Kernel-perfectness in digraphs obtained by some operations on digraphs

R. Lakshmi$^{\boxtimes}$, D. G. Sindhu

ABSTRACT

A kernel J of a digraph D is an independent set of vertices of D such that for every $z \in V(D) \backslash J$ there exists an arc from z to J. A digraph D is said to be kernel-perfect if every induced subdigraph of it has a kernel. We characterise kernel-perfectness in special families of digraphs, namely, the line digraph, the subdivision digraph, the middle digraph, the digraph $R(D)$ and the total digraph. We also obtain some results on kernel-perfectness in the generalised Mycielskian of digraphs. Moreover, we find some new classes of kernel-perfect digraphs by introducing a new product on digraphs.

Keywords: Kernel, semikernel, kernel-perfect, kernel-imperfect, generalised Mycielskian of a digraph

2020 Mathematics Subject Classification: Primary: 05C20, Secondary: 05C69

1. Introduction

Throughout this paper, $D = (V(D), A(D))$ denotes a finite digraph with no loops, no parallel arcs, and no isolates (isolates are vertices with in-degree and out-degree zero), where $V(D)$ (resp. $A(D)$) is the vertex set (resp. the arc set) of D. A walk, path, and cycle, respectively, denote a directed walk, directed path, and directed cycle. A walk, path, or cycle is called *even* (resp. *odd*) if its length is even (resp. odd). An arc $xy \in A(D)$ is *symmetric*, if $yx \in A(D)$. A digraph D is said to be *complete symmetric* if there is a symmetric arc between any pair of vertices of D. We call a digraph *bipartite* (resp. *connected*) if its underlying graph is bipartite (resp. connected). A digraph with no cycles is called *acyclic*.

Let $C = v_0 v_1 \ldots v_{n-1} v_0$ be a cycle of D. Two vertices joined by an arc of C are said to be *consecutive* on C. A *chord* of C is an arc of D of the form $v_i v_j$ with $j \not\equiv (i+1) \pmod{n}$. The chord $v_i v_j$ is *short* when $j \equiv (i+2) \pmod{n}$. The endpoints of chords of C are called *poles* of C.

For $X, Y \subseteq V(D)$, $[X, Y]$ denotes the set of all arcs of D with tail in X and head in Y; an arc (x, y) of D will be called an (X, Y)-*arc* whenever $x \in X$ and $y \in Y$. When $X = \{u\}$, we will simply write (u, Y)-arc and analogously we write (X, v)-arc if $Y = \{v\}$.

A *kernel* J of a digraph D is an independent set of vertices of D such that for every vertex $z \in V(D) \backslash J$ there exists an arc from z to a vertex in J.

A *semikernel* of a digraph D is an independent set S of vertices of D such that for every $z \in V(D) \backslash S$ for which there exists an (S, z)-arc there also exists an (z, S)-arc in D. Note that a kernel of D is a semikernel of D. Furthermore, for any digraph D, if $X_0 = \{x \in V(D) |$ there exists $y \in N_D^+(x)$ with $N_D^+(y) \subseteq N_D^+(x)\}$, then, for any semikernel S of D, $S \cap X_0 = \emptyset$.

A digraph D is said to be *kernel-perfect* (*KP-digraph*) if every induced subdigraph of D has a kernel. A digraph which is not kernel-perfect is called *kernel-imperfect*.

Many structural properties of digraphs that give rise to kernel-perfect digraphs have been explored widely. It is important to note that deciding the kernel-perfectness of a digraph is co-$\mathcal{NP}$-hard (see [1]).

We list some classical sufficient conditions for a digraph to be kernel-perfect.

Theorem 1.1 (Richardson [12])**.** *A digraph with no odd cycle is kernel-perfect.*

Below are some independent generalisations of Theorem 1.1.

