,n}(t;q) + \cdots + F_{k,0}(t;q)F_{k,1}(t;q)] \\
&\quad +\mathcal{S}_{k,0}[F_{k,n}(t;q)F_{k,n}(t;q) + \cdots + F_{k,0}(t;q)F_{k,0}(t;q)] \\
&= F_{k,n}(t;q)[(-1)^n\mathcal{S}_{k,n} + (-1)^{n-1}\mathcal{S}_{k,n-1}F_{k,1}(t;q) + \cdots + \mathcal{S}_{k,0}F_{k,n}(t;q)] \\
&\quad +F_{k,n-1}(t;q)[(-1)^{n-1}\mathcal{S}_{k,n-1} + \cdots + \mathcal{S}_{k,0}F_{k,n-1}(t;q)] + \cdots \\
&\quad +F_{k,1}(t;q)[-\mathcal{S}_{k,1} + \mathcal{S}_{k,0}F_{k,1}(t;q)] + 1.
\end{aligned}$$

We know $(-1)^n\mathcal{S}_{k,n} + (-1)^{n-1}\mathcal{S}_{k,n-1}F_{k,1}(t;q) + \cdots + \mathcal{S}_{k,0}F_{k,n}(t;q) = 0$ from the definition of $\mathcal{S}_{k,n}$, so $\overline{f}_{i,i} = 1$ for $i = 0, 1, 2, ..., k$. For $i > j$,

$$\begin{aligned}
\sum_{s=1}^{k} b_{i,s}m_{s,i} &= (-1)^i\mathcal{S}_{k,i}F_{k,j}(t;q) + (-1)^{i-1}\mathcal{S}_{k,i-1}[F_{k,1}(t;q)F_{k,j}(t;q) + F_{k,0}(t;q)F_{k,j-1}(t;q)] \\
&\quad +(-1)^{i-2}\mathcal{S}_{k,i-2}[F_{k,2}(t;q)F_{k,j}(t;q) + \cdots + F_{k,0}(t;q)F_{k,j-2}(t;q)] \\
&\quad + \cdots + \mathcal{S}_{k,0}[F_{k,i}(t;q)F_{k,j}(t;q) + \cdots + F_{k,i-j}(t;q)F_{k,0}(t;q)] \\
&= F_{k,j}(t;q)[(-1)^i\mathcal{S}_{k,i} + (-1)^{i-1}\mathcal{S}_{k,i-1}F_{k,1}(t;q) + \cdots + \mathcal{S}_{k,0}F_{k,i}(t;q)] \\
&\quad +F_{k,j-1}(t;q)[(-1)^{i-1}\mathcal{S}_{k,i-1} + (-1)^{i-2}\mathcal{S}_{k,i-2}F_{k,1}(t;q) + \cdots + \mathcal{S}_{k,0}F_{k,i-1}(t;q)] + \cdots \\
&\quad +F_{k,0}(t;q)[(-1)^{i-j}\mathcal{S}_{k,i-j} + (-1)^{i-j-1}\mathcal{S}_{k,i-j-1}F_{k,1}(t;q) + \cdots + \mathcal{S}_{k,0}F_{k,i-j}(t;q)].
\end{aligned}$$

We know $(-1)^n\mathcal{S}_{k,n} + (-1)^{n-1}\mathcal{S}_{k,n-1}F_{k,1}(t;q) + \cdots + \mathcal{S}_{k,0}F_{k,n}(t;q) = 0$ from the definition of $\mathcal{S}_{k,n}$, so $\overline{f}_{i,i} = 0$ for $i = 0, 1, 2, ..., k$. Finally, for $i < j$, equation $\overline{f}_{i,i} = F_{k,j-i}(t;q)$ is shown in a similar way. □

