Article

Super Edge-magic Total Strength of Some Unicyclic Graphs

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Abstract: Let G be a finite simple undirected (p, q)-graph, with vertex set $V(G)$ and edge set $E(G)$ such that $p = |V(G)|$ and $q = |E(G)|$. A super edge-magic total labeling f of G is a bijection $f: V(G) \cup E(G) \longrightarrow \{1, 2, \ldots, p + q\}$ such that for all edges $uv \in E(G)$, $f(u) + f(v) + f(uv) = c(f)$, where $c(f)$ is called a magic constant, and $f(V(G)) = \{1, \ldots, p\}$. The minimum of all $c(f)$, where the minimum is taken over all the super edge-magic total labelings f of G, is defined to be the super edge-magic total strength of the graph G. In this article, we work on certain classes of unicyclic graphs and provide shreds of evidence to conjecture that the super edge-magic total strength of a certain family of unicyclic (p, q)-graphs is equal to $2q + \frac{n+3}{2}$.

Keywords: Edge-magic total labeling, Unicyclic graph, Super edge-magic total labeling, Super edge-magic total strength

2020 Mathematics Subject Classification: 05C78

1. Introduction

Throughout this article, we only consider finite simple undirected graphs. Let G be a finite simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. Let us assume that $|V(G)| = p$ and $|E(G)| = q$, then G is called a (p, q)-*graph*. For any undefined terms and notations in this article, we follow [1].

A *magic valuation* of a graph G was first described by Kotzig and Rosa [2] as a bijection $f: V(G) \cup E(G) \longrightarrow \{1, 2, \ldots, p + q\}$ such that for all edges $uv \in E(G)$, the sum $f(u) + f(v) + f(uv)$ is a constant, called as the *magic constant* of f. A graph is said to be *magic* if it has a magic valuation. Ringel and Lladó [3] rediscovered this idea and gave it the name *edge-magic*. We shall use the phrase *edge-magic total*, as developed by Wallis [4] to distinguish this usage from that of other forms of labelings that utilize the word magic.

In [2], Kotzig and Rosa have demonstrated that the cycles C_n, $n \geq 3$ and the complete bipartite graphs $K_{m,n}$ with $m, n \geq 1$, are edge-magic total graphs. They also established that a complete graph K_n is an edge-magic total graph if and only if $n = 1, 2, 3, 5$, or 6. Further, they proved that the disjoint union of n copies of P_2 has an edge-magic total labeling if and only if n is odd. The bibliography section includes references to several pieces of pertinent literature, including [3, 5–8].

For studies related to magic constants for edge-magic total labelings of certain families of graphs such as certain cycles and crowns of cycles, see [9–11]. Avadayappan, Vasuki, and Jeyanthi [12] introduced the concept of *edge-magic total strength* of a graph G as the smallest magic constant over all edge-magic total labelings of G. Let us denote it by $em(G)$. Let f be an edge-magic total labeling

of G with magic constant $c(f)$. By definition,

$$em(G) = \min\{c(f)\colon f \text{ be an edge-magic total labeling of } G\}.$$

In this study, we explore an edge-magic total labeling f of the (p, q)-graph G, such that $f(V(G)) = \{1, 2, \ldots, p\}$. Enomoto, Llado, Nakamigawa, and Ringel [13] have defined this type of labeling as a *super edge-magic total labeling*. If a graph G contains a super edge-magic total labeling, then G is called a *super edge-magic total*. An important aspect of super edge-magic labelings that captivate researchers is the relationships they share with other well-studied labelings like graceful, harmonious, sequential, cordial labelings, and so on. For an in-depth exploration of these interconnections, references such as [14–16] provide valuable insights.

Enomoto, Llado, Nakamigawa, and Ringel [13] established that a complete graph K_n is super edge-magic total if and only if $n = 1, 2$, or 3 and also proved that a complete bipartite graph $K_{m,n}$ is super edge-magic total if and only if $m = 1$ or $n = 1$. As demonstrated in [13], cycles C_n are super edge-magic total if and only if n is odd. Some further results on the super edge-magic total graph can be found in [6, 17, 18].

A necessary and sufficient condition for a graph to be a super edge-magic total is stated in the following lemma.

Lemma 1 ([14]*Lemma 1). *A (p, q)-graph G is a super edge-magic total graph if and only if there exists a bijection $f\colon V(G) \longrightarrow \{1, 2, \ldots, p\}$, such that the set $\{f(u) + f(v)\colon uv \in E(G)\}$ is a set of q consecutive integers. And, f extends to a super edge-magic total labeling with magic constant $c(f) = p + q + \min\{f(u) + f(v)\colon uv \in E(G)\}$.*

In this case, the vertex labeling f can be extended to a super edge-magic total labeling of G by defining $f(uv) = p + q + \min\{f(\tilde{u}) + f(\tilde{v})\colon \tilde{u}\tilde{v} \in E(G)\} - f(u) - f(v)$, for every edge $uv \in E(G)$.

Acharya and Hedge introduced in [19] the concept of strongly indexable graphs and thanks to Lemma 1, it is very easy to see that the concepts of super edge-magic total graph and strongly indexable graph are equivalent.

For any regular super edge-magic total graph, we have the following result.

Lemma 2 ([14]*Lemma 4). *Let G be an r-regular (p, q)-graph, where $r > 0$. Let f be any super edge-magic total labeling of G. Then q is odd and $c(f) = \frac{4p+q+3}{2}$, for all super edge-magic total labeling f.*

By using Lemma 2, we can derive that for an odd cycle C_n and with any super edge-magic total labeling f of C_n,

$$c(f) = 2n + \min\{f(u) + f(v)\colon uv \in E(C_n)\} = 2n + \frac{n+3}{2}$$

$$\implies \min\{f(u) + f(v)\colon uv \in E(C_n)\} = \frac{n+3}{2}.$$

The article [20] demonstrates that by considering a super edge-magic total labeling of a super edge-magic total graph, we can add vertices and edges to the given graph such that the new graph constructed is also super edge-magic total.

