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Article

Sharp Bounds for the General Randić Index of R-Graph Products

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Abstract: A chemical structure specifies the molecular geometry of a given molecule or solid in the form of atom arrangements. A way to analyze its properties is to stimulate its formation as, product of two or more simpler graphs. In this article, we take this idea to find upper and lower bounds for the generalized Randić index $\mathcal{R}_\alpha$ of four types of graph products, by using combinatorial inequalities. We finish this paper providing the bounds for $\mathcal{R}_\alpha$ of a line graph and rooted product of graphs.

Keywords: The general Randić index, R-graphs, corona product, line graph, rooted product
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1. Introduction

In chemical graph theory, we identify chemical structures with graphs known as molecular graphs. For $v \in V_G$, the degree of v in G is $\deg_G(v) = N_G(v)$, where $N_G(v)$ denotes the set of neighbors of v in G. The degree of the vertices play an essential role to describe certain chemical properties of a corresponding structure. Every vertex v in a molecular graph G has degree smaller than or equal to four. We denote by p_G and q_G, the number of vertices and number edges of G, respectively. There are many degree-based indices that we associate with a molecular graph. These indices are numbers having an immense amount of applications in Quantitative Structure and Activity Relationships.

There are two oldest topological indices, the Zagreb indices [1, 2]. These indices have been used and examined to study the molecular complexity, chirality, ZE-isomerism, and hetero-systems. The first Zagreb index is defined as:

$$M_1(G) = \sum_{v\in V_G}(\deg_G(v))^2 = \sum_{uv\in E_G}[\deg_G(u) + \deg_G(v)]. \tag{1}$$

One of the most studied topological index, the Randić index, defined in 1975 by Milan Randić [3]. See also the survey by Li and Shi [4] and the recent paper Dalfó [5]. This index is essential to measure the level of carbon-atom skeleton in saturated hydrocarbons. In chemistry and pharmacology, the Randić index has significant applications. In particular, it describes boiling points, chromatographic retention times, and several other properties of alkanes.

For a graph G, the general Randić index $\mathcal{R}_\alpha(G)$ is defined as:

$$\mathcal{R}_\alpha(G) = \sum_{uv\in E(G)} (\deg_G(u)\deg_G(v))^\alpha, \tag{2}$$

where α is an arbitrary real number. In [3], Randić proposed both $\alpha = -1$ and $\alpha = -1/2$. In [6], Bollobás and Erdős generalized it by replacing $-1/2$ with any real number α. In 2017, the mathematical background of this index was improved. In [7], the hitherto unnoticed features of the Randić index were pointed out. Due to its innumerable applications, this index is enormously studied [8].

An effective way of better understanding a given chemical graph is to stimulate its formation as a product of two simpler graphs. Many graph operations such as, the Cartesian product, join of graphs, corona product, the edge corona product, the subdivision-vertex join, the subdivision edge join, the neighborhood corona, the subdivision vertex neighborhood corona, and the subdivision edge neighborhood corona are studied in the literature. For more details, we recommend the reader to see [9–17].

In [18], the corona product of two graphs was defined to address associated group isomorphisms. The corona product of two graphs G and H is defined as the graph $G \circ H$ obtained by having p_G copies $H_1, H_2, \ldots, H_{p_G}$ of H, and then joining by an edge ^{i}th vertex of G to every vertex of H_i. Thus, $p_{G\circ H} = p_G(1 + p_H)$ and $q_{G\circ H} = q_G + p_G(p_H + q_H)$. The degree of a vertex $v \in V_{G\circ H}$ is given by:

$$\deg_{G\circ H}(v) = \begin{cases} \deg_G(v) + p_H & \text{if } v \in V_G, \\ \deg_H(v) + 1 & \text{if } v \in V_H. \end{cases} \tag{3}$$

The main motivation of this paper is to find the minimum and maximum bounds for the general Randić index of different graph products. In [19], such bounds were computed in the case of the general sum-connectivity index. This paper is divided into four sections. After the introductory section, in Section 2, we calculate bounds on the general Randić index of R-graphs, including the R-vertex corona product, R-edge corona product, R-vertex neighborhood corona product, and R-edge neighborhood corona product. In Section 3, we compute the bounds on general Randić index of a line graph L^G of the graph G. In Section 4, we find the lower and upper bounds for the rooted product of graphs.

