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Forbidden trees with diameter six for 3-connected graphs

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ABSTRACT

For a family $\mathcal{F}$ of graphs, a graph G is said to be $\mathcal{F}$-free if G contains no member of $\mathcal{F}$ as an induced subgraph. We let $\mathcal{G}_3(\mathcal{F})$ be the family of 3-connected $\mathcal{F}$ -free graphs. Let P_n and C_n denote the path and the cycle of order n, respectively. Let T_0 be the tree of order nine obtained by joining a pendant edge to the central vertex of P_7. Let T_1 and T_2 be the trees of order ten obtained from T_0 by joining a new vertex to a vertex of P_7 adjacent to an endvertex, and to a vertex of P_7 adjacent to the central vertex, respectively. We show that $\mathcal{G}_3(\{C_3, C_4, T_1\})$ and $\mathcal{G}_3(\{C_3, C_4, T_2\})$ are finite families.

Keywords: forbidden subgraph, forbidden triple, 3-connected graph

2020 Mathematics Subject Classification: 05C75, 05C15.

1. Introduction

By a graph, we mean a finite, simple, undirected graph. Let G be a graph. We let $V(G)$ and $E(G)$ denote the *vertex set* and the *edge set* of G, respectively. For $u \in V(G)$, we let $N_G(u)$ and $\deg_G(u)$ denote the *neighborhood* and the *degree* of u in G, respectively; thus $\deg_G(u) = |N_G(u)|$. We let $\delta(G)$ and $\Delta(G)$ denote the *minimum degree* and the *maximum degree* of G, respectively. For $U \subseteq V(G)$, we set $N_G(U) = \bigcup_{u\in U} N_G(u)$, and let $G[U]$ denote the subgraph of G induced by U. For $U, U' \subseteq V(G)$ with $U \cap U' = \varnothing$, we let $E_G(U, U')$ be the set of edges of G joining a vertex in U and a vertex in U'. When G is connected, for $u, v \in V(G)$, we let $\mathrm{dist}(u, v)$ denote the distance of u and v in G, and let $\mathrm{diam}(G)$ denote the maximum of $\mathrm{dist}(u, v)$ as u and v range over $V(G)$. We let C_n and K_n denote the cycle and the complete graph of order n, respectively. We let K_{m_1,m_2}

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denote the complete bipartite graph with partite sets having cardinalities m_1 and m_2, respectively. For terms and symbols not defined here, we refer the reader to Diestel [2].

Let $n \geq 5$ be an integer, and let I, J be subsets of $\{2, 3, \cdots, n-1\}$ with $J \subseteq \{3, \cdots, n-2\}$ and $I \cap J = \varnothing$. We let $S_n(I, J)$ denote the tree obtained from a path $u_1u_2 \cdots u_n$ of order n by adding vertices v_i $(i \in I \cup J)$ and v_i' $(i \in J)$ and edges u_iv_i $(i \in I \cup J)$ and v_iv_i' $(i \in J)$. Also we let S^* denote the tree obtained from a path $u_1u_2 \cdots u_7$ of order 7 by adding vertices v_4, v_4', v_4'' and edges u_4v_4, v_4v_4', v_4v_4'' (see Figure 1).

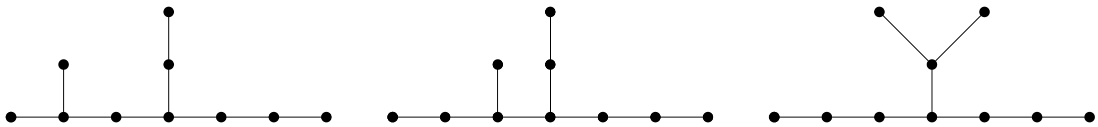

Fig. 1. Trees $S_n(I, J)$ and S^*

For two graphs G and H, we say that G is H-free if G does not contain an induced copy of H. For a family $\mathcal{F}$ of connected graphs, a graph G is said to be $\mathcal{F}$-free if G is H-free for every $H \in \mathcal{F}$. For an integer $k \geq 2$ and a family $\mathcal{F}$ of connected graphs, let $\mathcal{G}_k(\mathcal{F})$ denote the family of k-connected $\mathcal{F}$-free graphs. In this context, members of $\mathcal{F}$ are often referred to as forbidden subgraphs.

Let $k \geq 2$ be an integer. In this paper, we consider families $\mathcal{F}$ of connected graphs such that

$$\mathcal{G}_k(\mathcal{F}) \text{ is a finite family.} \tag{1}$$

Note that if a family $\mathcal{F}$ satisfies (1), then for any property P on graphs, although the proposition that all k-connected $\mathcal{F}$-free graphs satisfy P with finite exceptions holds, the proposition gives no information about P. Thus it is important to identify families $\mathcal{F}$ satisfying (1) in advance. With such a motivation, studies of $\mathcal{F}$ satisfying (1) have been started by Fujisawa, Plummer and Saito in [5]. In particular, it is known that if a finite family $\mathcal{F}$ of connected graphs satisfies (1), then $\mathcal{F}$ contains a complete graph, a complete bipartite graph and a tree. Based on this result, families $\mathcal{F}$ satisfying (1) which are written in the form $\mathcal{F} = \{K_n, K_{m_1,m_2}, T\}$ where $n \geq 3$, $2 \leq m_1 \leq m_2$ and T is a tree, have intensively been studied. For $k = 2$, such families are completely characterized in [5]. For $k = 3$, such families are characterized except for the case where $n = 3$ and $m_1 = m_2 = 2$ (see the references in [4]). For other related results, we refer the reader to Buelban et al. [1], Furuya and Okubo [6], Kotani [8], and Kotani and Nishiyama [7]. This paper is concerned with the case where $n = 3$ and $m_1 = m_2 = 2$.

The following conjecture is proposed in [4] (note that $K_3 = C_3$ and $K_{2,2} = C_4$).

Conjecture 1.1. *Let T be a tree. Then $\mathcal{G}_3(\{C_3, C_4, T\})$ is finite if and only if T is a subgraph of one of $S_9(\{2\}, \varnothing)$, $S_9(\{5\}, \varnothing)$, $S_9(\varnothing, \{3\})$, $S_8(\{2, 5\}, \varnothing)$, $S_8(\{7\}, \{3\})$, $S_8(\{4, 5, 6\}, \{3\})$, $S_8(\{4\}, \{3, 6\})$, $S_7(\{2\}, \{4\})$, $S_7(\{3\}, \{4\})$, $S_7(\{4\}, \{3, 5\})$, S^* and $S_6(\varnothing, \{3, 4\})$.*

The 'only if' part of the conjecture is proved in [4]. As for the 'if' part, it is proved in [3, 4] that $\mathcal{G}_3(\{C_3, C_4, T\})$ is finite if

$T = S_9(\{2\},\varnothing), S_9(\{5\},\varnothing), S_9(\varnothing,\{3\}), S_8(\{2,5\},\varnothing)$ or $S_8(\{7\},\{3\})$. In this paper, we prove the following theorem.

Theorem 1.2. *The families $\mathcal{G}_3(\{C_3, C_4, S_7(\{2\},\{4\})\})$ and $\mathcal{G}_3(\{C_3, C_4, S_7(\{3\},\{4\})\})$ are finite families.*

Together with the results mentioned above, this reduces Conjecture 1.1 to the following conjecture.

Conjecture 1.3. *Let T be a tree isomorphic to $S_8(\{4,5,6\},\{3\})$, $S_8(\{4\},\{3,6\})$, $S_7(\{4\},\{3,5\})$, S^* or $S_6(\varnothing,\{3,4\})$. Then $\mathcal{G}_3(\{C_3, C_4, T\})$ is a finite family.*

The following lemma is well-known (see [2, Proposition 1.3.3]).