Theorem 1.2 (Duchet [4])**.** *If D is a digraph such that every odd cycle possesses at least two symmetric arcs, then D is kernel-perfect.*

Theorem 1.3 (Duchet [4])**.** *If every cycle of a digraph D has a symmetric arc, then D is kernel-perfect.*

Theorem 1.4 (Duchet [4])**.** *If every odd cycle of a digraph D has two short chords and the directed triangles are symmetric, then D is kernel-perfect.*

Theorem 1.5 (Galeana-Sánchez and Neumann-Lara [7])**.** *If every odd cycle of a digraph D has two consecutive poles on the cycle, then D is kernel-perfect.*

Theorem 1.6 (Neumann-Lara [11])**.** *If every induced subdigraph of a digraph D has a nonempty semikernel, then D is kernel-perfect.*

For some more sufficient conditions, see [2], [7] and [8].

A digraph D is called *transitive*, if $uw \in A(D)$, whenever $uv, vw \in A(D)$. Some generalisations of transitive digraphs were introduced in [4] and [9]. Let D be a digraph; D is *quasi-transitive*, if $uv, vw \in A(D)$ implies that $uw \in A(D)$ or $wu \in A(D)$; D is

right-pretransitive (resp. *left-pretransitive*), if $uv, vw \in A(D)$ implies that $uw \in A(D)$ or $wv \in A(D)$ (resp. $uw \in A(D)$ or $vu \in A(D)$). Note that if D is transitive, then every induced subdigraph of D is also transitive. Similar property holds for quasi-transitive, left-pretransitive and right-pretransitive digraphs.

The *line digraph* $L(D)$ has vertex set $V(L(D)) = A(D)$ and arc set $A(L(D)) = \{ab : a, b \in V(L(D))$, the head of a coincides with the tail of b in $D\}$.

Let D be a digraph. For distinct elements $a, b \in V(D) \cup A(D)$, consider the following possibilities:

(1) $a \in V(D)$, $b \in V(D)$ and $ab \in A(D)$;

(2) $a \in V(D)$, $b \in A(D)$ and a coincides with the tail of arc b in D;

(3) $a \in A(D)$, $b \in V(D)$ and b coincides with the head of arc a in D;

(4) $a \in A(D)$, $b \in A(D)$ and $ab \in A(L(D))$.

With the vertex set as $V(D) \cup A(D)$, four operations on digraphs, namely, the *subdivision digraph* $S(D)$, the *middle digraph* $Q(D)$, the digraph $R(D)$, and the *total digraph* $T(D)$ are introduced in [13] with their arc sets defined as below:

- $ab \in A(S(D))$ if, and only if, the pair (a, b) satisfies (2) or (3);
- $ab \in A(Q(D))$ if, and only if, the pair (a, b) satisfies (2) or (3) or (4);
- $ab \in A(R(D))$ if, and only if, the pair (a, b) satisfies (1) or (2) or (3);
- $ab \in A(T(D))$ if, and only if, the pair (a, b) satisfies (1) or (2) or (3) or (4).

From the definitions, observe that $Q(D) = S(D) \cup L(D)$, $R(D) = D \cup S(D)$ and $T(D) = D \cup S(D) \cup L(D)$.

In [13], the author obtained some necessary or sufficient conditions for the existence or uniqueness of kernels in digraphs formed by the above four operations.

The *generalised Mycielskian* $\mu_m(D)$ of a loopless digraph $D = (V^0, A^0)$ is defined, in [6], as the digraph whose vertex set is $V^0 \cup V^1 \cup \ldots \cup V^m \cup \{u\}$, where $V^i = \{v^i : v^0 \in V^0\}$, $i = 1, 2, \ldots, m$, and arc set $A^0 \cup \left(\bigcup_{i=0}^{m-1} \{x^i y^{i+1}, \ x^{i+1} y^i : x^0 y^0 \in A^0\} \right) \cup \{x^m u, \ ux^m : x^m \in V^m\}$.

Given a digraph D_1, a symmetric digraph D_2, and a subset $Y \subseteq V(D_2)$, we define the product $D_1 \times_Y D_2$ as follows:

(i) $V(D_1 \times_Y D_2) = V(D_1) \times V(D_2)$;

(ii) $(u_1, v_1) \to (u_2, v_2)$ in $D_1 \times_Y D_2$ if $u_1 \to u_2$ in D_1 and $v_1 \to v_2$ in D_2; and

(iii) for every $v \in Y$, $(u_1, v) \to (u_2, v)$ in $D_1 \times_Y D_2$ if $u_1 \to u_2$ in D_1.