Theorem 1 ([20]*Theorem 2.4). *Let G_p be a connected super edge-magic total (p, q)-graph with $p \geq 3$. Let f be a super edge-magic total labeling of G_p and let us consider $F_f(G_p) = \{f(u) + f(v)\colon uv \in E(G_p)\}$. Let $\max(F_f(G_p)) = p + t$, and for some $a \in V(G_p)$, $f(a) = t$. We construct a graph $\widetilde{G_p}$, by taking each copy of G_p and mK_1, $m \geq 1$ and connecting all the vertices of mK_1 to the vertex $a \in V(G_p)$. Then, $\widetilde{G_p}$ is also a super edge-magic total graph.*

The magic constants for super edge-magic total labelings of crowns of cycles and cycles themselves, in general, have also been studied in many articles, for instance, [21, 22]. In [23], the concept of *super edge-magic total strength* of a graph G was introduced, where it is defined as the minimum magic constant, the minimum taken over all the super edge-magic total labelings of G. We denote it as $sm(G)$. So, we have

$$sm(G) = \min\{c(f) \colon f \text{ is a super edge-magic total labeling of } G\}.$$

The super edge-magic total strength of an odd cycle C_n was found to be $\frac{5n+3}{2}$, in [23]. More results regarding the super edge-magic total strength of certain families of graphs are demonstrated in [23, 24].

Let G be a super edge-magic total (p, q)-graph. For any $v \in V(G)$, the number of edges adjacent to vertex v is called the *degree* of v, denoted by $\deg(v)$. Let f be a super edge-magic total labeling of G, with magic constant $c(f)$. As noted in [12], each edge's magic constants are added together to produce the following result:

$$qc(f) = \sum_{v \in V(G)} \deg(v)f(v) + \sum_{e \in E(G)} f(e). \tag{1}$$

Also, since $\text{Im}(f) = \{1, 2, \ldots, p + q\}$ and $f(V(G)) = \{1, 2, \ldots, p\}$, we can derive that $p + q + 3 \leq sm(G) \leq 3p$.

In this paper, we are interested in the family of unicyclic graphs which consists of an odd cycle C_n and k_i pendant vertices adjacent to each $i \in V(C_n)$.

Let us consider $G(n; k_1, \ldots, k_n)$ to be the unicyclic (p, q)-graph consisting of an odd cycle $C_n = \{a_1, a_2, \ldots, a_n\}$ and k_i pendant vertices adjacent to the vertex a_i, $1 \leq i \leq n$. Swaminathan and Jeyanthi [24] established a range for the super edge-magic total strength of this family of unicyclic graphs.

Theorem 2 ([24]*Theorem 4). *The unicyclic (p, q)-graph $G(n; k_1, \ldots, k_n)$, where $n = 2s + 1$, is a super edge-magic total graph and*

$$2q + 2 + \frac{1}{q}\left(m_2 + 2m_3 + \cdots + (n - 1)m_n + \frac{n(n - 1)}{2}\right) \leq sm(G(n; k_1, \ldots, k_n))$$

$$\leq 2(k_1 + k_3 + \cdots + k_{2s+1}) + 3(k_2 + k_4 + \cdots + k_{2s}) + 2n + s + 2,$$

where $m_1 \geq m_2 \geq \cdots \geq m_n$ are integers such that

$$\{m_1, m_2, \ldots, m_n\} = \{k_1, k_2, \ldots, k_n\}.$$

Corollary 1 ([24]Corollary 4.1). *For any unicyclic graph $G(n; k_1, \ldots, k_n)$ such that $k_i = k$, for any $1 \leq i \leq n$,*

$$sm(G(n; k, \ldots, k)) = 2n(k + 1) + \frac{n + 3}{2}.$$

As stated earlier, in this study we will be investigating the super edge-magic total strength of the family of unicyclic graphs having an odd cycle C_n and k_i pendant vertices adjacent to each $i \in V(C_n)$. We will compute certain graphs' super edge-magic total strength and provide supporting evidence for the following conjecture.

Conjecture 1. *Let $G(n; k_1, \ldots, k_n)$ be the super edge-magic total unicyclic (p, q)-graph consisting of an odd cycle $C_n = \{a_1, a_2, \ldots, a_n\}$ and k_i number of pendant vertices adjacent to each a_i, $1 \leq i \leq n$. Then,*

$$sm(G(n; k_1, \ldots, k_n)) = 2q + \frac{n + 3}{2}.$$

In this article, we particularly examine three specific infinite families of graphs belonging to the family of unicyclic graphs $G(n; k_1, \ldots, k_n)$ and provide substantial evidence in favor of our Conjecture 1. Further, all the graphs under consideration have order and size equal. This makes the main results of the paper fascinating, since it is known that graphs of equal order and size which are super edge-magic total can be used as basis to generate and enumerate several other labelings (see, [15]).