Lemma 1.4. *Let $m \geq 2$ and $d \geq 3$ be integers, and let G be a graph with $\Delta(G) \leq m$ and $\mathrm{diam}(G) \leq d$. Then $|V(G)| \leq m^d$.*

Having Lemma 1.4 in mind, we establish Theorem 1.2 by bounding the diameter and the maximum degree of $\{C_3, C_4, T\}$-free graphs for $T = S_7(\{2\},\{4\})$ and $T = S_7(\{3\},\{4\})$. Specifically, we prove the following three propositions.

Proposition 1.5. *Let G be a 3-connected $\{C_3, C_4, T\}$-free graph, where $T = S_7(\{2\},\{4\})$ or $S_7(\{3\},\{4\})$. Then $\mathrm{diam}(G) \leq 24$.*

Proposition 1.6. *Let G be a 3-connected $\{C_3, C_4, S_7(\{2\},\{4\})\}$-free graph. Then $\Delta(G) < 3 \cdot 10^4$.*

Proposition 1.7. *Let G be a 3-connected $\{C_3, C_4, S_7(\{3\},\{4\})\}$-free graph. Then $\Delta(G) < 4.1 \cdot 10^5$.*

We prove Proposition 1.5 in Sections 2 and 3, and prove Propositions 1.6 and 1.7 in Sections 4, 5 and 6. We here mention that unlike in [3] or [4], bounding the diameter of graphs in $\mathcal{G}_3(\{C_3, C_4, S_7(\{2\},\{4\})\}) \cup \mathcal{G}_3(\{C_3, C_4, S_7(\{3\},\{4\})\})$ is nontrivial. We add that in Section 4, we use the fact that $R(3,3) = 6$ and $R(3,4) = 9$, where $R(s,t)$ denotes the usual Ramsey number, i.e., the minimum positive integer R such that any graph of order at least R contains a complete subgraph of order s or an independent set of cardinality t. We here state a corollary of a famous theorem of Turán [10], which appears as Theorem 7.1.1 in [2]. Let H be a graph, let k be the maximum cardinality of an independent set of H, and write $n = kq + r$, where q, r are integers with $0 \leq r \leq k-1$. Turán's theorem shows that $|E(H)| \geq r|E(K_{q+1})| + (k-r)|E(K_q)|$. By calculations, we obtain $2|E(H)| \geq |V(H)|((|V(H)|/k) - 1)$; i.e., if H has average degree at most d, then $d \geq (|V(H)|/k) - 1$. This shows that the following lemma holds (see also Nagayama [9]).

Lemma 1.8. *Let H be a graph with average degree at most d. Then H contains an* *independent set with cardinality greater than or equal to* $|V(H)|/(d+1)$.

2. Diameter

Throughout the rest of this paper, for simplicity, we let $S_7(\{2\},\{4\})$ and $S_7(\{3\},\{4\})$ be denote by T_1 and T_2, respectively.

In this section and the following section, we let G denote a 3-connected $\{C_3, C_4\}$-free graph with $\mathrm{diam}(G) \geq 25$. Let $d = \mathrm{diam}(G)$, take $u, v \in V(G)$ with $\mathrm{dist}(u, v) = d$, and let $P = u_0u_1 \cdots u_d$ be a shortest u-v path. The first two lemmas follow immediately from the fact that G is $\{C_3, C_4\}$-free and P is a shortest u-v path (we will make use of Lemma 2.1 without referring to it explicitly).

Lemma 2.1. *Let* $a \in N_G(V(P)) - V(P)$. *Then* $|N_G(a) \cap V(P)| = 1$.

Lemma 2.2. *Let* $0 \leq i, i' \leq d$, $a_i \in N_G(u_i) - V(P)$ *and* $a_{i'} \in N_G(u_{i'}) - V(P)$, *and suppose that* $a_ia_{i'} \in E(G)$. *Then* $|i - i'| = 2$ *or* 3.

Lemma 2.3. *Let* $0 \leq i, i' \leq d$, *and let* $a \in N_G(u_i) - V(P)$.

(i) *Let* $i \geq i' + 3$, *and suppose that* $\mathrm{dist}(u_{i'}, a) \leq i - i' - 1$. *Then for each* $y \in (N_G(a) - \{u_i\}) \cap N_G(V(P))$, *we have* $y \in N_G(u_{i-3})$ *or* $y \in N_G(u_{i-2})$.

(ii) *Let* $i \leq i' - 3$, *and suppose that* $\mathrm{dist}(u_{i'}, a) \leq i' - i - 1$. *Then for each* $y \in (N_G(a) - \{u_i\}) \cap N_G(V(P))$, *we have* $y \in N_G(u_{i+2})$ *or* $y \in N_G(u_{i+3})$.

Proof. Assume that $i \geq i' + 3$ and $\mathrm{dist}(u_{i'}, a) \leq i - i' - 1$. Let $y \in (N_G(a) - \{u_i\}) \cap V(P)$, and write $N_G(y) \cap V(P) = \{u_j\}$. By Lemma 2.2, $j \in \{i-3, i-2, i+2, i+3\}$. If $j \in \{i+2, i+3\}$, then we get $\mathrm{dist}(u_{i'}, u_j) \leq \mathrm{dist}(u_{i'}, a) + \mathrm{dist}(a, u_j) \leq (i - i' - 1) + 2 = (i+1) - i' < j - i'$ by assumption, which contradicts the fact that $\mathrm{dist}(u_{i'}, u_j) = j - i'$. Thus $j \in \{i-3, i-2\}$, which proves (i). By symmetry, (ii) is verified in a similar way. □

We now prove lemmas concerning the case where G is T_1-free.

Lemma 2.4. *Suppose that* G *is* T_1*-free. Let* $6 \leq i \leq d - 3$, $a_{i-3} \in N_G(u_{i-3}) - V(P)$ *and* $a_i \in N_G(u_i) - V(P)$, *and suppose that* $a_{i-3}a_i \in E(G)$. *Then the following hold.*

(i) *We have* $|N_G(u_{i-2}) - V(P)| = |N_G(a_i) - \{u_i, a_{i-3}\}| = 1$ *and* $N_G(u_{i-2}) - V(P) = N_G(a_i) - \{u_i, a_{i-3}\}$.

(ii) *We have* $|N_G(u_{i-1}) - V(P)| = |N_G(a_{i-3}) - \{u_{i-3}, a_i\}| = 1$ *and* $N_G(u_{i-1}) - V(P) = N_G(a_{i-3}) - \{u_{i-3}, a_i\}$.

Proof. Let x be an arbitrary vertex in $N_G(u_{i-2}) - V(P)$. Let $a_{i+2} \in N_G(u_{i+2}) - V(P)$. Since G is T_1-free, $\{x, u_{i-2}, u_{i-1}, u_i, a_i, a_{i-3}, u_{i+1}, u_{i+2}, a_{i+2}, u_{i+3}\}$ does not induce a copy of T_1. In view of Lemma 2.2, this implies $\{xa_i, a_ia_{i+2}\} \cap E(G) \neq \emptyset$. Applying Lemma 2.3 (i) with $i' = i - 3$ and $y = a_{i+2}$, we also get $a_{i+2} \notin N_G(a_i)$. Hence $xa_i \in E(G)$. Recall that G is $\{C_3, C_4\}$-free. Since $x \in N_G(u_{i-2}) - V(P)$ is arbitrary, it follows that $N_G(u_{i-2}) - V(P) = \{x\}$. Note that $Q = u_d \cdots u_ia_ia_{i-3}u_{i-3} \cdots u_0$ is a shortest v-u path.