The significance of introducing this product is to yield a new class of KP-digraphs $D_1 \times_Y D_2$ from certain known choices of KP-digraphs D_1. Also, note that: $D_1 \times_Y D_2$ is the tensor product $D_1 \times D_2$, when $Y = \emptyset$; $D_1 \times_Y D_2$ is $\mu_m(D_1) - u$, when D_2 is the symmetric path $v_0 v_1 \ldots v_m$ and $Y = \{v_0\}$.

For notation not defined here, see [3].

In Section 2, we prove some preliminary results that are used in Sections 4 and 5. In Section 3, kernel-perfectness is characterised in digraphs $L(D)$, $S(D)$, $Q(D)$, $R(D)$ and $T(D)$. In Section 4, we identify certain families of kernel-perfect digraphs D such that $\mu_m(D)$ is kernel-perfect. Specifically, in Theorem 4.2 (resp. in Theorems 4.5 and 4.6), we identify some special families of KP-digraphs D such that $\mu_m(D)$ belongs to the special

family of KP-digraphs to which the corresponding D belongs (resp. $\mu_m(D)$ is kernel-perfect but $\mu_m(D)$ does not belong to the special family of KP-digraphs to which the corresponding D belongs). In Section 5, we provide some sufficient conditions on D_1 for $D_1 \times_Y D_2$ to be kernel-perfect.

2. Preliminary results

In this section, we give some basic results that will be used in Sections 4 and 5.

Lemma 2.1. *In a quasi-transitive digraph, each vertex of a kernel (if exists) acts as a nonempty semikernel of it.*

Proof. Let D be a quasi-transitive digraph with a kernel J. Let $x \in J$. If $d_D^+(x) = 0$, then $\{x\}$ is a semikernel of D. So, assume $d_D^+(x) \geq 1$. Let $z \in N_D^+(x)$. Clearly, $z \in V(D) \backslash J$. As J is a kernel of D, there exists $y \in J$ such that $z \to y$. If $y \neq x$, then, as $x \to z \to y$, the quasi-transitivity implies that $x \to y$ or $y \to x$, a contradiction. Thus $y = x$ and so $z \in N_D^-(x)$. Hence $N_D^+(x) \subseteq N_D^-(x)$. Therefore, $\{x\}$ is a semikernel of D. $\qquad\square$

The above result is true for transitive digraphs and left-pretransitive digraphs. But it is not the case for right-pretransitive digraphs. To see this, consider the complete symmetric digraph K_n^* with $V(K_n^*) = \{x_1, x_2, \ldots, x_n\}$. Form D from K_n^* by adding new vertices $y_1, y_2, \ldots, y_n$ and arcs $y_1x_1, y_2x_2, \ldots, y_nx_n$. Clearly, D is right-pretransitive, but it is neither left-pretransitive nor quasi-transitive. Here, $J = \{x_1, y_2, y_3, \ldots, y_n\}$ is a kernel of D, but no vertex of $J \backslash \{x_1\}$ is a semikernel of D.

Lemma 2.2. *If D is right-pretransitive or left-pretransitive or transitive, then every cycle of D has a symmetric arc.*

Proof. First, let D be a right-pretransitive digraph. Suppose D has cycles with no symmetric arcs. Let C be a shortest such cycle. Let $C : v_0v_1v_2 \ldots v_{t-1}v_0$, where the suffixes in v_i are reduced modulo t. Since $v_0v_1, v_1v_2 \in A(D)$ and v_1v_2 is not symmetric, by the right-pretransitivity of D, we have $v_0v_2 \in A(D)$. If v_0v_2 is not symmetric, then $v_0v_2v_3 \ldots v_{t-1}v_0$ is a cycle of D with no symmetric arc, a contradiction to the choice of C. So v_0v_2 is a symmetric arc (i.e., v_2v_0 is also an arc of D). Now, since $v_2v_0, v_0v_1 \in A(D)$ and neither of v_0v_1 and v_1v_2 is symmetric, we have a contradiction to the right-pretransitivity of D.