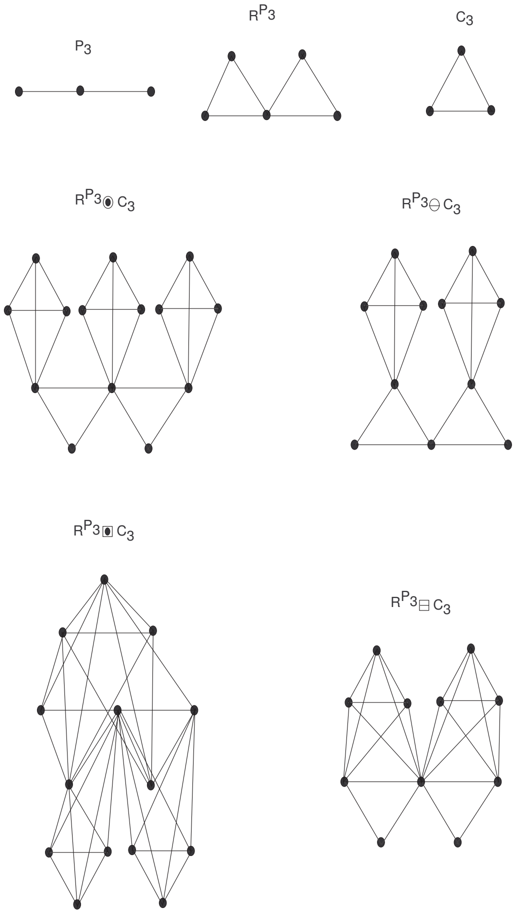

Figure 1. The Graphs $R^{P_3} \odot C_3$, $R^{P_3} \ominus C_3$, $R^{P_3} \boxdot C_3$ and $R^{P_3} \boxminus C_3$

2. The General Randić Index of *R*-graph Products

In this section, we provide results related to the general Randić index of different *R*-graph products. For a graph G, the maximum and minimum degrees of G are defined as $\Delta_G = \max\{\deg_G(v) \mid v \in V_G\}$ and $\delta_G = \min\{\deg_G(v) \mid v \in V_G\}$, respectively. Let R^G be the *R*-graph constructed from G as follows: to each edge e of G we add a new vertex v_e such that it shares one edge each with the end vertices of e. Thus, $p_{R^G} = p_G + q_G$. We denote by I_G the set of the newly added vertices to V_G.

2.1. R-vertex Corona Product of R^G and H

Let G and H be two connected and vertex-disjoint graphs. The *R*-vertex corona product of two graphs R^G and H is defined as the graph $R^G \odot H$ obtained by having p_G copies $H_1, H_2, \ldots, H_{p_G}$ of H and then joining by an edge ^{i}th vertex of G to every vertex of H_i. Thus, $p_{R^G \odot H} = p_G + q_G + p_G p_H$ and $q_{R^G \odot H} = 3q_G + p_G q_H + p_G p_H$. The degree of a vertex $v \in V_{R^G \odot H}$ is given by:

$$\deg_{R^G \odot H}(v) = \begin{cases} 2\deg_G(v) + p_H & \text{if } v \in V_G, \\ 2 & \text{if } v \in I_G, \\ \deg_H(v) + 1 & \text{if } v \in V_H. \end{cases} \tag{4}$$

The general Randić index of an *R*-vertex corona product of R^G and H is given in the following result.

Theorem 1. *Let G and H be two connected and vertex-disjoint graphs. Then, for $\alpha < 0$, the general Randić index of $R^G \odot H$ has the following minimum and maximum bounds*

$$\mathcal{R}_\alpha(R^G \odot H) \;\geq\; q_G(2\triangle_G + p_H)^\alpha[(2\triangle_G + p_H)^\alpha + 2^\alpha] + p_G q_H(\triangle_H + 1)^\alpha[(\triangle_H + 1)^\alpha + (2\triangle_G + p_H)^\alpha],$$

$$\mathcal{R}_\alpha(R^G \odot H) \leq q_G(2\delta_G + p_H)^\alpha[(2\delta_G + p_H)^\alpha + 2^\alpha] + p_G q_H(\delta_H + 1)^\alpha[(\delta_H + 1)^\alpha + (2\delta_G + p_H)^\alpha].$$

The above equalities hold if and only if both G and H are regular graphs.

Proof. Using (4) in Eq. (2), we get

$$\begin{aligned}
\mathcal{R}_\alpha(R^G \odot H) &= \sum_{uv\in E(R^G}(\deg_{R^G}(u)\deg_{R^G}(v))^\alpha + p_G\sum_{uv\in E_H}(\deg_H(u)\deg_H(v))^\alpha + \sum_{u\in V_{R^G}}\sum_{v\in V_H}(\deg_{R^G}(u)\deg_H(v))^\alpha \\
&= \sum_{\substack{uv\in E(R^G),\\ u,v\in V_G}}[(2\deg_G(u) + p_H)(2\deg_H(v) + p_H)]^\alpha + \sum_{\substack{uv\in E(R^G),\\ u\in V_G, v\in I_G}}[2(2\deg_G(u) + p_H)]^\alpha \\
&\quad + p_G\sum_{uv\in E_H}[(\deg_H(u) + 1)(\deg_H(v) + 1)]^\alpha + \sum_{\substack{u\in V_{R^G}\\ u\in V_G}}\sum_{v\in V_H}[(2\deg_G(u) + p_H)(\deg_H(v) + 1)]^\alpha \\
&\geq q_G(2\triangle_G + p_H)^{2\alpha} + 2^\alpha q_G(2\triangle_G + p_H)^\alpha + p_G q_H(\triangle_H + 1)^{2\alpha} + p_G p_H(2\triangle_G + p_H)^\alpha(\triangle_H + 1)^\alpha \\
&= q_G(2\triangle_G + p_H)^\alpha[(2\triangle_G + p_H)^\alpha + 2^\alpha] + p_G q_H(\triangle_H + 1)^\alpha[(\triangle_H + 1)^\alpha + (2\triangle_G + p_H)^\alpha].
\end{aligned} \tag{5}$$