A brief structure of this paper is as follows. Section 2 deals with the family of graphs $G(n; k_1, \ldots, k_n)$ with $k_i = k$, for any $1 \leq i \leq n - 1$ and $k_n = k + c$, where $1 \leq c < \frac{2n(k+1)}{n-3}$. We further prove that the super edge-magic total strength of this graph satisfies our main conjecture. In Section 3, we study $G(n; k, \ldots, k, k - c)$, where $1 \leq c \leq k$ and prove that the super edge-magic total strength of this family of graphs satisfies the conjecture. Section 4 is about the unicyclic graph $G(n, k_1, \ldots, k_n)$ with $k_i = k$, if $i \neq r, n - r$ and $k_r = k_{n-r} = k + 1$ for any odd number r, $1 \leq r < n$. Further, we prove that this family of graphs also satisfies our main conjecture. All of our conclusions from this study are included in Section 5.

2. Unicyclic Graph $G_{n,k,c}$

Let $G_{n,k,c} := G(n; k, \ldots, k, k + c)$, where $1 \leq c < \frac{2n(k+1)}{n-3}$. That is, $G_{n,k,c}$ is the unicyclic graph consisting of an odd cycle $C_n = \{a_1, a_2, \ldots, a_n\}$, with k pendant vertices adjacent to each of the vertices a_i, $1 \leq i \leq n - 1$ and $k + c$ pendant vertices adjacent to vertex a_n. For illustration, see Figure 1. Throughout this paper, let $V(C_n) = \{a_1, a_2, \ldots, a_n\}$ and $E(C_n) = \{a_i a_{i+1} \colon 1 \leq i \leq n-1\} \cup \{a_1 a_n\}$. The number of vertices and edges of the graph $G_{n,k,c}$ is $p = q = n(k+1) + c$. Let the vertex set $V(G_{n,k,c})$ be

$$V(C_n) \cup \{a_{i,j} \colon 1 \leq i \leq n - 1, 1 \leq j \leq k\} \cup \{a_{n,j} \colon 1 \leq j \leq k + c\}$$

and let the edge set $E(G_{n,k,c})$ be

$$E(C_n) \cup \{a_i a_{i,j} \colon 1 \leq i \leq n - 1, 1 \leq j \leq k\} \cup \{a_n a_{n,j} \colon 1 \leq j \leq k + c\}.$$
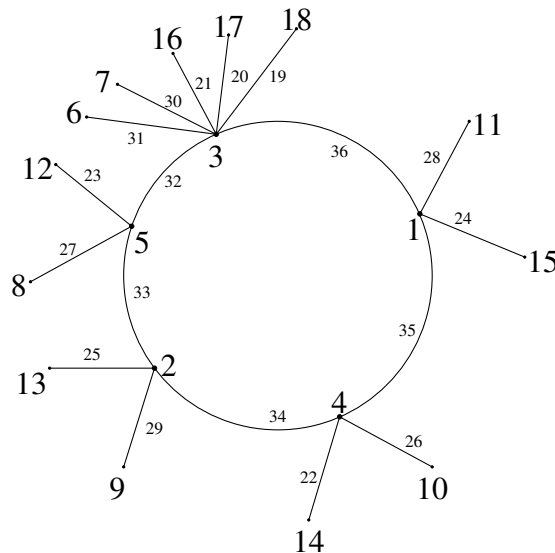

Figure 1. The Graph $G_{5,2,3}$

Theorem 3. *The unicyclic graph $G_{n,k,c}$ is a super edge-magic total graph with super edge-magic total strength given by*

$$sm(G_{n,k,c}) = 2n(k + 1) + 2c + \frac{n + 3}{2}.$$

Proof. By Theorem 2, the graph $G_{n,k,c} := G(n; k, \ldots, k, k + c)$, $1 \le c < \frac{2n(k+1)}{n-3}$, is super edge-magic total. From (1), for any super edge-magic total labeling f of $G_{n,k,c}$, we have

$$qc(f) = \sum_{v \in V(G_{n,k,c})} \deg(v)f(v) + \sum_{e \in E(G_{n,k,c})} f(e)$$

$$= q(2q + 1) + (k + 1) \sum_{a_i \in V(C_n)} f(a_i) + cf(a_n).$$

By assigning the smallest labels to vertices with higher degrees, we get the least possible value of $c(f)$ as

$$c(f) \ge 2q + 1 + \frac{(k + 1)n(n + 1)}{2q} + \frac{c}{q}.$$

Hence, we have

$$sm(G_{n,k,c}) \ge 2q + 1 + \frac{n(k + 1)(n + 1)}{2q} + \frac{c}{q}.$$

Since $sm(G_{n,k,c})$ is an integer, we consider the ceiling of $\left(\frac{n(k+1)(n+1)}{2q} + \frac{c}{q} \right)$. We have $\frac{n+1}{2} - \left(\frac{n(k+1)(n+1)}{2q} + \frac{c}{q} \right) = \frac{(n-1)c}{2(n(k+1)+c)}$. Since $1 \le c < \frac{2n(k+1)}{n-3}$, we have $(n-1)c < 2(n(k+1)+c)$. Therefore, $0 < \frac{(n-1)c}{2(n(k+1)+c)} < 1$. Hence,

$$sm(G_{n,k,c}) \ge 2q + 1 + \frac{(n + 1)}{2}$$

$$= 2n(k + 1) + 2c + \frac{n + 3}{2}.$$

That is,

$$sm(G_{n,k,c}) \ge 2n(k + 1) + 2c + \frac{n + 3}{2}. \tag{2}$$

By Theorem 1 and Theorem 2, the graph $G_{n,k,c}$ can be seen as the super edge-magic total graph constructed from the graph $G(n; k, \ldots, k)$, by connecting all vertices of cK_1 to the vertex a_n, with the super edge-magic total labeling f of $G(n; k, \ldots, k)$ defined as follows.