Hence applying the above argument with P, u_{i-3} and u_i replaced by Q, u_i and u_{i-3}, respectively, we obtain $N_G(a_i) - V(Q) = \{x\}$. Since $N_G(a_i) - V(Q) = N_G(a_i) - \{u_i, a_{i-3}\}$, this proves (i). Applying (i) to $u_0 \cdots u_{i-3}a_{i-3}a_iu_i \cdots u_d$, we see that (ii) holds. □

Lemma 2.5. *Suppose that G is T_1-free, let $0 \leq i, i' \leq d$, and let $a_i \in N_G(u_i) - V(P)$.*

(i) *Let $i' + 3 \leq i \leq d - 5$, and suppose that* $\mathrm{dist}(u_{i'}, a_i) \leq i - i' - 1$. *Then $N_G(a_i) \subseteq N_G(V(P))$.*

(ii) *Let $5 \leq i \leq i' - 3$, and suppose that* $\mathrm{dist}(u_{i'}, a_i) \leq i' - i - 1$. *Then $N_G(a_i) \subseteq N_G(V(P))$.*

Proof. Assume that $i' + 3 \leq i \leq d - 5$ and $\mathrm{dist}(u_{i'}, a_i) \leq i - i' - 1$. Suppose that $N_G(a_i) \not\subseteq N_G(V(P))$, and take $b \in N_G(a_i) - N_G(V(P))$. Let $a_{i+2} \in N_G(u_{i+2}) - V(P)$. Applying Lemma 2.3 (i) with $y = a_{i+2}$, we get $a_{i+2} \notin N_G(a_i)$. Since

$$\{u_{i-3}, u_{i-2}, u_{i-1}, u_i, a_i, b, u_{i+1}, u_{i+2}, a_{i+2}, u_{i+3}\},$$

does not induce a copy of T_1, it follows that $ba_{i+2} \in E(G)$. Let $a_{i+4} \in N_G(u_{i+4}) - V(P)$. Since

$$\{u_{i-1}, u_i, u_{i+1}, u_{i+2}, a_{i+2}, b, u_{i+3}, u_{i+4}, a_{i+4}, u_{i+5}\},$$

does not induce T_1, we see that $\{ba_{i+4}, a_{i+2}a_{i+4}\} \cap E(G) \neq \varnothing$. Consequently

$$\begin{aligned}\mathrm{dist}(u_{i'}, u_{i+4}) &\leq \mathrm{dist}(u_{i'}, a_i) + \mathrm{dist}(a_i, b) + \mathrm{dist}(b, a_{i+4}) + \mathrm{dist}(a_{i+4}, u_{i+4}) \\ &\leq (i - i' - 1) + 1 + 2 + 1 = i - i' + 3,\end{aligned}$$

a contradiction. This proves (i), and (ii) is verified in a similar way. □

We proceed to the case where G is T_2-free.

Lemma 2.6. *Suppose that G is T_2-free. Let $4 \leq i \leq d - 4$, and let $a_i \in N_G(u_i) - V(P)$. Then $N_G(a_i) \subseteq N_G(V(P))$.*

Proof. Since $d \geq 9$, we have $i \geq 5$ or $i \leq d - 5$. By symmetry, we may assume that $i \geq 5$. Suppose that $N_G(a_i) \not\subseteq N_G(V(P))$, and take $b \in N_G(a_i) - N_G(V(P))$. For each $j \in \{i-3, i-2, i-1, i+1, i+2\}$, let $a_j \in N_G(u_j) - V(P)$. Since

$$\{u_{i-3}, u_{i-2}, u_{i-1}, u_i, a_i, b, u_{i+1}, a_{i+1}, u_{i+2}, u_{i+3}\},$$

does not induce T_2, it follows from Lemma 2.2 that $ba_{i+1} \in E(G)$. Since $i + 1 \leq d - 3$, applying the same argument with i and $i + 1$ replaced by $i + 1$ and $i + 2$, respectively, we get $ba_{i+2} \in E(G)$. Since $i - 2 \geq 3$, we similarly get $ba_{i-1} \in E(G)$, $ba_{i-2} \in E(G)$, $ba_{i-3} \in E(G)$. Consequently $u_{i-3}a_{i-3}ba_{i+2}u_{i+2}$ is a $u_{i-3} - u_{i+2}$ path of length four, a contradiction. □

The following lemma shows that the conclusion of Lemma 2.4 holds for T_2-free graphs as well.

Lemma 2.7. *Suppose that G is T_2-free. Let $7 \le i \le d-4$, $a_{i-3} \in N_G(u_{i-3}) - V(P)$ and $a_i \in N_G(u_i) - V(P)$, and suppose that $a_{i-3}a_i \in E(G)$. Then the following hold.*

(i) *We have $|N_G(u_{i-2}) - V(P)| = |N_G(a_i) - \{u_i, a_{i-3}\}| = 1$ and $N_G(u_{i-2}) - V(P) = N_G(a_i) - \{u_i, a_{i-3}\}$.*

(ii) *We have $|N_G(u_{i-1}) - V(P)| = |N_G(a_{i-3}) - \{u_{i-3}, a_i\}| = 1$ and $N_G(u_{i-1}) - V(P) = N_G(a_{i-3}) - \{u_{i-3}, a_i\}$.*

Proof. Let $x \in N_G(a_i) - \{u_i, a_{i-3}\}$. By Lemma 2.6, $x \in N_G(V(P))$. Applying Lemma 2.3 (i) with $i' = i-3$, we get $x \in N_G(u_{i-3}) \cup N_G(u_{i-2})$. Since $a_{i-3} \in N_G(u_{i-3}) \cap N_G(a_i)$ and G is $\{C_3, C_4\}$-free, we have $x \notin N_G(u_{i-3})$. Hence $x \in N_G(u_{i-2})$. Since $x \in N_G(a_i) - \{u_i, a_{i-3}\}$ is arbitrary and G is $\{C_3, C_4\}$-free, it follows that $N_G(a_i) - \{u_i, a_{i-3}\} = \{x\}$. Applying the above argument with P, u_{i-3} and u_i replaced by $u_d \cdots u_i a_i a_{i-3} u_{i-3} \cdots u_0$, u_i and u_{i-3}, respectively, we also obtain $N_G(u_{i-2}) - V(P) = \{x\}$. Thus (i) is proved. Applying (i) to $u_0 \cdots u_{i-3} a_{i-3} a_i u_i \cdots u_d$, we see that (ii) holds. □

3. Proof of Proposition 1.5

In this section, we prove Proposition 1.5. Thus let $T = T_1$ or T_2, and let G be a 3-connected $\{C_3, C_4, T\}$-free graph and, by way of contradiction, suppose that $d = \mathrm{diam}(G) \ge 25$. Let $P = u_0 \cdots u_d$ be as in Section 2. We derive a contradiction by proving several claims.

Claim 1. Let $9 \le i \le d-5$ and, for $j \in \{i-4, i-2, i\}$, let $a_j \in N_G(u_j) - V(P)$. Then $a_{i-4}a_{i-2} \notin E(G)$ or $a_{i-2}a_i \notin E(G)$.