Similar proof holds if D is left-pretransitive, considering the arcs v_1v_2 and v_2v_0. As every transitive digraph is both right-pretransitive and left-pretransitive, it is evident that every cycle of a transitive digraph has a symmetric arc. $\qquad\square$

The above result need not be true for quasi-transitive digraphs. To see this, let D be a digraph with $V(D) = \{x, y, v_1, v_2, \ldots, v_{n-2}\}$ and $A(D) = \{yx\} \cup \{xv_i, v_iy : i \in \{1, 2, \ldots, n-2\}\}$. Clearly, D is quasi-transitive but none of the cycles of D has a symmetric arc.

The following result is a consequence of Lemma 2.2.

Theorem 2.3. *Let H_1 and H_2 be digraphs ($A(H_1) \cap A(H_2)$ may be non-empty). If $D = H_1 \cup H_2$ with H_1 and H_2 as its induced subdigraphs, then, in any of the following cases, every cycle of D has a symmetric arc:*
   (1) *H_1 is right-pretransitive and H_2 is left-pretransitive;*
   (2) *both H_1 and H_2 are left-pretransitive or right-pretransitive;*
   (3) *every cycle of H_1 has a symmetric arc and H_2 is left-pretransitive or right-pretransitive.*

Proof. (1) Proof is by contradiction. Suppose D has cycles with no symmetric arcs. Let C be a shortest such cycle. Let $C : v_0 v_1 v_2 \ldots v_{t-1} v_0$, where the suffixes in v_i are reduced modulo t. If C has two consecutive arcs in H_1 or in H_2, then, from the proof of Lemma 2.2, we have a contradiction. So t is even and C alternates its arcs in H_1 and H_2. As $v_i \in V(H_1) \cap V(H_2)$ for every $i \in \{0, 1, 2, \ldots, t-1\}$ and H_1, H_2 are induced subdigraphs of D, we have that C is contained in H_1 and in H_2, a contradiction.

For (2), the proof is similar to that of (1).

(3) Assume that every cycle of H_1 has a symmetric arc. First, we prove this result if H_2 is left-pretransitive. Proof is by contradiction. Let C be a shortest cycle in D such that no arc of C is symmetric. Let $C : v_0 v_1 v_2 \ldots v_{t-1} v_0$, where the suffixes in v_i are reduced modulo t. As in the proof of Lemma 2.2, if C has two consecutive arcs in H_2, then we have a contradiction. So no two consecutive arcs of C are in H_2. As $v_i \in V(H_1)$ for every $i \in \{0, 1, 2, \ldots, t-1\}$, we have that C is in H_1, a contradiction.

Proof for right-pretransitive digraph H_2 follows in a similar way. □

3. Characterisation of kernel-perfectness in digraphs obtained by some unary operations

Regarding the presence of kernels in digraphs obtained by some unary operations from other digraphs, many authors had previously investigated. For example, see [5], [10] and [13]. In this section, we characterise the kernel-perfectness of digraphs $L(D)$, $S(D)$, $Q(D)$, $R(D)$ and $T(D)$.

It merits attention that the line digraph is a well-known and extensively studied operation. Harminc characterised the existence of a kernel in $L(D)$ as below:

Theorem 3.1 (Harminc [10]). *D has a kernel if, and only if, $L(D)$ has a kernel.*

As a natural progression to the above theorem, we characterise the kernel-perfectness of $L(D)$ in Theorem 3.4, which carries a deep significance but its proof is delightfully simple.

Observation 3.2. *If D has an odd cycle, then $L(D)$ has an induced odd cycle.*

Observation 3.3. *If $L(D)$ has an odd cycle, then D has an odd cycle.*

Consider a kernel-perfect digraph D with an odd cycle. From Observation 3.2, we see that $L(D)$ has an induced odd cycle, which is a kernel-less induced subdigraph. Hence, even if D is kernel-perfect, $L(D)$ need not be. From Observations 3.2 and 3.3 along with Theorem 1.1, we have:

Theorem 3.4. $L(D)$ *is kernel-perfect if, and only if, D has no odd cycles.*

Observation 3.5. *For any digraph D, $S(D)$ is bipartite and hence is kernel-perfect.*

While considering $Q(D)$ and $R(D)$, take $W = V(D)$ and $Z = V(L(D))$. Here, $\{W, Z\}$ is a partition of each of $V(Q(D))$ and $V(R(D))$.