For $1 \le i \le n$,

$$f(a_i) = \begin{cases} \dfrac{i + 1}{2} & \text{if } i \text{ is odd,} \\[2mm] \dfrac{n + i + 1}{2} & \text{if } i \text{ is even.} \end{cases}$$

$$f(a_{i,j}) = n(k + 1) - (n - 1)(j - 1) - (i - 1), \ 1 \le i \le n - 1, \ 1 \le j \le k.$$

$$f(a_{n,j}) = n + j, \ 1 \le j \le k.$$

Now, we extend f to a vertex labeling $f : V(G_{n,k,c}) \longrightarrow \{1, \ldots, p\}$ by defining

$$f(a_{n,j}) = n(k + 1) + j - k, \ k + 1 \le j \le k + c.$$

As per the above labeling, for any $uv \in E(G_{n,k,c})$ we observe the following.

- If $u, v \in V(G_{n,k,c}) \cap V(G(n; k, \ldots, k))$, since f is a super edge-magic total labeling of the graph $G(n; k, \ldots, k)$, the set $\{f(u) + f(v)\}$ is a consecutive sequence with highest element $n(k + 1) + \frac{n+1}{2}$.
- If $u = a_n$ and $v = a_{n,j}$, for $k + 1 \le j \le k + c$, then $\{f(u) + f(v)\} = \{\frac{n+1}{2} + n(k + 1) + 1, \ldots, \frac{n+1}{2} + n(k + 1) + c\}$ is a consecutive sequence.

Therefore, we observe that $\{f(u) + f(v) : uv \in E(G_{n,k,c})\}$ is a consecutive sequence and we have $\min\{f(u) + f(v) : uv \in E(G_{n,k,c})\} = \frac{n+3}{2}$. By Lemma 1, the vertex labeling f extends to a super edge-magic total labeling of $G_{n,k,c}$ with $c(f) = 2n(k + 1) + 2c + \frac{n+3}{2}$. Hence,

$$sm(G_{n,k,c}) \leq 2n(k + 1) + 2c + \frac{n + 3}{2}. \tag{3}$$

From (2) and (3), we have $sm(G_{n,k,c}) = 2n(k + 1) + 2c + \frac{n+3}{2}$.   □

Example 1. *Super edge-magic total labeling of the graph $G_{5,2,3}$ with strength $sm(G_{5,2,3}) = 40$, is illustrated in Figure 1.*

Example 2. *Super edge-magic total labeling of $G_{9,3,4}$ with $sm(G_{9,3,4}) = 86$ is illustrated in Figure 2.*
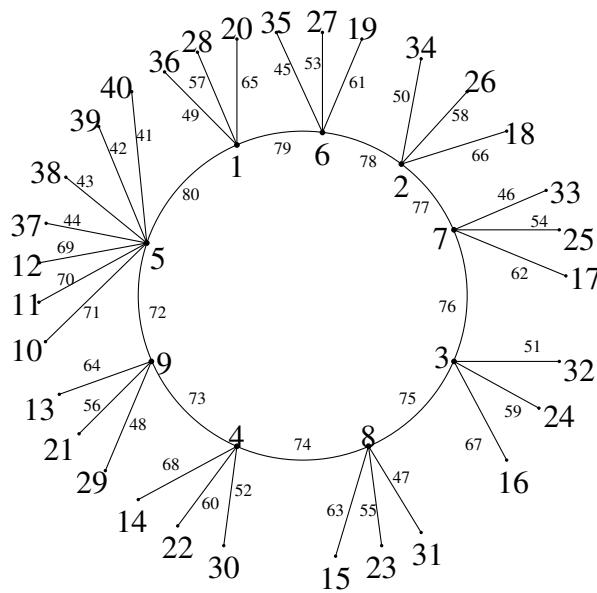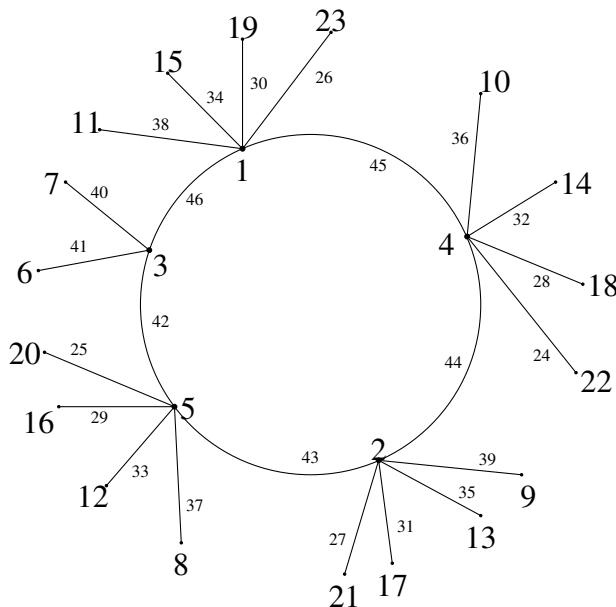

Figure 2. The Graph $G_{9,3,4}$

3. Unicyclic Graph $G_{n,k,-c}$

Let us consider the unicyclic graph $G_{n,k,-c} := G(n; k, \ldots, k, k - c)$, $1 \leq c \leq k$. For example, see Figure 3. For $G_{n,k,-c}$, the number of vertices and edges are $p = q = n(k + 1) - c$.

Let $V(G_{n,k,-c}) = V(C_n) \cup \{a_{i,j} : 1 \leq i \leq n - 1, 1 \leq j \leq k\} \cup \{a_{n,j} : 1 \leq j \leq k - c\}$ and let the edge set be equal to

$$E(C_n) \cup \{a_i a_{i,j} : 1 \leq i \leq n - 1, 1 \leq j \leq k\} \cup \{a_n a_{n,j} : 1 \leq j \leq k - c\},$$

where $1 \leq c \leq k$.