Definition 10. *Let $n \geq 1$ be an integer. Then (k, q, t)-symmetric generalized Lucas matrix $\mathcal{P}_{k,n}^{(t,q)} := [n_{i,j}]_{0\leq i,j\leq n}$ is defined by*

$$n_{i,j} = \begin{cases} \sum_{l=0}^{i} L_{k,l}(t;q)L_{k,l}(t;q), & \text{if } i = j, \\ \sum_{l=0}^{i} L_{k,i-l}(t;q)L_{k,j-l}(t;q), & \text{if } i+1 \leq j. \end{cases}$$

Theorem 6. *The Cholesky factorization of $\mathcal{P}_{k,n}^{(t,q)}$ is given by*

$$\mathcal{P}_{k,n}^{(t,q)} = \mathcal{L}_{k,n}^{(t;q)}(\mathcal{L}_{k,n}^{(t;q)})^T$$

Proof. The proof runs like in Theorem 5. □

4. The Determinantal and Permanental Representations

In this section, we obtain any term of q-generalized Fibonacci and Lucas polynomials using determinants and permanents of Hessenberg matrices.

Theorem 7. *Let $n \geq 1$ be an integer, $F_{k,n}(t;q)$ be the nth q-generalized Fibonacci polynomial and ${}_{-}U_{k,n}^{(q)} = [u_{i,j}]_{i,j=1,2,...,n}$ be the $n \times n$ Hessenberg matrix defined as*

$$u_{ij} = \begin{cases} (-1)^{i-j}\mathcal{S}_{k,i-j+1}, & \text{if } i-j \geq 0, \\ -1, & \text{if } i+1 = j, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\det({}_{-}U^{(q)}_{k,n}) = F_{k,n}(t;q)$.

Proof. We proceed by induction on m. The result clearly holds for $m = 1$, since $\det({}_{-}U^{(q)}_{k,1}) = \mathcal{S}_{k,1} = t_1 = F_{k,1}(t;q)$. Now, suppose that the result is true for all positive integers less than or equal to m. We prove it for $(m+1)$. In fact, using Lemma 3 we have

$$\begin{aligned}
\det({}_{-}U^{(q)}_{k,m+1}) &= u_{m+1,m+1}\det({}_{-}U^{(q)}_{k,m}) + \sum_{i=1}^{m}\left[(-1)^{m+1-i}u_{m+1,i}\prod_{j=i}^{m}u_{j,j+1}\det({}_{-}U^{(q)}_{k,i-1})\right]\\
&= t_1\det({}_{-}U^{(q)}_{k,m}) + \sum_{i=1}^{m-k+1}\left[(-1)^{m+1-i}u_{m+1,i}\prod_{j=i}^{m}u_{j,j+1}\det({}_{-}U^{(q)}_{k,i-1})\right]\\
&\quad + \sum_{i=m-k+2}^{m}\left[(-1)^{m+1-i}u_{m+1,i}\prod_{j=i}^{m}u_{j,j+1}\det({}_{-}U^{(q)}_{k,i-1})\right]\\
&= t_1\det({}_{-}U^{(q)}_{k,m}) + \sum_{i=m-k+2}^{m}\left[u_{m+1,i}\det({}_{-}U^{(q)}_{k,i-1})\right]\\
&= t_1\det({}_{-}U^{(q)}_{k,m}) - \mathcal{S}_{k,2}\det({}_{-}U^{(q)}_{k,m-1}) + \cdots + (-1)^{k-1}\mathcal{S}_k\det({}_{-}U^{(q)}_{k,m-k+1}).
\end{aligned}$$

From the hypothesis of induction and Eq. (3), we obtain

$$\det({}_{-}U^{(q)}_{k,m+1}) = \sum_{j=1}^{k}(-1)^{j-1}\mathcal{S}_{k,j}F_{k,m+1-j}(t;q) = F_{k,m+1}(t;q).$$

Therefore, $\det({}_{-}U^{(q)}_{k,n}) = F_{k,n}(t;q)$ holds for all positive integers n. □