Proof. Suppose that $a_{i-4}a_{i-2}, a_{i-2}a_i \in E(G)$. Then $\mathrm{dist}(u_{i-4}, a_i) \le 3$. Take $b_i \in N_G(a_i) - \{u_i, a_{i-2}\}$. By Lemma 2.5 (i) or 2.6, $b_i \in N_G(V(P))$. Hence by Lemma 2.3 (i), $b_i \in N_G(u_{i-3}) \cup N_G(u_{i-2})$. Since $a_{i-2} \in N_G(u_{i-2}) \cap N_G(a_i)$ and G is $\{C_3, C_4\}$-free, $b_i \notin N_G(u_{i-2})$. Hence $b_i \in N_G(u_{i-3})$. Now take $b_{i-4} \in N_G(a_{i-4}) - \{u_{i-4}, a_{i-2}\}$. It follows from Lemma 2.5 (ii) or 2.6 and Lemma 2.3 (ii) that $b_{i-4} \in N_G(u_{i-1})$. Since $b_i \in N_G(u_{i-3})$, $a_i \in N_G(u_i)$, $b_i a_i \in E(G)$, $b_{i-4} \in N_G(u_{i-1})$ and $a_{i-2} \in N_G(u_{i-2})$, it follows from (ii) of Lemma 2.4 or 2.7 that $b_i b_{i-4} \in E(G)$, $N_G(u_{i-1}) = \{u_{i-2}, u_i, b_{i-4}\}$ and $N_G(b_i) = \{u_{i-3}, a_i, b_{i-4}\}$, and it follows from (i) of Lemma 2.4 or 2.7 that $N_G(u_{i-2}) = \{u_{i-3}, u_{i-1}, a_{i-2}\}$ and $N_G(a_i) = \{u_i, a_{i-2}, b_i\}$ (see Figure 2).

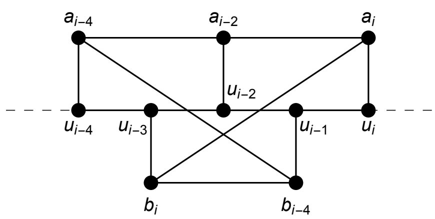

Fig. 2. Claim 1

Similarly, applying Lemma 2.4 or 2.7 with u_{i-3} and u_i replaced by u_{i-4} and u_{i-1}, we also get $N_G(a_{i-4}) = \{u_{i-4}, a_{i-2}, b_{i-4}\}$, $N_G(u_{i-3}) = \{u_{i-4}, u_{i-2}, b_i\}$ and $N_G(b_{i-4}) = \{u_{i-1}, a_{i-4}, b_i\}$. Note that $Q = u_0 \cdots u_{i-3} b_i a_i u_i \cdots u_d$ is a shortest u-v path, and we have $a_{i-4} \in N_G(u_{i-4}) - V(Q)$, $a_{i-2} \in N_G(a_i) - V(Q)$ and $a_{i-4}a_{i-2} \in E(G)$. Consequently, applying (i) of Lemma 2.4 or 2.7 with P, u_{i-3} and u_i replaced by Q, u_{i-4} and a_i, we obtain $N_G(a_{i-2}) = \{u_{i-2}, a_{i-4}, a_i\}$. Therefore $\{u_{i-4}, u_i\}$ separates $\{u_{i-3}, u_{i-2}, u_{i-1}, a_{i-4}, a_{i-2}, a_i, b_{i-4}, b_i\}$ from the rest. This contradicts the assumption that G is 3-connected, which completes the proof of Claim 1. □

We here prove two technical claims concerning the case where $T = T_1$.

Claim 2. Let $T = T_1$. Let $4 \leq i \leq d-2$. For $j \in \{i-4, i-2, i, i+2\}$, let $a_j \in N_G(u_j) - V(P)$, and suppose that $a_{i-2}a_i \in E(G)$.

(i) Let $7 \leq i \leq d-7$. Suppose that $N_G(a_i) \not\subseteq N_G(P)$, and let $b \in N_G(a_i) - N_G(P)$. Then $ba_{i+2} \in E(G)$.

(ii) Let $9 \leq i \leq d-5$. Suppose that $N_G(a_{i-2}) \not\subseteq N_G(P)$, and let $b \in N_G(a_{i-2}) - N_G(P)$. Then $ba_{i-4} \in E(G)$.

Proof. Assume that $7 \leq i \leq d-7$ and $N_G(a_i) \not\subseteq N_G(P)$, and let $b \in N_G(a_i) - V(P)$. Since $\{u_{i-3}, u_{i-2}, u_{i-1}, u_i, a_i, b, u_{i+1}, u_{i+2}, a_{i+2}, u_{i+3}\}$ does not induce T_1, $\{a_ia_{i+2}, ba_{i+2}\} \cap E(G) \neq \varnothing$. By Claim 1, $a_ia_{i+2} \notin E(G)$. Hence $ba_{i+2} \in E(G)$. This proves (i), and (ii) is verified in a similar way. □

Claim 3. Let $T = T_1$. Let $12 \leq i \leq d-10$, $a_{i-2} \in N_G(u_{i-2}) - V(P)$, and $a_i \in N_G(u_i) - V(P)$, and suppose that $a_{i-2}a_i \in E(G)$. Then $N_G(a_{i-2}) \subseteq N_G(V(P))$ and $N_G(a_i) \subseteq N_G(V(P))$.

Proof. Suppose that $N_G(a_i) \not\subseteq N_G(V(P))$, and take $b_i \in N_G(a_i) - N_G(V(P))$. For $j \in \{i+2, i+3, i+4, i+5\}$, let $a_j \in N_G(u_j) - V(P)$. By Claim 2 (i), $b_ia_{i+2} \in E(G)$. Since $\{u_{i-1}, u_i, u_{i+1}, u_{i+2}, a_{i+2}, b_i, u_{i+3}, u_{i+4}, a_{i+4}, u_{i+5}\}$ does not induce T_1, $\{a_{i+2}a_{i+4}, b_ia_{i+4}\} \cap E(G) \neq \varnothing$. If $b_ia_{i+4} \in E(G)$, then $\mathrm{dist}(u_{i-2}, u_{i+4}) \leq 5$, a contradiction. Thus $b_ia_{i+4} \notin E(G)$. Hence $a_{i+2}a_{i+4} \in E(G)$, which implies $\mathrm{dist}(u_{i-2}, a_{i+4}) \leq 5$. Take $b_{i+4} \in N_G(a_{i+4}) - \{u_{i+4}, a_{i+2}\}$. By Lemma 2.5 (i), $b_{i+4} \in N_G(V(P))$. Hence by Lemma 2.3 (i), $b_{i+4} \in N_G(u_{i+1}) \cup N_G(u_{i+2})$. Since $a_{i+2} \in N_G(u_{i+2}) \cap N_G(a_{i+4})$ and G is $\{C_3, C_4\}$-free, $b_{i+4} \notin N_G(u_{i+2})$. Consequently $b_{i+4} \in N_G(u_{i+1})$. Applying Lemma 2.4 (ii) with $u_{i-3}a_{i-3}a_iu$ replaced by $u_{i+1}b_{i+4}a_{i+4}u_{i+4}$, we see that $a_{i+3}b_{i+4} \in E(G)$ (see Figure 3).