Figure 3. The Graph $G_{5,4,-2}$

Theorem 4. *The unicyclic graph $G_{n,k,-c}$ is a super edge-magic total graph with super edge-magic total strength*

$$sm(G_{n,k,-c}) = 2n(k+1) - 2c + \frac{n+3}{2}.$$

Proof. By Theorem 2, the graph $G_{n,k,-c}$ is super edge-magic total and the lower bound of its super edge-magic total strength is:

$$sm(G_{n,k,-c}) \geq 2q + 2 + \frac{1}{q}\left(\frac{nk(n-1)}{2} + \frac{n(n-1)}{2} - c(n-1)\right)$$

$$= 2n(k+1) - 2c + 2 + \frac{1}{n(k+1) - c}\left(\frac{n(n-1)(k+1)}{2} - c(n-1)\right)$$

$$= 2n(k+1) - 2c + 2 + \frac{n-1}{2}\left(\frac{n(k+1) - 2c}{n(k+1) - c}\right).$$

Since $sm(G_{n,k,-c})$ is an integer, we consider the ceiling of $\frac{n-1}{2}\left(\frac{n(k+1)-2c}{n(k+1)-c}\right)$. We have $\frac{n-1}{2} - \frac{n-1}{2}\left(\frac{n(k+1)-2c}{n(k+1)-c}\right) = \frac{(n-1)c}{2(n(k+1)-c)}$. Since $c \leq k$, we observe that $(n-1)c < 2(n(k+1) - c)$. Therefore,

$$0 < \frac{n-1}{2} - \frac{n-1}{2}\left(\frac{n(k+1) - 2c}{n(k+1) - c}\right) < 1.$$

Hence for $G_{n,k,-c}$, we have

$$sm(G_{n,k,-c}) \geq 2n(k+1) - 2c + 2 + \frac{n-1}{2}$$

$$= 2n(k+1) - 2c + \frac{n+3}{2}.$$

That is,

$$sm(G_{n,k,-c}) \geq 2n(k+1) - 2c + \frac{n+3}{2}. \tag{4}$$

Recall that, our graph $G_{n,k,-c}$ is the unicyclic graph $G(n; k, \ldots, k, k-c)$, with k pendant vertices adjacent to each of the vertices a_i, $1 \leq i \leq n-1$ and $k-c$ pendant vertices adjacent to vertex a_n, where $1 \leq c \leq k$. Now, we define a vertex labeling $f : V(G_{n,k,-c}) \longrightarrow \{1, \ldots, p\}$ as follows:

For $1 \le i \le n$,

$$f(a_i) = \begin{cases} \dfrac{i+1}{2} & \text{if } i \text{ is odd,} \\[2ex] \dfrac{n+i+1}{2} & \text{if } i \text{ is even.} \end{cases} \tag{5}$$

$$f(a_{i,j}) = n(k+2) - c - (n-1)j - i, \; 1 \le i \le n-1, \; 1 \le j \le k.$$

$$f(a_{n,j}) = n + j, \; 1 \le j \le k - c.$$

As per the labeling defined in (5), for any $uv \in E(G_{n,k,-c})$ we observe the following.

- If $u, v \in V(C_n)$ then, $\{f(u) + f(v)\} = \{1 + \frac{n+1}{2}, \ldots, n + \frac{n+1}{2}\}$ is a consecutive sequence.
- If $u = a_n$ and $v = a_{n,j}$, $1 \le j \le k - c$, then we have $\{f(u) + f(v)\} = \{n + \frac{n+3}{2}, \ldots, n + k - c + \frac{n+1}{2}\}$, a consecutive sequence.
- If $u = a_i$ and $v = a_{i,j}$, for $1 \le i \le n-1$, $1 \le j \le k$, then we have $\{f(u) + f(v)\} = \{n + k - c + \frac{n+3}{2}, \ldots, n(k+1) - c + 2\}$, which is a consecutive sequence.

Thus we observe that $\{f(u) + f(v) : uv \in E(G_{n,k,-c})\}$ is a consecutive sequence with $\min\{f(u) + f(v) : uv \in E(G_{n,k,-c})\} = \frac{n+3}{2}$. Therefore by Lemma 1, the vertex labeling f extends to a super edge-magic total labeling of $G_{n,k,-c}$ with a magic constant $c(f) = 2n(k+1) - 2c + \frac{n+3}{2}$. Hence,

$$sm(G_{n,k,-c}) \le 2n(k+1) - 2c + \frac{n+3}{2}. \tag{6}$$

From (4) and (6), we have $sm(G_{n,k,-c}) = 2n(k+1) - 2c + \frac{n+3}{2}$. $\qquad\square$

Example 3. *Super edge-magic total labeling of the graph $G_{5,4,-2}$ with super edge-magic total strength $sm(G_{5,4,-2}) = 50$, is illustrated in Figure 3.*