Theorem 8. *Let $n \geq 1$ be an integer, $F_{k,n}(t;q)$ be the nth q-generalized Fibonacci polynomial and ${}_{+}V^{(q)}_{k,n} = [v_{s,t}]_{i,j=1,2,\ldots,n}$ be the $n \times n$ Hessenberg matrix defined as*

$$v_{s,t} = \begin{cases} i^{3(s-t)}\mathcal{S}_{k,i-j+1}, & \text{if } s-t \geq 0,\\ i, & \text{if } s+1 = t,\\ 0, & \text{otherwise.}\end{cases}$$

Then, $\det({}_{+}V^{(q)}_{k,n}) = F_{k,n}(t;q)$.

Proof. Since the proof is similar to the proof of Theorem 7 using Lemma 3, we omit the details. □

Theorem 9. *Let $n \geq 1$ be an integer, $L_{k,n}(t;q)$ be the nth q-generalized Lucas polynomial, ${}_{-}W^{(q)}_{k,n} = [w_{i,j}]_{i,j=1,2,\ldots,n}$ and ${}_{+}Y^{(q)}_{k,n} = [y_{i,j}]_{i,j=1,2,\ldots,n}$ be the $n \times n$ Hessenberg matrices defined as*

$$w_{i,j} = \begin{cases} (-1)^{i-j}\mathcal{T}_{k,i-j+1}, & \text{if } i-j \geq 0,\\ -1, & \text{if } i+1 = j,\\ 0, & \text{otherwise,}\end{cases} \quad \text{and} \quad y_{s,t} = \begin{cases} i^{3(s-t)}\mathcal{T}_{k,i-j+1}, & \text{if } s-t \geq 0,\\ i, & \text{if } s+1 = t,\\ 0, & \text{otherwise.}\end{cases}$$

Then, $\det({}_{-}W^{(q)}_{k,n}) = \det({}_{+}Y^{(q)}_{k,n}) = L_{k,n}(t;q)$.

Proof. Since the proof is similar to the proof of Theorem 7 using Lemma 3, we omit the details. □

The permanent of an n-square matrix is defined by $\text{per}A = \sum_{\sigma\in S_n}\prod_{i=1}^{n}a_{i\sigma(i)}$, where the summation extends over all permutations σ of the symmetric group S_n (cf. [38]). There is a relation between permanent and determinant of a Hessenberg matrix (cf. [33, 39]). Then, it is clear that the following corollaries hold.

Corollary 7. *Let $n \geq 1$ be an integer, $F_{k,n}(t;q)$ be the nth q-generalized Fibonacci polynomial, ${}_{+}U_{k,n}^{(q)} = [\overline{u}_{i,j}]_{i,j=1,2,\ldots,n}$ and ${}_{-}V_{k,n}^{(q)} = [\overline{v}_{s,t}]_{s,t=1,2,\ldots,n}$ be the $n \times n$ Hessenberg matrix defined as*

$$\overline{u}_{ij} = \begin{cases} (-1)^{i-j}\mathcal{S}_{k,i-j+1}, & \text{if } i-j \geq 0, \\ 1, & \text{if } i+1=j, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \overline{v}_{s,t} = \begin{cases} i^{3(s-t)}\mathcal{S}_{k,i-j+1}, & \text{if } s-t \geq 0, \\ -i, & \text{if } s+1=t, \\ 0, & \text{otherwise}, \end{cases}$$

where $i = \sqrt{-1}$. Then, $per({}_{+}U_{k,n}^{(q)}) = per({}_{-}V_{k,n}^{(q)}) = F_{k,n}(t;q)$.