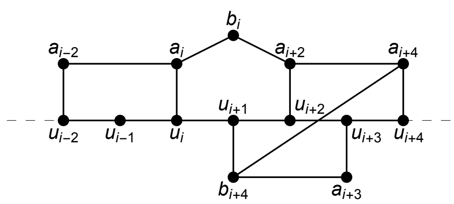

Fig. 3. Claim 3

Note that $Q = u_0 \cdots u_{i+1}b_{i+4}a_{i+4}u_{i+4} \cdots u_d$ is shortest u-v path. We have already shown that $b_ia_{i+4} \notin E(G)$. If $b_ib_{i+4} \in E(G)$, then we get a conradiction by applying Claim 1 with P, u_{i-4}, u_{i-2} and u_i replaced by Q, u_{i-2}, u_i and b_{i+4}, respectively. Thus $b_ib_{i+4} \notin E(G)$. Hence $b_i \notin N_G(V(Q))$. Note that $a_{i+3} \in N_G(b_{i+4})$. Applying Claim 2 (i) to Q, we now see that $b_i a_{i+3} \in E(G)$. We already know that $b_{i+4} \in N_G(u_{i+1}) - V(P)$, $a_{i+3} \in N_G(u_{i+3}) - V(P)$ and $b_{i+4}a_{i+3} \in E(G)$. We have just shown that $b_i a_{i+3} \in E(G)$. Since $b_i \notin N_G(V(P))$ by the choice of b_i, it follows that $b_i \in N_G(a_{i+3}) - N_G(V(P))$. Therefore we can apply Claim 2 (i) with a_{i-2}, a_i and b replaced by b_{i+4}, a_{i+3} and b_i, respectively, to obtain $b_i a_{i+5} \in E(G)$, which implies $\mathrm{dist}(u_{i-2}, u_{i+5}) \leq 5$, a contradiction. Thus $N_G(a_i) \subseteq N_G(V(P))$. By symmetry, we similarly obtain $N_G(a_{i-2}) \subseteq N_G(V(P))$. □

Claim 4. Let $12 \leq i \leq d-10$, and let $a_{i-2} \in N_G(u_{i-2}) - V(P)$ and $a_i \in N_G(u_i) - V(P)$. Then $a_{i-2}a_i \notin E(G)$.

Proof. Suppose that $a_{i-2}a_i \in E(G)$. Take $b_i \in N_G(a_i) - \{u_i, a_{i-2}\}$. By Claim 3 or Lemma 2.6, $b_i \in N_G(V(P))$. Hence $b_i \in N_G(\{u_{i-3}, u_{i-2}, u_{i+2}, u_{i+3}\})$ by Lemma 2.2. If $b_i \in N_G(u_{i+3})$, we get a contradiction by applying Lemma 2.3 (ii) with $i' = i+3$. Thus $b_i \notin N_G(u_{i+3})$. By Claim 2 $b_i \notin N_G(u_{i+2})$. Since $a_{i-2} \in N_G(u_{i-2}) \cap N_G(a_i)$ and G is $\{C_3, C_4\}$-free, $b_i \notin N_G(u_{i-2})$. Consequently $b_i \in N_G(u_{i-3})$. Let $a_{i-1} \in N_G(u_{i-1}) - V(P)$. By (ii) of Lemma 2.4 or 2.7, $a_{i-1}b_i \in E(G)$. Now take $b_{i-2} \in N_G(a_{i-2}) - \{u_{i-2}, a_i\}$. It follows from Claim 3 or Lemma 2.6, Lemma 2.2, Lemma 2.3 (i) and Claim 2 that $b_{i-2} \in N_G(u_{i+1})$. Hence by (i) of Lemma 2.4 or 2.7, $a_{i-1}b_{i-2} \in E(G)$. Therefore $b_i \in N_G(u_{i-3}) - V(P)$, $a_{i-1} \in N_G(u_{i-1}) - V(P)$, $b_{i-2} \in N_G(u_{i+1}) - V(P)$ and $b_i a_{i-1}, a_{i-1}b_{i-2} \in E(G)$, which contradicts Claim 1, completing the proof of the claim. □

Claim 5. Let $12 \leq i \leq d-10$, and let $a_{i-3} \in N_G(u_{i-3}) - V(P)$ and $a_i \in N_G(u_i) - V(P)$. Then $a_{i-3}a_i \notin E(G)$.

Proof. Suppose that $a_{i-3}a_i \in E(G)$. Let $a_{i-2} \in N_G(u_{i-2}) - V(P)$. By (i) of Lemma 2.4 or 2.7, we get $a_{i-2}a_i \in E(G)$, which contradicts Claim 4. □

We can now complete the proof of Proposition 1.5. Let $a_{12} \in N_G(u_{12}) - V(P)$. Take $b, b' \in N_G(a_{12}) - \{u_{12}\}$ with $b \neq b'$. By Claim 4, $b, b' \notin N_G(\{u_{10}, u_{14}\})$. By Claim 5, $b, b' \notin N_G(\{u_9, u_{15}\})$. Hence $b, b' \notin N_G(V(P))$ by Lemma 2.2. If $T = T_2$, this contradicts Lemma 2.6. Thus $T = T_1$. Let $a_{10} \in N_G(u_{10}) - V(P)$. Since G is $\{C_3, C_4\}$-free, we have $a_{10}b \notin E(G)$ or $a_{10}b' \notin E(G)$. We may assume that $a_{10}b \notin E(G)$. By Claim 4, $a_{10}a_{12} \notin E(G)$. Therefore $\{u_9, u_{10}, a_{10}, u_{11}, u_{12}, a_{12}, b, u_{13}, u_{14}, u_{15}\}$ induces a copy of T_1, which is a contradiction.

This completes the proof of Proposition 1.5.

4. Paths of order four

Throughout the rest of this paper, we fix a 3-connected $\{C_3, C_4\}$-free graph G and, for $u \in V(G)$ and $U \subseteq V(G)$, we write $N(u)$ and $N(U)$ for $N_G(u)$ and $N_G(U)$.

In this section and the following section, we study relation between induced paths joining two given vertices and the existence of an induced tree. For an integer $k \geq 4$ and two nonadjacent vertices v, w of G, we let

$$M_k^w(v)$$
$$= \{x \in N(v) | \text{there exists an induced } v\text{-}w \text{ path } P \text{ of order } k \text{ such that } N_P(v) = \{x\}\}.$$

In the remainder of this section and the following section, we let v, w be nonadjacent vertices of G. In this section, we deal with the case where $M_4^w(v)$ is large. We first consider T_1.

Lemma 4.1. *Suppose that* $|M_4^w(v)| \geq 16$. *Then* G *contains an induced copy of* T_1.

Proof. Take $a_1, \cdots, a_{16} \in M_4^w(v)$. For each $i \in \{1, \cdots, 16\}$, let va_ib_iw be an induced v-w path. Since G is $\{C_3, C_4\}$-free, $\{a_i, b_i\} \cap \{a_j, b_j\} = \varnothing$ for any i, j with $i \neq j$, and

$$E(G[\{v, w\} \cup \{a_i, b_i | 1 \leq i \leq 16\}]) = \{va_i, a_ib_i, b_iw | 1 \leq i \leq 16\}. \quad (2)$$

For each $i \in \{1, \cdots, 16\}$, take $x_i \in N(a_i) - \{v, b_i\}$. By (2), $\{x_i | 1 \leq i \leq 16\} \cap (\{v, w\} \cup \{a_i, b_i | 1 \leq i \leq 16\}) = \varnothing$. Since G is $\{C_3, C_4\}$-free, $x_1, \cdots, x_{16}$ are distinct,

$$x_ia_j \notin E(G) \text{ for any } i, j \in \{1, \cdots, 16\} \text{ with } i \neq j, \quad (3)$$

and

$$x_iv, x_iw, x_ib_i \notin E(G) \text{ for every } i \in \{1, \cdots, 16\}. \quad (4)$$

Now let D be the digraph on $\{1, \cdots, 16\}$ obtained by joining i to j $(i \neq j)$ if and only if $x_ib_j \in E(G)$, and let H be the (simple) graph obtain by ignoring the direction of the edges of D. Since G is $\{C_3, C_4\}$-free, each $i \in \{1, \cdots, 16\}$ has outdegree at most one in D. Hence $|E(H)| \leq |V(H)|$, which means that the average degree of H is at most two. In view of Lemma 1.8, we may assume that $\{1, \cdots, 6\}$ is independent in H. Then by (3) and (4),

$$E_G(\{x_i | 1 \leq i \leq 16\}, \{v, w\} \cup \{a_i | 1 \leq i \leq 16\} \cup \{b_i | 1 \leq i \leq 6\}) = \{x_ia_i | 1 \leq i \leq 6\}. \quad (5)$$