Theorem 3.6. $Q(D)$ *is kernel-perfect if, and only if, D has no odd cycles.*

Proof. From the definition of $Q(D)$, $L(D)$ is an induced subdigraph of $Q(D)$. By Theorem 3.4, for $Q(D)$ to be kernel-perfect, it is necessary that D must be free of odd cycles. So the necessity follows. To prove the sufficiency, we prove that every induced subdigraph of $Q(D)$ has a kernel, when D has no odd cycles. Observe that $L(D)$ is kernel-perfect. Let H be an induced subdigraph of $Q(D)$. If $Z \cap V(H)$ is empty, then H has no arcs and therefore it has a kernel. Let $X = W \cap V(H)$ and $Y = W \setminus V(H)$. If X is empty, then H is an induced subdigraph of the kernel-perfect digraph $L(D)$, and therefore H has a kernel. Hence, assume that both X and $Z \cap V(H)$ are nonempty.

Let

$$
\begin{aligned}
Z_{WX}^H &= \{(w, x) \in Z \cap V(H) \mid w \in W \text{ and } x \in X\}, \\
Z_{XY}^H &= \{(x, y) \in Z \cap V(H) \mid x \in X \text{ and } y \in Y\}, \text{ and} \\
Z_{YY}^H &= \{(y, y') \in Z \cap V(H) \mid y, y' \in Y\}.
\end{aligned}
$$

Note that $\{Z_{WX}^H, Z_{XY}^H, Z_{YY}^H\}$ is a partition of $Z \cap V(H)$. In H (and also in $Q(D)$), the sets X and Z_{XY}^H are independent and the arc sets $\left[X, Z_{YY}^H\right]$, $\left[Z_{WX}^H, Z_{YY}^H\right]$, $\left[Z_{XY}^H, X\right]$, $\left[Z_{YY}^H, X\right]$ and $\left[Z_{YY}^H, Z_{XY}^H\right]$ are empty.

Consider $H\left[Z_{YY}^H\right]$, an induced subdigraph of H. As it is an induced subdigraph of the kernel-perfect digraph $L(D)$, it has a kernel, say, K_1. Clearly, K_1 absorbs $Z_{YY}^H \setminus K_1$. Now, let A_1 be the set of vertices in Z_{XY}^H that are absorbed by K_1 in H (A_1 may be empty). Let $K_2 = Z_{XY}^H \setminus A_1$. Since Z_{XY}^H is independent and $\left[Z_{YY}^H, Z_{XY}^H\right] = \emptyset$, we have $K_1 \cup K_2$ is independent in H. Also, it absorbs $(Z_{YY}^H \setminus K_1) \cup A_1$. Next, let A_2 be the set of vertices in X that are absorbed by K_2 in H (A_2 may be empty). Take $K_3 = X \setminus A_2$. Note that, the set $K = K_1 \cup K_2 \cup K_3$ is independent in H, as X is independent and the arc sets $\left[X, Z_{YY}^H\right]$, $\left[Z_{YY}^H, X\right]$ and $\left[Z_{XY}^H, X\right]$ are empty. Clearly, K absorbs $(Z_{YY}^H \setminus K_1) \cup A_1 \cup A_2$ in H. Further, if we show that K absorbs Z_{WX}^H, then K will become a kernel of H. Let $(w, x) \in Z_{WX}^H$, where $w \in W$ and $x \in X$. Clearly, $(w, x) \to x$ in H. As X is a disjoint union of K_3 and A_2, either $x \in K_3$ or $x \in A_2$. If $x \in K_3$, then, in H, (w, x) is absorbed by x, a vertex of K. If $x \in A_2$, then by the definition of A_2, there exists $y \in Y$ such that $x \to (x, y)$ and $(x, y) \in K_2$, where $K_2 \subseteq Z_{XY}^H$. Therefore, in H, we have $(w, x) \to (x, y)$, where $(x, y) \in K$. This completes the proof. $\qquad \square$