Similarly we have the following inequality.

$$\mathcal{R}_\alpha(R^G \odot H) \leq q_G(2\delta_G + p_H)^\alpha[(2\delta_G + p_H)^\alpha + 2^\alpha] + p_G q_H(\delta_H + 1)^\alpha[(\delta_H + 1)^\alpha + (2\delta_G + p_H)^\alpha]. \tag{6}$$

The above equalities hold if and only if G and H are regular graphs. This completes the proof. □

2.2. R-edge Corona Product of R^G and H

The R-edge corona product of R^G and H is defined as the graph $R^G \ominus H$ obtained by having q_G copies $H_1, H_2, \ldots, H_{q_G}$ of H, and then joining by an edge ^{i}th vertex of I_G to every vertex of H_i. Thus, $p_{R^G\ominus H} = p_G + q_G + q_G p_H$ and $q_{R^G\ominus H} = 3q_G + q_G q_H + q_G p_H$. The degree of a vertex $v \in V_{R^G\ominus H}$ is given by:

$$\deg_{R^G\ominus H}(v) = \begin{cases} 2\deg_G(v) & \text{if } v \in V_G, \\ 2 + p_H & \text{if } v \in I_G, \\ \deg_H(v) + 1 & \text{if } v \in V_H. \end{cases} \tag{7}$$

The general Randć index of R-edge corona product of an R^G and H is given in the following result.

Theorem 2. *Let G and H be two connected and vertex-disjoint graphs. Then, for $\alpha < 0$, the general Randić index of $R^G \ominus H$ has the following minimum and maximum bounds*

$$\mathcal{R}_\alpha(R^G \ominus H) \geq 4^\alpha q_G \triangle_G^{2\alpha} + 2^{\alpha+1} q_G \triangle_G^\alpha (p_H + 2)^\alpha + q_G q_H(\triangle_H + 1)^{2\alpha} + q_G p_H(p_H + 2)^\alpha(\triangle_H + 1)^\alpha,$$
$$\mathcal{R}_\alpha(R^G \ominus H) \leq 4^\alpha q_G \delta_G^{2\alpha} + 2^{\alpha+1} q_G \delta_G^\alpha (p_H + 2)^\alpha + q_G q_H(\delta_H + 1)^{2\alpha} + q_G p_H(p_H + 2)^\alpha(\delta_H + 1)^\alpha.$$

The above equalities hold if and only if both G and H are regular graphs.

Proof. Using (7) in Eq. (2), we get

$$\begin{aligned}
\mathcal{R}_\alpha(R^G \ominus H) &= \sum_{uv\in E_{R^G}}(\deg_{R^G}(u)\deg_{R^G}(v))^\alpha + q_G\sum_{uv\in E_H}(\deg_H(u)\deg_H(v))^\alpha + \sum_{u\in V_{R^G}}\sum_{v\in V_H}(\deg_{R^G}(u)\deg_H(v))^\alpha \\
&= 4^\alpha\sum_{\substack{uv\in E_{R^G},\\ u,v\in V_G}}(\deg_G(u)\deg_H(v))^\alpha + \sum_{\substack{uv\in E_{R^G},\\ u\in V_G, v\in I_G}}(2\deg_G(u)(p_H + 2))^\alpha \\
&\quad + q_G\sum_{uv\in E_H}[(\deg_H(u) + 1)(\deg_H(v) + 1)]^\alpha + \sum_{\substack{u\in V_{R^G}\\ u\in I_G}}\sum_{v\in V_H}[(p_H + 2)(\deg_H(v) + 1)]^\alpha \\
&\geq 4^\alpha q_G \triangle_G^{2\alpha} + 2^{\alpha+1} q_G \triangle_G^\alpha (p_H + 2)^\alpha + q_G q_H(\triangle_H + 1)^{2\alpha} + q_G p_H(p_H + 2)^\alpha(\triangle_H + 1)^\alpha.
\end{aligned} \tag{8}$$

Similarly, we have the following inequality.

$$\mathcal{R}_\alpha(R^G \ominus H) \leq 4^\alpha q_G \delta_G^{2\alpha} + 2^{\alpha+1} q_G \delta_G^\alpha (p_H + 2)^\alpha + q_G q_H (\delta_H + 1)^{2\alpha} + q_G p_H (p_H + 2)^\alpha (\delta_H + 1)^\alpha. \tag{9}$$

The above equalities hold if and only if G and H are regular graphs. This completes the proof. □