Note that $R(3, 3) = 6$. Since G is $\{C_3, C_4\}$-free, we may assume that

$$\{x_1, x_2, x_3\} \text{ is independent.} \quad (6)$$

Since G is $\{C_3, C_4\}$-free, we can take $z_1 \in N(x_1) - \{a_1\}$ so that $z_1w \notin E(G)$. By (5) and (6), $z_1 \notin \{w, v, x_1, x_2, x_3\} \cup \{a_i | 1 \leq i \leq 16\} \cup \{b_i | 1 \leq i \leq 6\}$. Since G is $\{C_3, C_4\}$-free, we have $z_1a_2 \notin E(G)$ or $z_1a_3 \notin E(G)$. We may assume that $z_1a_2 \notin E(G)$. Since G is $\{C_3, C_4\}$-free, it follows that

$$z_1v, z_1w, z_1a_1, z_1a_2, z_1b_1 \notin E(G). \quad (7)$$

Since G is $\{C_3, C_4\}$-free, $|N_G(z_1) \cap \{b_4, b_5, b_6\}| \leq 1$ and $|N_G(z_1) \cap \{a_7, a_8\}| \leq 1$. We may assume that $z_1b_4, z_1b_5, z_1a_7 \notin E(G)$. Now if $z_1x_2 \in E(G)$, then it follows from (2), (5), (6) and (7) that $\{x_2, z_1, x_1, a_1, v, a_7, b_1, w, b_4, b_5\}$ induces a copy of T_1; if $z_1x_2 \notin E(G)$, then it follows from (2), (5), (6) and (7) that $\{x_2, a_2, v, a_1, x_1, z_1, b_1, w, b_4, b_5\}$ induces a copy of T_1 (see Figure 4). This completes the proof of Lemma 4.1.

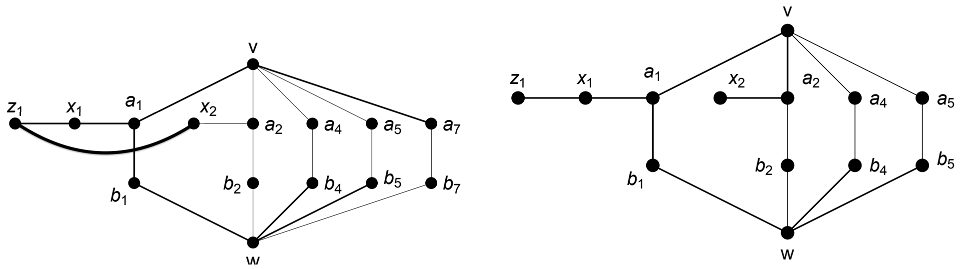

Fig. 4. Lemma 4.1

□

Next we consider T_2.

Lemma 4.2. *Suppose that* $|M_4^w(v)| \geq 41$. *Then* G *contains an induced copy of* T_2.

Proof. Take $a_1, \cdots, a_{41} \in M_4^w(v)$. For each $i \in \{1, \cdots, 41\}$, let va_ib_iw be an induced v-w path. Since G is $\{C_3, C_4\}$-free, $\{a_i, b_i\} \cap \{a_j, b_j\} = \varnothing$ for any i, j with $i \neq j$, and

$$E(G[\{v, w\} \cup \{a_i, b_i | 1 \leq i \leq 41\}]) = \{va_i, a_ib_i, b_iw | 1 \leq i \leq 41\}. \tag{8}$$

For each $i \in \{1, \cdots, 41\}$, take $x_i \in N(a_i) - \{v, b_i\}$ and $y_i \in N(b_i) - \{w, a_i\}$. By (2), $\{x_i, y_i | 1 \leq i \leq 41\} \cap (\{v, w\} \cup \{a_i, b_i | 1 \leq i \leq 41\}) = \varnothing$. Since G is $\{C_3, C_4\}$-free, $x_i \neq y_i$ for each i, $x_1, \cdots, x_{41}$ are distinct, $y_1, \cdots, y_{41}$ are distinct,

$$x_ia_j, y_ib_j \notin E(G) \text{ for any } i, j \in \{1, \cdots, 41\} \text{ with } i \neq j, \tag{9}$$

and

$$x_iv, y_iv, x_ib_i, y_ia_i, x_iy_i \notin E(G) \text{ for every } i \in \{1, \cdots, 41\}. \tag{10}$$

Now let D be the digraph on $\{1, \cdots, 41\}$ obtained by joining i to j ($i \neq j$) if and only if we have $x_ib_j \in E(G)$ or $y_ia_j \in E(G)$, and let H be the (simple) graph obtained by ignoring the direction of the edges of D. Since G is $\{C_3, C_4\}$-free, each $i \in \{1, \cdots, 41\}$ has outdegree at most two in D, which means that the average degree of H is at most four. In view of Lemma 1.8, we may assume that $\{1, \cdots, 9\}$ is independent in H. Then $\{x_i | 1 \leq i \leq 9\} \cap \{y_i | 1 \leq i \leq 9\} = \varnothing$ and, by (9) and (10),

$$E_G(\{x_i, y_i | 1 \leq i \leq 9\}, \{v\} \cup \{a_i, b_i | 1 \leq i \leq 9\}) = \{x_ia_i, y_ib_i | 1 \leq i \leq 9\}. \tag{11}$$

Note that $R(3, 4) = 9$. Since G is C_3-free, we may assume that $\{y_1, y_2, y_3, y_4\}$ is independent. Since G is $\{C_3, C_4\}$-free, there exist $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$ such that $x_iy_j \notin E(G)$. We may assume that $x_1y_2 \notin E(G)$. We now see from (8), (10) and (11) that $\{y_1, b_1, a_1, x_1, v, a_3, b_3, a_2, b_2, y_2\}$ induces a copy of T_2, as desired (see Figure 5). □

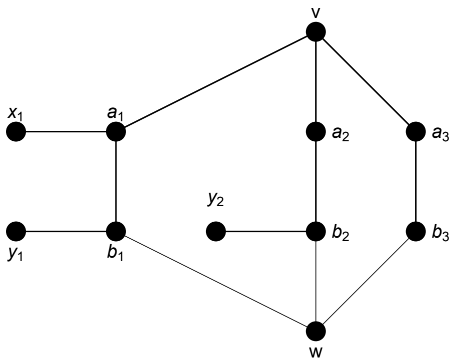

Fig. 5. Lemma 4.2

5. Paths of order five

We continue with the notation of the preceding section. In this section, we consider the case where $|M_5^w(v)|$ is large.

Lemma 5.1. *Suppose that* $|M_4^w(v) \cup M_5^w(v)| \geq 511$. *Then* G *contains an induced copy of* T_1.