Lemma 3.7. *If D_0 is a digraph having an independent set I such that (i) for any three vertices $x \notin I$, $y \in I$, $z \notin I$, if $x \to y \to z$, then $x \to z$; and (ii) $D = D_0 \setminus I$ is kernel-perfect, then D_0 is kernel-perfect.*

Proof. Let H be an induced subdigraph of D_0. If $V(H) \cap I$ is empty, then H is an induced subdigraph of the kernel-perfect digraph D, and therefore H has a kernel. If $V(H) \cap V(D)$ is empty, then H has no arcs and hence it has a kernel. So, assume that both $V(H) \cap I$ and $V(H) \cap V(D)$ are nonempty.

Let $K'_1 = \{v \in V(H) \cap I : d^+_H(v) = 0\} \cup \{v \in V(H) \cap I : d^-_H(v) \geq 1 \text{ and } d^+_H(v) \geq 1\}$. As $K'_1 \subseteq I$, K'_1 is independent. Let A_1 be the set of vertices of $D \cap H$ that are absorbed by K'_1 (A_1 may be empty). Consider the subdigraph $(D \cap H) \setminus A_1$. As it is an induced subdigraph of the kernel-perfect digraph D, it has a kernel, say, K_2. Clearly, $[K_2, K'_1] = \emptyset$. Consider $[K'_1, K_2]$. Let K''_1 be the set of vertices in K'_1 that are absorbed by K_2 (note that $K''_1 = \emptyset$ when $[K'_1, K_2] = \emptyset$). Set $K_1 = K'_1 \setminus K''_1$. As $[K_1, K_2] = \emptyset$, $K_1 \cup K_2$ is independent. We claim that the set of vertices of A_1 that are absorbed by K''_1 is absorbed by K_2 also. To see this, let x be a vertex of A_1 that is absorbed by a vertex y of K''_1. From the definition of K''_1, there exists $z \in K_2$ such that $y \to z$ in H. By (i), $x \to z$ in H which proves the claim. Therefore, A_1 is absorbed by $K_1 \cup K_2$.

Now, $K_1 \cup K_2$ absorbs $K''_1 \cup (V(D \cap H) \setminus K_2)$, since K_2 is a kernel of $(D \cap H) \setminus A_1$. Let A_2 be the set of vertices of $(V(H) \cap I) \setminus K'_1$ that are absorbed by K_2. Set $K_3 = (V(H) \cap I) \setminus (K'_1 \cup A_2)$. From the definition of K_3, $K_1 \cup K_2 \cup K_3$ is independent and hence is a kernel of H. $\square$

Theorem 3.8. *$R(D)$ is kernel-perfect if, and only if, D is kernel-perfect.*

Proof. As D is an induced subdigraph of $R(D)$, for $R(D)$ to be kernel-perfect, it is necessary that D is kernel-perfect. To prove the sufficiency, take $D_0 = R(D)$ and I to be the set of vertices of $R(D)$ corresponding to the arcs of D, and apply Lemma 3.7. $\square$

Theorem 3.9. *$T(D)$ is kernel-perfect if, and only if, D is acyclic.*

Proof. Observe that $T(D)$ has a cycle, if, and only if, D has a cycle. So, if D is acyclic, then $T(D)$ is also acyclic and hence is kernel-perfect.

Conversely, suppose that D has a cycle, say, C. Let $C : v_0 v_1 v_2 \ldots v_n v_0$ and let $e_i = v_{i-1} v_i$, $i \in \{1, 2, \ldots, n\}$, and $e_{n+1} = v_n v_0$. Clearly, $e_1 e_2 \ldots e_{n+1} e_1$ is an induced $(n+1)$-cycle in $L(D)$ and so in $T(D)$ also. As $T(D)$ is kernel-perfect, n must be odd. If at least one arc of C is symmetric, say e_i is symmetric, then let $e'_i = v_i v_{i-1}$. Now, the subdigraph induced by $\{v_{i-1}, v_i, e_i, e'_i\}$ in $T(D)$ is kernel-less, a contradiction. Hence, assume that every arc of C is asymmetric. Therefore, we have the induced odd cycle $v_0 v_1 e_2 e_3 \ldots e_n e_{n+1} v_0$ of length $(n+2)$ in $T(D)$, which is again a contradiction to the kernel-perfectness of $T(D)$. $\square$

4. Kernel-perfectness in $\mu_m(D)$

In this section, let the vertex set (resp. arc set) of D be denoted by V^0 (resp. A^0) and the vertex set (resp. arc set) of $\mu_m(D)$ be denoted by V (resp. A).