2.3. *R-vertex Neighborhood Corona Product of R^G and H*

The R-vertex neighborhood corona product of R^G and H is defined as the graph $R^G \boxdot H$ obtained by having p_G copies $H_1, H_2, \dots, H_{p_G}$ of H and then joining by an edge the neighbors of a vertex of G in R^G that is on the i-th position in R^G to every vertex of H_i. Thus $p_{R^G \boxdot H} = p_G + q_G + p_G p_H$ and $q_{R^G \boxdot H} = 3q_G + p_G q_H + 4q_G p_H$ number of edges. The degree of vertices of $R^G \boxdot H$ are given by:

$$\deg_{R^G \boxdot H}(u) = \begin{cases} 2(p_H + 1), & \text{if } u \in V_G \\ \frac{5a-4}{2}, & \text{if } u \in I_G \end{cases}$$

$$\deg_{R^G \boxdot H}(v) = \deg_H(v) + 2\deg_G(u) \quad \text{if } v \in V_H, u \in V_G. \tag{10}$$

Here $v \in V_{H_i}$ corresponds to the ^{i}th vertex $v \in V_G$ in R^G.

The general Randić index of an R-vertex neighborhood corona product of R^G and H is given in the following result.

Theorem 3. *Let G and H be two connected and vertex-disjoint graphs. Then, for $\alpha < 0$, the general Randić index of $R^G \boxdot H$ has the following minimum and maximum bounds*

$$\begin{aligned}\mathcal{R}_\alpha(R^G \boxdot H) \geq\ & q_G(p_H + 2)^{2\alpha} \triangle_G^{2\alpha} + 2^{\alpha+1} q_G (p_H + 2)^\alpha (p_H + 1)^\alpha (\triangle_G + 2)^\alpha \\ & + p_G q_H (\triangle_H + 2\triangle_G)^{2\alpha} + p_H (p_H + 2)^\alpha \sum_{\substack{u \in V_G, \\ w_i \in N_G(u), w_i \in V_G}} \triangle_G^\alpha (\triangle_H + 2\triangle_G)^\alpha \\ & + 2^\alpha p_H (p_H + 1)^\alpha \sum_{\substack{u \in V_G, \\ w_i \in N_G(u), w_i \in I_G}} (\triangle_H + 2\triangle_G)^\alpha\end{aligned}$$

$$\begin{aligned}\mathcal{R}_\alpha(R^G \boxdot H) \leq\ & q_G(p_H + 2)^{2\alpha} \delta_G^{2\alpha} + 2^{\alpha+1} q_G (p_H + 2)^\alpha (p_H + 1)^\alpha (\delta_G + 2)^\alpha + p_G q_H (\delta_H + 2\delta_G)^{2\alpha} \\ & + p_H (p_H + 2)^\alpha \sum_{\substack{u \in V_G, \\ w_i \in N_G(u), w_i \in V_G}} \delta_G^\alpha (\delta_H + 2\delta_G)^\alpha + 2^\alpha p_H (p_H + 1)^\alpha \sum_{\substack{u \in V_G, \\ w_i \in N_G(u), w_i \in I_G}} (\delta_H + 2\delta_G)^\alpha.\end{aligned}$$

The equalities hold if and only if both G and H are regular graphs.

Proof. Using (10) in Eq. (2), we get

$$
\begin{aligned}
\mathcal{R}_\alpha(R^G \boxdot H) &= \sum_{uv\in E_{R^G}} (\deg_{R^G}(u)\deg_{R^G}(v))^\alpha + p_G \sum_{uv\in E_H} (\deg_H(u)\deg_H(v))^\alpha + \sum_{u\in V_{R^G}}\sum_{v\in V_H} (\deg_{R^G}(u)\deg_H(v))^\alpha \\
&= \sum_{\substack{uv\in E_{R^G},\\ u,v\in V_G}} [(\deg_G(u)(p_H+2))(\deg_H(v)(p_H+2))]^\alpha + \sum_{\substack{uv\in E_{R^G},\\ u\in V_G, v\in I_G}} [(\deg_G(u)(p_H+2))(2(p_H+1))]^\alpha \\
&\quad + p_G \sum_{uv\in E_H, w_i\in V_G} [(\deg_H(u)+2\deg_G(w_i))(\deg_H(v)+2\deg_G(w_i))]^\alpha \\
&\quad + \sum_{\substack{u\in V_G\\ w_i\in N_G(u), w_i\in V_G}} \sum_{v\in V_H} [(\deg_G(w_i)(p_H+2))(\deg_H(v)+2\deg_G(u))]^\alpha \\
&\quad + \sum_{\substack{u\in V_G,\\ w_i\in N_G(u), w_i\in I_G}} \sum_{h\in V_H} [(2(p_H+1)(\deg_H(v)+2\deg_G(u))]^\alpha \\
&\geq q_G(p_H+2)^{2\alpha} \triangle_G^{2\alpha} + 2^{\alpha+1} q_G(p_H+2)^\alpha (p_H+1)^\alpha (\triangle_G+2)^\alpha \\
&\quad + p_G q_H(\triangle_H + 2\triangle_G)^{2\alpha} \\
&\quad + p_H(p_H+2)^\alpha \sum_{\substack{u\in V_G\\ w_i\in N_G(u), w_i\in V_G}} \triangle_G^\alpha(\triangle_H+2\triangle_G)^\alpha + 2^\alpha p_H(p_H+1)^\alpha \sum_{\substack{u\in V_G\\ w_i\in N_G(u), w_i\in I_G}} (\triangle_H+2\triangle_G)^\alpha.
\end{aligned}
\tag{11}
$$