Proof. In view of Lemma 4.1, we may assume that $|M_4^w(v)| \leq 15$. Then $|M_5^w(v) - M_4^w(v)| \geq 496$. Take $a_1, \cdots, a_{496} \in M_5^w(v) - M_4^w(v)$. For each $i \in \{1, \cdots, 496\}$, let $va_ib_ic_iw$ be an induced v-w path. If there exists $c \in \{c_1, \cdots, c_{496}\}$ such that $|\{i \in \{1, \cdots, 496\} | c_i = c\}| \geq 16$, then $|M_4^c(v)| \geq 16$, and hence the desired conclusion follows from Lemma 4.1. Thus we may assume that $|\{i \in \{1, \cdots, 496\} | c_i = c\}| \leq 15$ for each $c \in \{c_1, \cdots, c_{496}\}$. Then $|\{c_1, \cdots, c_{496}\}| \geq \lceil 496/15 \rceil = 34$. We may assume that $c_1, \cdots, c_{34}$ are distinct. Since G is $\{C_3, C_4\}$-free, we see that $\{a_i, b_i, c_i\} \cap \{a_j, b_j, c_j\} = \varnothing$ for any $i, j \in \{1, \cdots, 34\}$ with $i \neq j$. Since $a_1, \cdots, a_{34} \notin M_4^w(v)$, $a_ic_j \notin E(G)$ for any $i, j \in \{1, \cdots, 34\}$ with $i \neq j$. Since G is $\{C_3, C_4\}$-free, it follows that

$$\begin{aligned} E(G[\{v,w\} \cup \{a_i, b_i, c_i | 1 \leq i \leq 34\}]) - \{va_i, a_ib_i, b_ic_i, c_iw | 1 \leq i \leq 34\} \\ = E(G[\{b_i | 1 \leq i \leq 34\}]). \end{aligned} \tag{12}$$

Since $b_1w \notin E(G)$ by (12), we have $N(b_1) \cap \{b_i | 2 \leq i \leq 34\} \subseteq M_4^w(b_1)$. Thus by Lemma 4.1, we may assume that $|N(b_1) \cap \{b_i | 2 \leq i \leq 34\}| \leq 15$. We assume that

$$b_1b_i \notin E(G) \text{ for every } i \in \{2, \cdots, 19\}. \tag{13}$$

Take $y_1 \in N(b_1) - \{a_1, c_1\}$. By (12) and (13), $y_1 \notin \{v, w\} \cup \{a_i, b_i, c_i | 1 \leq i \leq 19\}$. Since G is $\{C_3, C_4\}$-free,

$$y_1v, y_1w, y_1a_1, y_1c_1 \notin E(G). \tag{14}$$

Since $y_1w \notin E(G)$ by (14), we have $b_1 \in M_4^w(y_1)$ and $N(y_1) \cap \{b_i | 2 \leq i \leq 19\} \subseteq M_4^w(y_1)$. Thus by Lemma 4.1, we may assume that $|N(y_1) \cap \{b_i | 2 \leq i \leq 19\}| \leq 14$. We may assume that

$$y_1b_i \notin E(G) \text{ for every } i \in \{2, \cdots, 5\}. \tag{15}$$

Since G is $\{C_3, C_4\}$-free, $|N(y_1)\cap\{a_2, \cdots, a_5\}|\leq 1$. We may assume that $y_1a_2, y_1a_3, y_1a_4 \notin E(G)$. Since G is C_3-free, some two of b_2, b_3 and b_4, say b_2 and b_3, are nonadjacent. Since G is $\{C_3, C_4\}$-free, we have $y_1c_2 \notin E(G)$ or $y_1c_3 \notin E(G)$. We may assume that $y_1c_2 \notin E(G)$. It now follows from (12) through (15) that $\{c_1, b_1, y_1, a_1, v, a_3, b_3, a_2, b_2, c_2\}$ induces a copy of T_1, as desired. □

Lemma 5.2. *Suppose that* $|M_4^w(v) \cup M_5^w(v)|\geq 3361$. *Then* G *contains an induced copy of* T_2.

Proof. By Lemma 4.2, we may assume that $|M_5^w(v) - M_4^w(v)|\geq 3361 - 40 = 3321$. Take $a_1, \cdots, a_{3321} \in M_5^w(v) - M_4^w(v)$. For each $i \in \{1, \cdots, 3321\}$, let $va_ib_ic_iw$ be an induced v-w path. By Lemma 4.2, we may assume that $|\{c_1, \cdots, c_{3321}\}|\geq \lceil 3321/40\rceil = 84$. We may assume that $c_1, \cdots, c_{84}$ are distinct. As in the proof of Lemma 5.1, we see that $\{a_i, b_i, c_i\} \cap \{a_j, b_j, c_j\} = \varnothing$ for any $i, j \in \{1, \cdots, 84\}$ with $i \neq j$, and

$$\begin{aligned} E(G[\{v, w\}\cup\{a_i, b_i, c_i|1 \leq i \leq 84\}]) &- \{va_i, a_ib_i, b_ic_i, c_iw|1 \leq i \leq 84\} \\ &= E(G[\{b_i|1 \leq i \leq 84\}]). \end{aligned} \tag{16}$$

Take $x_1 \in N(a_1) - \{v, b_1\}$. By (16), $x_1 \notin \{v, w\} \cup \{a_i, b_i, c_i|1 \leq i \leq 84\}$. Since $a_1 \notin M_4^w(v)$,

$$x_1w \notin E(G). \tag{17}$$

Since G is $\{C_3, C_4\}$-free,

$$x_1a_i \notin E(G) \text{ for every } i \in \{2, \cdots, 84\} \text{ and } x_1v, x_1b_1, x_1c_1 \notin E(G). \tag{18}$$

We have $N(b_1) \cap \{b_2, \cdots, b_{84}\} \subseteq M_4^w(b_1)$ by (16), and $N(x_1) \cap \{b_2, \cdots, b_{84}\} \subseteq M_4^w(x_1)$ by (17). By Lemma 4.2, we may assume that $b_1b_i, x_ib_i \notin E(G)$ for every $i \in \{2, 3, 4\}$. Since G is $\{C_3, C_4\}$-free, we may assume that $b_2b_3 \notin E(G)$ and that $x_1c_2 \notin E(G)$. It now follows from (16) and (18) that $\{c_1, b_1, a_1, x_1, v, a_3, b_3, a_2, b_2, c_2\}$ induces a copy of T_2, as desired. □

6. Proof of Propositions 1.6 and 1.7

In this section, we prove Propositions 1.6 and 1.7.

Recall that G denotes a 3-connected $\{C_3, C_4\}$-free graph. In this section, we fix a vertex $w \in V(G)$ with $\deg_G(w) = \Delta(G)$.

For a vertex $u \in V(G)$ and a nonnegative integer d, let $N_d(u)$ be the set of vertices of G such that $\mathrm{dist}(u, v) = d$, and let $N_{\leq d}(u) = \bigcup_{0\leq i\leq d} N_i(u)$ and $N_{\geq d}(u) = \bigcup_{i\geq d} N_i(u)$; thus $N_0(u) = \{u\}$ and $N_1(u) = N(u)$. Clearly $N(w)$ is independent, $|N(x) \cap N_2(w)|\geq 2$ for every $x \in N(w)$ and, since $|N(y) \cap N(w)|\leq 1$ for every $y \in N_{\geq 2}(w)$, we have $\delta(G[N_{\geq 2}(w)]) \geq 2$. As in [4], for $U \subseteq V(G)$, we let $L(U)$ denote the set of those vertices $v \in N_2(w) \cup N_3(w)$ for which there exists a v-w path of order four avoiding U. The following three lemmas hold (see [3, Section 5]).