For $X^0 \subseteq V^0$, set $X^i = \{x^i : x^0 \in X^0\} \subseteq V^i$, for each $i \in \{1, 2, \ldots, m\}$.

As D is an induced subdigraph of $\mu_m(D)$, if D is kernel-imperfect, then so is $\mu_m(D)$. So, in what follows, we assume that D is kernel-perfect.

Even if D is kernel-perfect, $\mu_m(D)$ may be kernel-imperfect. For example, let $D' = C + \{v_0^0 v_2^0, v_1^0 v_3^0\}$, where $C : v_0^0 v_1^0 \ldots v_{2k}^0 v_0^0$, $k \geq 3$. By Theorem 1.4, D' is kernel-perfect. For $m \geq 2$, $v_0^0 v_1^1 v_2^2 v_3^1 v_4^0 v_5^0 v_6^0 \ldots v_{2k-1}^0 v_{2k}^0 v_0^0$ is an induced odd cycle, and hence $\mu_m(D')$ is kernel-imperfect. Here, D' also serves as an example to prove that neither of the sufficient conditions of Theorems 1.4 and 1.5 in D' is sufficient for $\mu_m(D')$ to be kernel-perfect.

First, we shall see some sufficient conditions for kernel-perfectness of $\mu_m(D)$.

Lemma 4.1. *If D is a digraph such that every odd cycle in D possesses at least two symmetric arcs, then, in $\mu_m(D)$ also, every odd cycle possesses at least two symmetric arcs.*

Proof. Let C be an odd cycle in $\mu_m(D)$. If C contains u, then C has at least two symmetric arcs (arcs incident at u). If C does not contain u, then C induces a closed odd walk, say, C' in D. As every closed odd walk contains an odd cycle, C' contains an odd cycle, say, C'' in D. By hypothesis, C'' has at least two symmetric arcs in D. The arcs in C corresponding to the two symmetric arcs of C'' are symmetric in $\mu_m(D)$ and hence C has at least two symmetric arcs in $\mu_m(D)$.   □

From Lemma 4.1 and Theorems 1.2 and 1.3, we have:

Theorem 4.2. *For a digraph D, $\mu_m(D)$ is kernel-perfect in each of the following cases:*
  (1) *every odd cycle of D possesses at least two symmetric arcs;*
  (2) *every cycle of D has a symmetric arc.*

Corollary 4.3. *If D is transitive or right-pretransitive or left-pretransitive, then $\mu_m(D)$ is kernel-perfect.*

Proof. Follows from Lemma 2.2 and (2) of Theorem 4.2.   □

Corollary 4.4. *$\mu_m(D)$ is kernel-perfect, if D has no odd cycles. So does $\mu_m(D)$ if D is bipartite.*

Observe that $\mu_m(D)$ is neither transitive nor quasi-transitive since $V^m \to u \to V^m$ and V^m is independent.

Theorem 4.5. *If D is a kernel-perfect quasi-transitive digraph, then $\mu_m(D)$ is kernel-perfect.*

Proof. Let H be an induced subdigraph of $\mu_m(D)$. We prove that H has a nonempty semikernel. If $\delta^+(H) = 0$ (resp. $u \in V(H)$), then $\{v \in V(H) : d_H^+(v) = 0\}$ (resp. $\{u\}$) is a nonempty semikernel of H. Therefore, assume $\delta^+(H) \geq 1$ and $u \notin V(H)$. Let H^0 denote the subdigraph of D induced by $\{v^0 : v^i \in V(H)$ for some $i \in \{0, 1, 2, \ldots, m\}\}$, which is clearly a nonempty set. As H^0 is an induced subdigraph of D, H^0 has a kernel J^0. Observe that $\delta^+(H^0) \geq 1$ and H^0 is quasi-transitive. Let $x^0 \in J^0$. By Lemma 2.1, $\{x^0\}$ is a semikernel of H^0 and so, we have $N_{H^0}^+(x^0) \subseteq N_{H^0}^-(x^0)$. As $x^0 \in V(H^0)$, there exists $i \in \{0, 1, 2, \ldots, m\}$ such that $x^i \in V(H)$, which implies that $N_H^+(x^i) \subseteq N_H^-(x^i)$. So $\{x^i\}$ is a semikernel of H. Therefore, by Theorem 1.6, $\mu_m(D)$ is kernel-perfect. $\square$