Similarly, we have the following inequality.

$$
\begin{aligned}
\chi_\alpha(R^G \boxdot H) \leq{}& q_G(p_H+2)^{2\alpha}\delta_G^{2\alpha} + 2^{\alpha+1} q_G(p_H+2)^\alpha(p_H+1)^\alpha(\delta_G+2)^\alpha + p_G q_H(\delta_H+2\delta_G)^{2\alpha} \\
&+ p_H(p_H+2)^\alpha \sum_{\substack{u\in V_G,\\ w_i\in N_G(u), w_i\in V_G}} \delta_G^\alpha(\delta_H+2\delta_G)^\alpha + 2^\alpha p_H(p_H+1)^\alpha \sum_{\substack{u\in V_G,\\ w_i\in N_G(u), w_i\in I_G}} (\delta_H+2\delta_G)^\alpha.
\end{aligned}
\tag{12}
$$

The above equalities hold if and only if G and H are regular graphs. This completes the proof. □

2.4. R-edge Neighborhood Corona Product of R^G and H

The R-edge neighborhood corona product of R^G and H is defined as the graph $R^G \boxminus H$ obtained by having q_G copies $H_1, H_2, \ldots, H_{q_G}$ of H, and then joining by and edge the neighbors of a vertex of I_G in R^G that is on the ^{i}th position in R^G, to every vertex in H_i. Thus, $p_{R^G\boxminus H} = p_G + q_G + q_G p_H$ and $q_{R^G\boxminus H} = 3q_G + q_G q_H + 2q_G p_H$. The degree of a vertex $v \in V_{R^G\boxminus H}$ is given by:

$$
\deg_{R^G\boxminus H}(v) = \begin{cases} \deg_G(v)(2+p_H) & \text{if } v\in V_G, \\ 2 & \text{if } v\in I_G, \\ \deg_H(v)+2 & \text{if } v\in V_H. \end{cases}
\tag{13}
$$

The general Randić index of an R-edge neighborhood corona product of R^G and H is given in the following result.

Theorem 4. *Let G and H be two connected and vertex-disjoint graphs. Then for $\alpha < 0$, the general Randić index of $R^G \boxminus H$ has the following minimum and maximum bounds*

$$
\begin{aligned}
\mathcal{R}_\alpha(R^G \boxminus H) \geq{}& q_G(p_H+2)^{2\alpha} \triangle_G^{2\alpha} + 2^\alpha q_G(p_H+2)^\alpha \triangle_G^\alpha + q_G q_H(\triangle_H+2)^{2\alpha} \\
&+ p_H(p_H+2)^\alpha \sum_{\substack{u\in I_G,\\ w_i\in N_G(u), w_i\in V_G}} \triangle_G^\alpha(\triangle_H+2)^\alpha, \\
\mathcal{R}_\alpha(R^G \boxminus H) \leq{}& q_G(p_H+2)^{2\alpha}\delta_G^{2\alpha} + 2^\alpha q_G(p_H+2)^\alpha\delta_G^\alpha + q_G q_H(\delta_H+2)^{2\alpha} \\
&+ p_H(p_H+2)^\alpha \sum_{u\in I_G,\ w_i\in N_G(u), w_i\in V_G} \delta_G^\alpha(\delta_H+2)^\alpha.
\end{aligned}
$$

The equalities hold if and only if both G and H are regular graphs.

Proof. Using (13) in Eq. (2), we get

$$\begin{aligned}
\mathcal{R}_\alpha(R^G \boxminus H) &= \sum_{uv\in E_{R^G}} (\deg_{R^G}(u)\deg_{R^G}(v))^\alpha + q_G \sum_{uv\in E_H} (\deg_H(u)\deg_H(v))^\alpha + \sum_{u\in V_{R^G}}\sum_{v\in V_H} (\deg_{R^G}(u)\deg_H(v))^\alpha \\
&= \sum_{\substack{uv\in E_{R^G},\\ u,v\in V_G}} [(\deg_G(u)(p_H+2))(\deg_G(v)(p_H+2))]^\alpha + \sum_{\substack{uv\in E_{R^G},\\ u\in V_G, v\in I_G}} (2\deg_G(u)(p_H+2))^\alpha \\
&\quad + q_G \sum_{uv\in E_H} [(\deg_H(u)+2)(\deg_H(v)+2)]^\alpha \\
&\quad + \sum_{\substack{u\in I_G,\\ w_i\in N_G(u), w_i\in V_G}} \sum_{v\in V_H} [(\deg_G(w_i)(p_H+2))(\deg_H(u)+2)]^\alpha \\
&\geq q_G(p_H+2)^{2\alpha} \triangle_G^{2\alpha} + 2^\alpha q_G(p_H+2)^\alpha \triangle_G^\alpha + q_G q_H(\triangle_H+2)^{2\alpha} \\
&\quad + p_H(p_H+2)^\alpha \sum_{\substack{u\in I_G,\\ w_i\in N_G(u), w_i\in V_G}} \triangle_G^\alpha(\triangle_H+2)^\alpha.
\end{aligned} \tag{14}$$