Example 4. *Super edge-magic total labeling of the graph $G_{5,8,-6}$ with super edge-magic total strength $sm(G_{5,8,-6}) = 82$, is illustrated in Figure 4.*
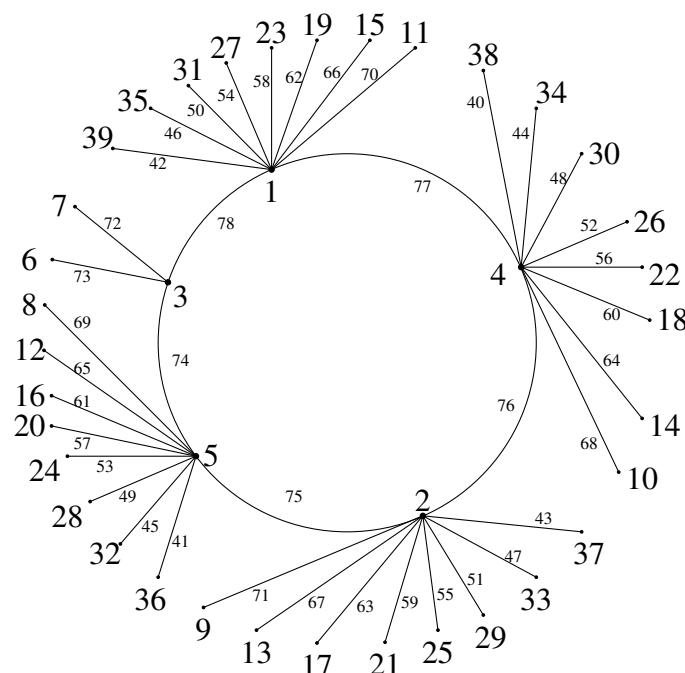
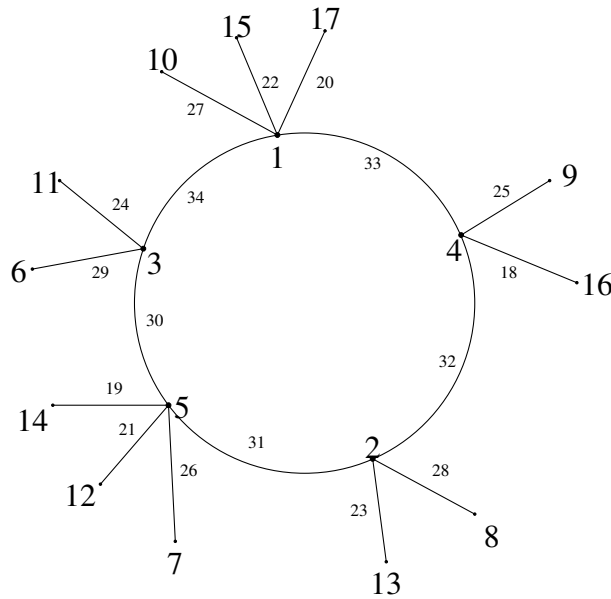**Figure 4.** The Graph $G_{5,8,-6}$

4. Unicyclic Graph $G(n; k, r)$

Let $G(n; k, r)$ be the unicyclic graph $G(n, k_1, \ldots, k_n)$ with $k_i = k$, if $i \neq r, n - r$ and $k_r = k_{n-r} = k + 1$ for any odd number r, $1 \leq r < n$. See Figure 5. Let $p = q = n(k + 1) + 2$, be the number of vertices and edges of $G(n; k, r)$. Let

$$V(G(n; k, r)) = V(C_n) \cup \{a_{i,j} : 1 \leq i \leq n, 1 \leq j \leq k\} \cup \{a_{r,k+1}, a_{n-r,k+1}\},$$

and the edge set $E(G(n; k, r))$ be

$$E(C_n) \cup \{a_i a_{i,j} : 1 \leq i \leq n, 1 \leq j \leq k\} \cup \{a_i a_{i,k+1} : i \in \{r, n - r\}\}.$$

Figure 5. The Graph $G(5; 2, 1)$

Theorem 5. *The unicyclic graph $G(n; k, r)$, where r is any odd number such that $1 \leq r < n$, admits a super edge-magic total labeling and has a super edge-magic total strength*

$$sm(G(n; k, r)) = 2n(k + 1) + 4 + \frac{n + 3}{2}.$$

Proof. By Theorem 2, the unicyclic graph $G(n; k, r)$ is a super edge-magic total graph with

$$sm(G(n; k, r)) \geq 2q + 2 + \frac{1}{q}\left((k + 1) + 2k + \cdots + (n - 1)k + \frac{n(n - 1)}{2}\right)$$

$$= 2n(k + 1) + 6 + \frac{1}{n(k + 1) + 2}\left(\frac{nk(n - 1)}{2} + \frac{n(n - 1)}{2} + 1\right)$$

$$= 2n(k + 1) + 6 + \frac{n - 1}{2}\left(\frac{n(k + 1)}{n(k + 1) + 2}\right) + \frac{1}{n(k + 1) + 2}.$$

That is, we have $sm(G(n; k, r)) \geq 2n(k + 1) + 6 + \frac{n-1}{2}\left(\frac{n(k+1)}{n(k+1)+2}\right) + \frac{1}{n(k+1)+2}$. We know that $sm(G(n; k, r))$ is an integer and we can observe that

$$\frac{n - 1}{2} - \left(\frac{n - 1}{2}\left(\frac{n(k + 1)}{n(k + 1) + 2}\right) + \frac{1}{n(k + 1) + 2}\right)$$

$$= \frac{n - 1}{2}\left(\frac{2}{n(k + 1) + 2}\right) - \frac{1}{n(k + 1) + 2}$$

$$= \frac{n - 2}{n(k + 1) + 2}$$

$$< 1.$$

Hence, we observe that the ceiling of $\frac{n-1}{2}\left(\frac{n(k+1)}{n(k+1)+2}\right) + \frac{1}{n(k+1)+2}$ is $\frac{n-1}{2}$. And we have

$$sm(G(n;k,r)) \geq 2n(k+1) + 6 + \frac{n-1}{2} = 2n(k+1) + 4 + \frac{n+3}{2}.$$

Therefore, we can express

$$sm(G(n;k,r)) \geq 2n(k+1) + 4 + \frac{n+3}{2}. \tag{7}$$

Now, if we prove that there exists a super edge-magic total labeling f of $G(n;k,r)$ with magic constant $c(f) = 2n(k+1) + 4 + \frac{n+3}{2}$, then our proof is complete.