Corollary 8. *Let $n \geq 1$ be an integer, $L_{k,n}(t;q)$ be the nth q-generalized Lucas polynomial, ${}_{+}W_{k,n}^{(q)} = [\overline{w}_{i,j}]_{i,j=1,2,\ldots,n}$ and ${}_{-}Y_{k,n}^{(q)} = [\overline{y}_{s,t}]_{i,j=1,2,\ldots,n}$ be the $n \times n$ Hessenberg matrices defined as*

$$\overline{w}_{i,j} = \begin{cases} (-1)^{i-j}\mathcal{T}_{k,i-j+1}, & \text{if } i-j \geq 0, \\ 1, & \text{if } i+1=j, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \overline{y}_{s,t} = \begin{cases} i^{3(s-t)}\mathcal{T}_{k,i-j+1}, & \text{if } s-t \geq 0, \\ -i, & \text{if } s+1=t, \\ 0, & \text{otherwise}. \end{cases}$$

Then, $per({}_{+}W_{k,n}^{(q)}) = per({}_{-}Y_{k,n}^{(q)}) = L_{k,n}(t;q)$.

Conflict of Interest

The author declares that they have no conflicts of interest.

References

1. MacHenry, T., 1999. A Subgroup of the Group of Units in the Ring of Arithmetic Functions. *Rocky Mountain Journal of Mathematics, 29*(3), pp.1055-1065.
2. MacHenry, T. and Wong, K., 2012. A Correspondence Between the Isobaric Ring and Multiplicative Arithmetic Functions. *Rocky Mountain Journal of Mathematics, 42*(4), pp.1247–1290.
3. Li, H. and MacHenry, T., 2013. Permanents and Determinants, Weighted Isobaric Polynomials, and Integer Sequences. *Journal of Integer Sequences, 16*, Article 13.3.5.
4. MacHenry, T. and Tudose, G., 2005. Reflections on Symmetric Polynomials and Arithmetic Functions. *Rocky Mountain Journal of Mathematics, 35*(3), pp.901-926.
5. MacHenry, T. and Li, H., 2013. The Convolution Ring of Arithmetic Functions and Symmetric Polynomials. *Rocky Mountain Journal of Mathematics, 43*(4), pp.1-33.
6. MacHenry, T. and Wong, K., 2011. Degree k Linear Recursions mod(p) and Number Fields. *Rocky Mountain Journal of Mathematics, 41*(4), pp.1303-1327.
7. Al-Salam, W.A., 1959. q-Bernoulli numbers and polynomials. *Mathematische Nachrichten, 17*, pp.239–260.
8. Andrews, G.E.,2004. Fibonacci numbers and Rogers-Ramanujan identities. *Fibonacci Quart., 42*(1), pp.3-19.
9. Belbachir, H. and Benmezai, A., 2014. An alternative approach to Cigler's q-Lucas polynomials. *Applied Mathematics and Computation, 226*, pp.691–698.
10. Belbachir, H. and Benmezai, A., 2014. q analogue for binomial coefficients and generalized Fibonacci sequence. *Comptes Rendus de l'Académie des Sciences - Series I, 352*(3), pp.167–171
11. Carlitz, L., 1974. Fibonacci Notes-3: q-Fibonacci Numbers. *Fibonacci Quart., 12*(7), pp.317-322.
12. Carlitz, L., 1975. Fibonacci notes 4. *Fibonacci Quart., 12*, pp.97-102.