Similarly, we have following bound.

$$\begin{aligned}
\mathcal{R}_\alpha(R^G \boxminus H) \leq & q_G(p_H+2)^{2\alpha}\delta_G^{2\alpha} + 2^\alpha q_G(p_H+2)^\alpha\delta_G^\alpha + q_G q_H(\delta_H+2)^{2\alpha} \\
& + p_H(p_H+2)^\alpha \sum_{\substack{u\in I_G,\\ w_i\in N_G(u), w_i\in V_G}} \delta_G^\alpha(\delta_H+2)^\alpha.
\end{aligned} \tag{15}$$

The above equalities hold if and only if G and H are regular graphs, which completes the proof. □

3. The general Randić Index of a Line Graph

Let G be a simple graph with vertex set $V_G = \{v_1, \ldots, v_{p_G}\}$ and edge set $E_G = \{e_1, \ldots, e_{p_G}\}$. We construct a line graph L^G to G as follows. The set of vertices of L^G is the same as the set of edges of G. Thus, $V_{L^G} = E_G$. Also, $E_{L^G} = \{\{e_i, e_j\} \mid e_i, e_j$ share a common vertex in $G\}$. Thus $p_{L^G} = q_G$ and $q_{L^G} = \frac{1}{2}\sum_{i=1}^{p_G}(\deg(v_i))^2 - q_G$. Thus using (1), $q_{L^G} = \frac{1}{2}M_1(G) - q_G$. The degree of a vertex $v \in L^G$ is given by:

$$\deg_{L^G}(v) = \deg_G(v_i) + \deg_G(v_j) - 2 \quad \text{if } v = v_iv_j,\ \forall\, v_i, v_j \in V_G. \tag{16}$$

We compute the lower and upper bounds on the general Randić index of L^G in the following result.

Theorem 5. *Let G be a connected graph. Then, for $\alpha < 0$, the bounds for the general Randić index of L^G are given by*

$$2^\alpha(\Delta_G - 1)^{2\alpha}\left(\frac{1}{2}M_1(G) - q_G\right) \leq \mathcal{R}_\alpha(L^G) \leq 2^\alpha(\delta_G - 1)^{2\alpha}\left(\frac{1}{2}M_1(G) - q_G\right).$$

The equality holds if and only if G is a regular graph.

Proof. Using (16) in Eq. (2), we get

$$\begin{aligned}
\mathcal{R}_\alpha(L^G) &= \sum_{uv\in E_{L^G}} (\deg_{L^G}(u)\deg_{L^G}(v))^\alpha = \sum_{\substack{u=w_iw_j\\ v=w_jw_k\in E_G}} [(\deg_G(w_i)\deg_G(w_j) - 2)(\deg_G(w_j) + \deg_G(w_k) - 2)]^\alpha \\
&\geq 2^\alpha(\triangle_G - 1)^{2\alpha}\left(\frac{1}{2}M_1(G) - q_G\right).
\end{aligned} \tag{17}$$

Similarly, we have the following inequality.

$$\mathcal{R}_\alpha(L^G) \le 2^\alpha(\delta_G - 1)^{2\alpha}\left(\frac{1}{2}M_1(G) - q_G\right). \tag{18}$$

The above equalities hold if and only if G is a regular graph. □

4. The General Randić Index of the Rooted Products of Graphs

A rooted graph is a labeled graph with one vertex labeled as the root. Let Y be a labeled graph with p_Y number of vertices. Let G be a sequence of p_Y rooted graphs $G_1, G_2, \ldots, G_{p_Y}$. The rooted product of Y and G is defined as the graph Y^G obtained by having p_Y copies of G and identifying the rooted vertex of G_i $(1 \le i \le p_Y)$ with i-th vertex of Y. Thus, $p_{Y^G} = p_G = p_{G_1} + p_{G_2} + \cdots + p_{G_{p_Y}}$ and $q_{Y^G} = q_G + q_Y$. Let w_i be the rooted vertex of G_i. The degree of a vertex $v \in V_{Y^G}$ is given by

$$\deg_{Y^G}(v) = \begin{cases} \deg_Y(v) + \deg_{G_i}(w_i) & \text{if } v \in V_Y, \\ \deg_Y(v) + \deg_{G_i}(w_i) & \text{if } v \in V_G, v = w_i, \\ \deg_{G_i}(v) & \text{if } v \in V_G, v \ne w_i. \end{cases} \tag{19}$$