Let us define a vertex labeling $f: V(G(n;k,r)) \longrightarrow \{1, \ldots, p\}$ as follows. For $1 \leq i \leq n$,

$$f(a_i) = \begin{cases} \dfrac{i+1}{2} & \text{if } i \text{ is odd}, \\[3mm] \dfrac{n+i+1}{2} & \text{if } i \text{ is even}. \end{cases} \tag{8}$$

And,

$$f(a_{i,j}) = n(k-j+2) - 2f(a_i) + 2, \; 1 \leq i \leq n, \; 1 \leq j \leq k-1,$$

$$f(a_{i,k}) = \begin{cases} 2n + 2 - 2f(a_i) & \text{if } 1 \leq f(a_i) \leq \frac{n+1}{2}, \\ n(k+2) + 4 - 2f(a_i) & \text{if } \frac{n+3}{2} \leq f(a_i) \leq n - f(a_r), \\ n(k+2) + 2 - 2f(a_i) & \text{if } n - f(a_r) + 1 \leq f(a_i) \leq n, \end{cases} \tag{9}$$

$$f(a_{r,k+1}) = n(k+1) + 2,$$

$$f(a_{n-r,k+1}) = nk + r + 3.$$

As per the above labeling, for $uv \in E(G(n;k,r))$ we observe that:

- For $u, v \in V(C_n)$, $\{f(u) + f(v)\} = \{1 + \frac{n+1}{2}, \ldots, n + \frac{n+1}{2}\}$ is a consecutive sequence.
- Let us consider $u = a_i$ and $v = a_{i,j}$, for any $1 \leq i \leq n$, $1 \leq j \leq k-1$. Then $\{f(u) + f(v)\} = \{n(k-j+2) - f(a_i) + 2: 1 \leq i \leq n, \; 1 \leq j \leq k-1\}$ is a consecutive sequence with minimal element $2n+2$ and maximal element $n(k+1) + 1$.
- Let $u = a_i$ and $v = a_{i,k}$, $1 \leq i \leq n$.
    - If $1 \leq f(a_i) \leq \frac{n+1}{2}$, then $\{f(a_i) + f(a_{i,k})\} = \{n + \frac{n+3}{2}, \ldots, 2n+1\}$, is a consecutive sequence.
    - If $\frac{n+3}{2} \leq f(a_i) \leq n - f(a_r)$, then $\{f(a_i) + f(a_{i,k})\}$ is consecutive and equals $\left\{n(k+1) + 4 + f(a_r), \ldots, n(k+1) + 4 + \frac{n-3}{2}\right\}$.
    - If $n - f(a_r) + 1 \leq f(a_i) \leq n$, then we see that the set $\{f(a_i) + f(a_{i,k})\} = \left\{n(k+1) + 2, \ldots, n(k+1) + 1 + f(a_r)\right\}$, is consecutive.
- For $u = a_r$ and $v = a_{r,k+1}$, $f(u) + f(v) = n(k+1) + 2 + f(a_r)$.
- If $u = a_{n-r}$ and $v = a_{n-r,k+1}$, then $f(u) + f(v) = n(k+1) + 3 + f(a_r)$.

Therefore, we observe that $\{f(u) + f(v): uv \in E(G(n;k,r))\}$ is a consecutive sequence whose minimum element is $\frac{n+3}{2}$. Hence by Lemma 1, the vertex labeling f extends to a super edge-magic total labeling of $G(n;k,r)$ with magic constant $c(f) = 2n(k+1) + 4 + \frac{n+3}{2}$. Hence,

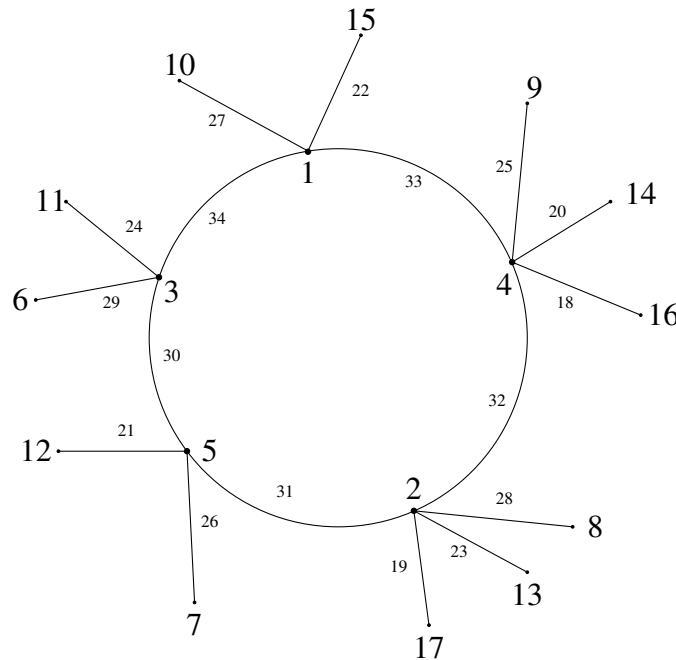$$sm(G(n;k,r)) \leq 2n(k+1) + 4 + \frac{n+3}{2}. \tag{10}$$

From (7) and (10), we have

$$2n(k+1) + 4 + \frac{n+3}{2} \leq sm(G(n;k,r)) \leq 2n(k+1) + 4 + \frac{n+3}{2}.$$