13. Cigler, J., 2003. q-Fibonacci polynomials. *Fibonacci Quart., 41*, pp.31-40.
14. Cigler, J., 2002. Einige q-Analoga der Lucas und Fibonacci-Polynome. *Sitzungsber. Abt. II., 211*, pp.3–20.
15. Cigler, J., 2003. Some algebraic aspects of Morse code sequences. *Discrete Mathematics & Theoretical Computer Science, 6*, pp.055–068.
16. Cigler, J., 2003. A new class of q-Fibonacci polynomials. *The Electronic Journal of Combinatorics, 10*, R19.
17. Tauraso, R., 2012. q-Analogs of some congruences involving Catalan numbers. *Advances in Applied Mathematics, 48*, pp.603-614.
18. Brawer, R. and Pirovino, M., 1992. The linear algebra of the Pascal matrix. *Linear Algebra and its Applications , 174*, pp.13–23.
19. Falcon, S., 2011. The k-Fibonacci matrix and the Pascal matrix. *Central European Journal of Mathematics,* 9(6), pp.1403–1410.
20. Lee, G.Y., Kim, J.S. and Lee, S.G., 2002. Factorizations and eigenvalues of Fibonacci and symmetric Fibonacci matrices. *Fibonacci Quart., 40*(3), pp.203–211.
21. Lee, G.Y., Kim, J.S., 2003. The linear algebra of the k-Fibonacci matrix. *Linear Algebra Appl., 373*, pp.75–87.
22. Sahin, A.,2016. On the q-analogue of Fibonacci and Lucas matrices and Fibonacci polynomials. *Utilitas Mathematica, 100*, pp.113–125.
23. Sahin, A., 2020. On (p,q)-Rogers-Szegö matrices. *Adıyaman University Journal of Science, 10*(1), pp.285-294.
24. Infante, G.M., Ramirez, J.L. and Sahin, A., 2017. Some Results on q Analogue of the Bernoulli Euler and Fibonacci Matrices, *Mathematical Reports, 19*(69), pp.399–417.
25. Stanica, P., 2005. Cholesky Factorizations of Matrices Associated with r-Order Recurrent Sequences. *Integers, 5*(2), A16.
26. Miles, E.P., 1960. Generalized Fibonacci Numbers and Associated Matrices. *The American Mathematical Monthly, 67*, pp.745–752.
27. Amdeberhan, T., Chen, X., Moll, V.H. and Sagan, B.E., 2014. Generalized Fibonacci polynomials and Fibonomial coefficients. *Annals of Combinatorics, 18*, pp.541–562.
28. Cahill, N.D., D'Errico, J.R., Narayan, D.A. and Narayan, J.Y., 2002. Fibonacci determinants. *The College Mathematics Journal*, 33, pp.221–225.
29. Cheon, G.-S., Kim, H. and Shapiro, L.W., 2009. A generalization of Lucas polynomial sequence. *Discrete Applied Mathematics, 157*, pp.920–927.
30. Jína, J. and Trojovský, P., 2013. On determinants of some tridiagonal matrices connected with Fibonacci numbers. *International Journal of Pure and Applied Mathematics, 88*(4), pp.569–575.
31. Kaygısız, K. and Şahin, A., 2013. A new method to compute the terms of generalized order-k Fibonacci numbers. *Journal of Number Theory, 133*, pp.3119–3126.
32. Kaygısız, K. and Şahin, A., 2012. Generalized bivariate Lucas p-Polynomials and Hessenberg Matrices. *Journal of Integer Sequences, 15*, Article 12.3.4.
33. Kaygısız, K. and Şahin, A., 2013. Determinant and permanent of Hessenberg matrix and generalized Lucas polynomials. *Bulletin of the Iranian Mathematical Society, 39*(6), pp.1065–1078.
34. Kılıc, E. and Stakhov, A.P., 2009. On the Fibonacci and Lucas p-numbers, their sums, families of bipartite graphs and permanents of certain matrices. *Chaos, Solitons & Fractals, 40*, pp.2210–2221.

35. Kılıc, E. and Tasci, D., 2010. On the generalized Fibonacci and Pell sequences by Hessenberg matrices. *Ars Combinatoria, 94*, pp.161–174.

36. Kılıc, E., Tasci, D. and Haukkanen, P., 2010. On the generalized Lucas sequences by Hessenberg matrices. *Ars Combinatoria, 95*, pp.383–395.

37. Li, H-C., 2012. On Fibonacci-Hessenberg matrices and the Pell and Perrin numbers. *Applied Mathematics and Computation, 218*(17), pp.8353–8358.

38. Minc, H., 1978. *Encyclopedia of Mathematics and its Applications, Permanents, 6*, Addison-Wesley Publishing Company.

39. Gibson, P.M., 1969. An identity between permanents and determinants. *The American Mathematical Monthly, 76*, pp.270–271.