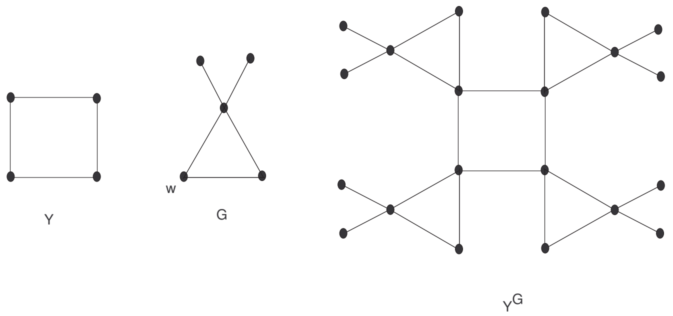

Figure 2. The Graph Y^G

In the following result, the bounds on the general Randić index of a rooted product of given graphs are computed.

Theorem 6. *Let Y be a labeled graph and G be a sequence of rooted graphs $G_1, \ldots, G_{p_Y}$. Then, for $\alpha < 0$, we have*

$$\mathcal{R}_\alpha(Y^G) \ge \sum_{ij \in E_Y} (\triangle_Y + \omega_i)^\alpha(\triangle_Y + \omega_j)^\alpha + \sum_{i=1}^{p_Y} \mathcal{R}_\alpha(G_i) + \sum_{i=1}^{p_Y} |N_{G_i}(w_i)|\, \triangle_{G_i}^\alpha \left[(\triangle_Y + \triangle_{G_i})^\alpha - \triangle_{G_i}^\alpha\right],$$

$$\mathcal{R}_\alpha(Y^G) \le \sum_{ij \in E_Y} (\delta_Y + \omega_i)^\alpha(\delta_Y + \omega_j)^\alpha + \sum_{i=1}^{p_Y} \mathcal{R}_\alpha(G_i) + \sum_{i=1}^{p_Y} |N_{G_i}(w_i)|\delta_{G_i}^\alpha \left[(\delta_Y + \delta_{G_i})^\alpha - \delta_{G_i}^\alpha\right].$$

The equalities hold if and only if Y and G_i are regular graphs.

Proof. The number of neighbors of w_i is denoted by $|N_{G_i}(w_i)|$ and the degree of w_i is denoted by ω_i in G_i. Using (19) in Eq. (2), we get

$$\mathcal{R}_\alpha(Y^G) = \sum_{ij \in E_Y} [(\deg_Y(i) + \omega_i)(\deg_Y(j) + \omega_j)]^\alpha + \sum_{i=1}^{p_Y} \left[\sum_{\substack{uv \in E_{G_i} \\ u,v \ne w_i}} (\deg_{G_i}(u) \deg_{G_i}(v))^\alpha \right]$$

$$+ \sum_{i=1}^{p_Y} \left[\sum_{\substack{uv \in E_{G_i} \\ u \in V_{G_i}, v = w_i}} [\deg_{G_i}(u)(\deg_Y(i) + \omega_i)]^\alpha \right]$$

$$= \sum_{ij \in E_Y} [(\deg_Y(i) + \omega_i)(\deg_Y(j) + \omega_j)]^\alpha + \sum_{i=1}^{p_Y} \left[\mathcal{R}_\alpha(G_i) - \sum_{\substack{uv \in E_{G_i} \\ u \in V_{G_i}, v = w_i}} (\deg_{G_i}(u)\omega_i)^\alpha \right]$$

$$+ \sum_{i=1}^{p_Y} \left[\sum_{\substack{uv \in E_{G_i} \\ u \in V_{G_i}, v = w_i}} [\deg_{G_i}(u)(\deg_Y(i) + \omega_i)]^\alpha \right]$$

$$\geq \sum_{ij \in E_Y} (\triangle_Y + \omega_i)^\alpha (\triangle_Y + \omega_j)^\alpha + \sum_{i=1}^{p_Y} \mathcal{R}_\alpha(G_i) + \sum_{i=1}^{p_Y} |N_{G_i}(w_i)| \triangle_{G_i}^\alpha \left[(\triangle_Y + \triangle_{G_i})^\alpha - \triangle_{G_i}^\alpha \right].$$

Similarly, we can show that

$$\mathcal{R}_\alpha(Y^G) \leq \sum_{ij \in E_Y} (\delta_Y + \omega_i)^\alpha (\delta_Y + \omega_j)^\alpha + \sum_{i=1}^{p_Y} \mathcal{R}_\alpha(G_i) + \sum_{i=1}^{p_Y} |N_{G_i}(w_i)| \delta_{G_i}^\alpha \left[(\delta_Y + \delta_{G_i})^\alpha - \delta_{G_i}^\alpha \right].$$

The above equalities hold if and only if Y and G_i are regular graphs. This completes the proof. □

Let the sequence of graphs $G_1, G_2, G_3, \ldots, G_{p_H}$ be isomorphic to a graph G. Then, we denote the rooted product of H and G by $H\{G\}$. This rooted product is called a cluster of H and G. Thus, using Theorem 6, we get at the following result.