This implies, $sm(G(n;k,r)) = 2n(k+1) + 4 + \frac{n+3}{2}$. □

Example 5. *Super edge-magic total labeling of the graph $G(5;2,1)$ with super edge-magic total strength $sm(G(5;2,1)) = 38$ is illustrated in Figure 5.*

Example 6. *Super edge-magic total labeling of the graph $G(5;2,3)$ with super edge-magic total strength $sm(G(5;2,3)) = 38$ is illustrated in Figure 6.*

Figure 6. The Graph $G(5;2,3)$

5. Conclusions

In this study, we determine the super edge-magic total strength of three variations of $G(n;k_1,\ldots,k_n)$, unicyclic (p,q)-graphs. All three of them have super edge-magic total strength equal to $2q + \frac{n+3}{2}$. These results can be considered as the preliminary steps to provide evidence in proving the Conjecture 1.
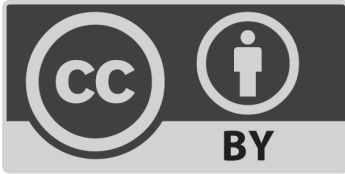
Acknowledgment

Conflict of Interest

The authors declare no conflict of interest

References

1. Bondy, J. A. and Murty, U. S. R., 2008. *Graph Theory*. New York, NY: Springer.

2. Kotzig, A. and Rosa, A., 1970. Magic valuations of finite graphs. *Canadian Mathematical Bulletin*, 13, pp.451–461.

3. Ringel, G. and Lladó, A., 1996. Another tree conjecture. *Bulletin of the Institute of Combinatorics and its Applications*, 18, pp.83–85.

4. Wallis, W. D., 2001. *Magic Graphs*. Boston, MA: Birkhäuser.

5. Babujee, J. B. and Rao, N., 2002. Edge-magic trees. *Indian Journal of Pure and Applied Mathematics*, 33, pp.1837–1840.

6. Gallian, J. A., 2021. A dynamic survey of graph labeling. *Electronic Journal of Combinatorics, 6*(25), pp.4-623.

7. Ngurah, A. A. G. and Baskoro, E. T., 2003. On magic and antimagic total labelings of generalized Petersen graph. *Utilitas Mathematica*, 63, pp.97–107.

8. Slamin, Bača, M., Lin, Y., Miller, M. and Simanjuntak, R., 2002. Edge-magic total labeling of wheels, fans, and friendship graphs. *Bulletin of the Institute of Combinatorics and its Applications*, 35, pp.89–98.

9. López, S. C., Muntaner-Batle, F. A. and Rius-Font, M., 2014. Perfect edge-magic graphs. *Bulletin of the Mathematical Society of the Sciences of Mathematics Roumanie*, 57(105)(1), pp.81–91.

10. López, S. C., Muntaner-Batle, F. A. and Rius-Font, M., 2014. A problem on edge-magic labelings of cycles. *Canadian Mathematical Bulletin*, 57(2), pp.375–380.

11. McQuillan, D., 2009. Edge-magic and vertex-magic total labelings of certain cycles. *Ars Combinatoria*, 91, pp.257–266.

12. Avadayappan, S., Jeyanthi, P. and Vasuki, R., 2000. Magic strength of a graph. *Indian Journal of Pure and Applied Mathematics*, 31(7), pp.873–883.

13. Enomoto, H., Llado, A. S., Nakamigawa, T. and Ringel, G., 1998. Super edge-magic graphs. *SUT Journal of Mathematics*, 34, pp.105–109.

14. Figueroa-Centeno, R. M., Ichishima, R. and Muntaner-Batle, F. A., 2001. The place of super edge-magic labelings among other classes of labelings. *Discrete Mathematics*, 231, pp.153–168.

15. López, S. C., Muntaner-Batle, F. A. and Rius-Font, M., 2013. Labeling constructions using digraph products. *Discrete Applied Mathematics*, 161(18), pp.3005-3016.

16. Figueroa-Centeno, R. M., Ichishima, R., Muntaner-Batle, F. A. and Rius-Font, M., 2008. Labeling generating matrices. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 67, pp.189–216.

17. Figueroa-Centeno, R. M., Ichishima, R. and Muntaner-Batle, F. A., 2002. Magical coronations of graphs. *Australasian Journal of Combinatorics*, 26, pp.199–208.

18. Figueroa-Centeno, R. M., Ichishima, R. and Muntaner-Batle, F. A., 2002. On super edge-magic graphs. *Ars Combinatoria*, 64, pp.81–95.

19. Acharya, B. D. and Hegde, S. M., 1991. Strongly indexable graphs. *Discrete Mathematics*, 93(2-3), pp.123–129.

20. Darmaji, Wahyudi, S., Rinurwati, Saputro, S. W., 2022. On the construction of super edge-magic total graphs. *Electronic Journal of Graph Theory and Applications*, 10(1), pp.301–309.

21. López, S. C., Muntaner-Batle, F. A. and Prabu, M., 2017. Perfect (super) edge-magic crowns. *Results in Mathematics*, 71, pp.1459–1471.

22. López, S. C., Muntaner-Batle, F. A. and Rius-Font, M., 2012. Perfect super edge-magic graphs. *Bulletin of the Mathematical Society of the Sciences of Mathematics Roumanie*, 55(103)(2), pp.199–208.

23. Avadayappan, S., Jeyanthi, P. and Vasuki, R., 2001. Super magic strength of a graph. *Indian Journal of Pure and Applied Mathematics*, 32(11), pp.1621–1630.

24. Swaminathan, V. and Jeyanthi, P., 2006. Super edge-magic strength of fire crackers, banana trees and unicyclic graphs. *Discrete Mathematics*, 306, pp.1624-1636.