Lemma 6.1. *Let* $X \subseteq N_{\geq 2}(w)$*, and set* $Y_1 = (X \cup N(X)) \cap N_2(w)$*,* $Y_2 = N(Y_1) \cap N(w)$*,* $Z_1 = N(X) \cap L(X)$*,* $Z_2 = (X \cup N(X \cup Z_1)) \cap N_2(w)$ *and* $Z_3 = N(Z_2) \cap N(w)$*. Then the following hold.*

(i) *If* $a \in N(w) - Y_2$*, then* $E_G(X, N_{\leq 1}(a)) = \varnothing$*.*

(ii) *If* $a \in N(w) - Z_3$*, then* $E_G(X, N_{\leq 2}(a) - Z_3) = \varnothing$*.*

Lemma 6.2. *Let* $X \subseteq N_{\geq 2}(w)$*. Then* $N(X) \cap N_2(w) \subseteq \bigcup_{u \in X} M_4^w(u)$ *and* $N(X) \cap L(X) \subseteq \bigcup_{u \in X} M_5^w(u)$*.*

Lemma 6.3. *Let* $X \subseteq N_{\geq 2}(w)$*, and let* Y_1, Y_2, Z_1, Z_2, Z_3 *be as in Lemma 6.1. Then*

(i) $|Y_2| \leq |Y_1| \leq |X - N(X)| + \sum_{u \in X} |M_4^w(u)|$*,*

(ii) $|Z_1| \leq \sum_{u \in X} |M_5^w(u)|$*, and*

(iii) $|Z_3| \leq |Z_2| \leq |X - N(X \cup Z_1)| + \sum_{u \in X \cup Z_1} |M_4^w(u)|$*.*

Proof of Proposition 1.6. By way of contradiction, suppose that G is T_1-free and $\Delta(G) \geq 3 \cdot 10^4$. By Lemmas 4.1 and 5.1,

$$|M_4^w(u)| \leq 15 \text{ and } |M_5^w(u)| \leq |M_4^w(u) \cup M_5^w(u)| \leq 510 \text{ for every } u \in N_{\geq 2}(w). \tag{19}$$

We seek for a contradiction. Let $a_1 \in N(w)$. Take $a_2 \in N(a_1) \cap N_{\geq 2}(w)$ and $a_3, a_3' \in N(a_2) \cap N_{\geq 2}(w)$ with $a_3 \neq a_3'$. Set $X = \{a_2, a_3, a_3'\}$. Then $E(G[X \cup \{a_1\}]) = \{a_3a_2, a_3'a_2, a_2a_1\}$. Let Z_1, Z_3 be as in Lemma 6.1. We have $X - N(X \cup Z_1) = X - N(X) = \varnothing$. By Lemma 6.3 (ii), (iii) and (19), $|Z_1| \leq 3 \cdot 510 < 1600$ and $|Z_3| < 1603 \cdot 15 < 3 \cdot 10^4 \leq |N(w)|$. Let $b_1 \in N(w) - Z_3$, and take $b_2 \in N(b_1) \cap N_{\geq 2}(w)$. Since $\delta(G[N_{\geq 2}(w)]) \geq 2$, we can take $b_3 \in N(b_2) \cap N_{\geq 2}(w)$ so that $a_1b_3 \notin E(G)$ (see Figure 6). Then $E(G[\{a_1, w, b_1, b_2, b_3\}]) = \{a_1w, wb_1, b_1b_2, b_2b_3\}$. Since $E_G(X, \{w, b_1, b_2, b_3\}) = \varnothing$ by Lemma 6.1 (ii), it follows that

$$E(G[X \cup \{a_1, w, b_1, b_2, b_3\}]) = \{a_3a_2, a_3'a_2, a_2a_1, a_1w, wb_1, b_1b_2, b_2b_3\}.$$

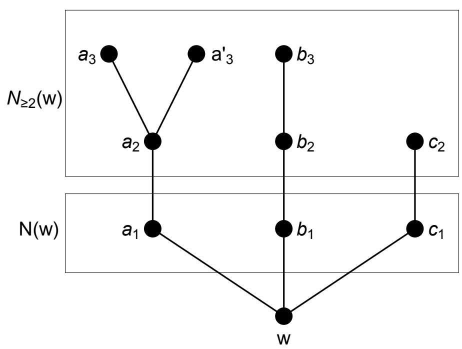

Fig. 6. Proposition 1.6

Let now $X' = X \cup \{b_2, b_3\}$. We have $X' - N(X') = \varnothing$. Let Y_2 be as in Lemma 6.1 with X replaced by X'. By Lemma 6.3 (i) and (19), $|Y_2| \leq 5 \cdot 15 < |N(w)|$. Let $c_1 \in N(w) - Y_2$, and take $c_2 \in N(c_1) \cap N_{\geq 2}(w)$. Then $E(G[\{a_1, b_1, w, c_1, c_2\}]) = \{a_1w, b_1w, wc_1, c_1c_2\}$. Since $E_G(X', \{w, c_1, c_2\}) = \varnothing$ by Lemma 6.1 (i), we therefore see that $X \cup \{a_1, w, b_1, b_2, b_3, c_1, c_2\}$

induces a copy of T_1. This contradicts the assumption that G is T_1-free, and completes the proof of Proposition 1.6.

□

Proof of Proposition 1.7. Suppose that G is T_2-free and $\Delta(G) \geq 4.1 \cdot 10^5$. By Lemmas 4.2 and 5.2,

$$|M_4^w(u)| \leq 40 \text{ and } |M_5^w(u)| \leq |M_4^w(u) \cup M_5^w(u)| \leq 3360 \text{ for every } u \in N_{\geq 2}(w). \quad (20)$$

Let $a_1 \in N(w)$. Take $a_2, a_2' \in N(a_1) \cap N_{\geq 2}(w)$ with $a_2 \neq a_2'$, and take $a_3 \in N(a_2) \cap N_{\geq 2}(w)$. Set $X = \{a_2, a_2', a_3\}$. Then $E(G[X \cup \{a_1\}]) = \{a_3a_2, a_2a_1, a_2'a_1\}$. Let Z_1, Z_3 be as in Lemma 6.1. We have $X - N(X \cup Z_1) \subseteq X - N(X) = \{a_2'\}$. By Lemma 6.3 (ii), (iii) and (20), $|Z_1| \leq 3 \cdot 3360 < 10100$ and $|Z_3| < 1 + 10103 \cdot 40 < 4.1 \cdot 10^5 \leq |N(w)|$. Let $b_1 \in N(w) - Z_3$, take $b_2 \in N(b_1) \cap N_{\geq 2}(w)$, and take $b_3 \in N(b_2) \cap N_{\geq 2}(w)$ with $a_1b_3 \notin E(G)$ (see Figure 7). Then $E(G[\{a_1, w, b_1, b_2, b_3\}]) = \{a_1w, wb_1, b_1b_2, b_2b_3\}$. Since $E_G(X, \{w, b_1, b_2, b_3\}) = \varnothing$ by Lemma 6.1 (ii), it follows that $E(G[X \cup \{a_1, w, b_1, b_2, b_3\}]) = \{a_3a_2, a_2a_1, a_2'a_1, a_1w, wb_1, b_1b_2, b_2b_3\}$. Let now $X' = X \cup \{b_2, b_3\}$. We have $X' - N(X') = \{a_2'\}$. Let Y_2 be as in Lemma 6.1 with X replaced by X'. By Lemma 6.3 (i) and (20), $|Y_2| \leq 1 + 5 \cdot 40 < |N(w)|$. Let $c_1 \in N(w) - Y_2$, and take $c_2 \in N(c_1) \cap N_{\geq 2}(w)$. Then $E(G[\{a_1, b_1, w, c_1, c_2\}]) = \{a_1w, b_1w, wc_1, c_1c_2\}$. Since $E_G(X', \{w, c_1, c_2\}) = \varnothing$ by Lemma 6.1 (i), we therefore see that $X \cup \{a_1, w, b_1, b_2, b_3, c_1, c_2\}$ induces a copy of T_2. This contradicts the assumption that G is T_2-free, and completes the proof of Proposition 1.7.

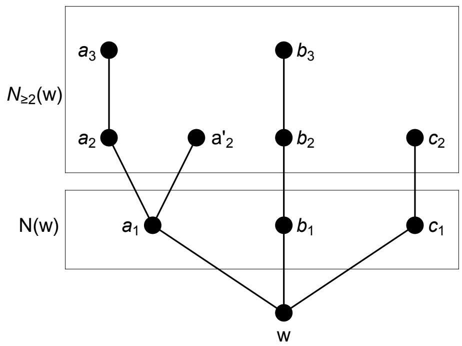

Fig. 7. Proposition 1.7

□

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