Corollary 1. *Let H be a labeled graph, and G be a sequence of isomorphic rooted graphs $G_1, G_2, \ldots, G_{p_H}$. Then, for $\alpha < 0$, we have*

$$\mathcal{R}_\alpha(H\{G\}) \geq q_H(\triangle_Y + \omega)^{2\alpha} + p_H \mathcal{R}_\alpha(G) + p_H |N_G(w)| \triangle_{G_i}^\alpha \left[(\triangle_H + \triangle_{G_i})^\alpha - \triangle_{G_i}^\alpha \right],$$

$$\mathcal{R}_\alpha(H\{G\}) \leq q_H(\delta_Y + \omega)^{2\alpha} + p_H \mathcal{R}_\alpha(G) + p_H |N_G(w)| \delta_{G_i}^\alpha \left[(\delta_H + \delta_{G_i})^\alpha - \delta_{G_i}^\alpha \right],$$

where $\omega = \deg_G(w)$.

5. Conclusions

We have obtained the lower, and upper bounds for the general Randić index $\mathcal{R}_\alpha$ of four types of graph operations involving an R-graph. Additionally, we have determined the bounds for the general Randić index of a line graph, and rooted product of graphs.

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Conflict of Interest

The authors declare no conflict of interest

References

1. Khalifeh, M. H., Ashrafi, A. R. and Azari, H. Y., 2009. The first and second Zagreb indices of some graphs operations. *Discrete Applied Mathematics, 157*(7), pp.804-811.
2. Zhou, B., 2004. Zagreb indices. *MATCH Communications in Mathematical and Computer Chemistry, 52*(1), pp.113-118.
3. Randic, M., 1975. On characterization of molecular branching. *Journal of the American Chemical Society, 97*(23), pp.6609-6615.
4. Li, X. and Shi, Y., 2008. A survey on the Randic index. *MATCH Communications in Mathematical and Computer Chemistry, 59*(1), pp.127-156.
5. Dalfó, C., 2019. On the Randić index of graphs. *Discrete Mathematics,342*(12), pp.2792–2796.
6. Bollobás, B. and Erdős, P., 1998. Graphs of extremal weights. *Ars Combinatoria,50*, pp.225–233.
7. Gutman, I., Furtula, B. and Katanic, V., 2017. Randić index and information. *AKCE International Journal of Graphs and Combinatorics,11*, pp.78-88.
8. Gutman, I., 2013. Degree-based topological indices. *Croatica Chemica Acta,86*(2), pp.351–361.
9. Godsil, C. and McKay, B., 1978. A new graph product and its spectrum. *Bulletin of the Australian Mathematical Society,18*(1), pp.21–28.
10. Lan, J. and Zhou, B., 2015. Spectra of graph operations based on *R*-graph. *Linear Algebra and its Applications, 63*(7), pp.1401-1422.
11. Shao, Z., Amjadi, J. S., Sheikholeslami, M. and Valinavaz, M., 2019. On the total double Roman domination. *IEEE Access, 7*, pp.52035-52041.
12. Shao, Z., Liang, M. and Xu, X., 2012. Some new optimal generalized Sidon sequences. *Ars Combinatoria, 107*, pp.369-378.
13. Shao, Z., Siddiqui, M. K. and Muhammad, M. H., 2018. Computing Zagreb indices and Zagreb polynomials for symmetrical nanotubes. *Symmetry, 10*(7), p.244.
14. Shao, Z., Wu, P., Zhang, X., Dimitrov, D., Liu, J. B., (2018. On the maximum ABC index of graphs with prescribed size and without pendent vertices. *IEEE Access, 6*, pp.604-616.
15. Shao, Z., Wu, P., Gao, Y., Gutman, I. and Zhang, X., 2018. On the maximum ABC index of graphs without pendent vertices. *Applied Mathematics and Computation, 315*, pp.298-312.
16. Wang, H., Liu, J., Wang, S., Gao, W., Akhter, S., Imran, M., ... and Farahani, M. R., 2017. Sharp bounds for the general sum-connectivity index of transformation graphs. *Discrete Dynamics in Nature and Society, 2017*, pp.1-11.
17. Yang, H., Rashid, M. A., Ahmad, S., Khan, S. S. and Siddiqui, M. K., 2019. On Molecular Descriptors of Face-Centred Cubic Lattice. *Processes, 7*(1), pp.1-15.
18. Frucht, R. and Harary, F., 1970. On the corona of two graphs. *Aequationes Mathematicae,4*(3), pp.322–325.
19. Akhter, S., Imran, M. and Raza, Z., 2017. Bounds for the general sum-connectivity index of composite graphs. *Journal of Inequalities and Applications,76*, pp.331-341.