Now, we consider right- or left- pretransitive digraphs. Observe that, for a right- (resp. left-) pretransitive digraph D with an asymmetrical arc, $\mu_m(D)$ is not right- (resp. left-) pretransitive. Since, for an asymmetrical arc $ab \in A(D)$, in $\mu_m(D)$ we have: $u \to a^m \to b^{m-1}$ with neither $u \to b^{m-1}$ nor $b^{m-1} \to a^m$, if D is right-pretransitive; $a^{m-1} \to b^m \to u$ with neither $a^{m-1} \to u$ nor $b^m \to a^{m-1}$, if D is left-pretransitive. Thus observe that $\mu_m(D)$ does not preserve the property of D. Hence, by Theorems 2.3 and 4.2 (2), we have:

Theorem 4.6. *Let H_1 and H_2 be digraphs $(A(H_1) \cap A(H_2)$ may be nonempty). If $D = H_1 \cup H_2$ with H_1 and H_2 as its induced subdigraphs, then $\mu_m(D)$ is kernel-perfect in any of the following cases:*
  (1) H_1 is right-pretransitive and H_2 is left-pretransitive;
  (2) both H_1 and H_2 are left-pretransitive or right-pretransitive;
  (3) every cycle of H_1 has a symmetric arc and H_2 is left-pretransitive or right-pretransitive.

Next, we concentrate on giving a necessary condition for $\mu_m(D)$ to be kernel-perfect. By Corollary 4.4, it is enough to consider digraphs with odd cycles. For an odd cycle $C : v_0^0 v_1^0 \ldots v_{2k}^0 v_0^0$ in a digraph D, and for each $i \in \{0, 1, \ldots, 2k\}$, let $C_i = \{(v_{i-\ell}^0, v_{i+\ell}^0) : \ell = 1, 2, \ldots, k-1\}$, where the subscripts in v_j^0 are reduced modulo $2k+1$ and the set C_i is a collection of arcs that are not necessarily present in D.

Theorem 4.7. *Let D be a digraph with odd cycles, let c be the length of a longest odd cycle in D, and $m \geq \frac{c-1}{2}$. If $\mu_m(D)$ is kernel-perfect, then, for every odd cycle $C : v_0^0 v_1^0 \ldots v_{2k}^0 v_0^0$ of D, an arc of C is symmetric or for every $i \in \{0, 1, \ldots, 2k\}$, $A^0 \cap (C_{i-1} \cup C_i) \neq \emptyset$.*

Proof. Suppose D has an odd cycle $C : v_0^0 v_1^0 \ldots v_{2k}^0 v_0^0$ such that no arc of C is symmetric and $A^0 \cap (C_{i-1} \cup C_i) = \emptyset$ for some $i \in \{0, 1, \ldots, 2k\}$. We obtain a contradiction by finding an induced odd cycle in $\mu_m(D)$. Corresponding to C, by the definition of $\mu_m(D)$, we have an odd cycle $C' : v_i^0 v_{i+1}^1 v_{i+2}^2 \ldots v_{i+k-1}^{k-1} v_{i+k}^k v_{i+k+1}^{k-1} v_{i+k+2}^{k-2} \ldots v_{i+2k-1}^1 v_{i+2k}^0 v_i^0$ in $\mu_m(D)$. Note that any chord of C' is of the form $(v_{i-\ell}^{\ell-1}, v_{i+\ell}^\ell)$, $(v_{i+\ell}^\ell, v_{i-\ell}^{\ell-1})$, $(v_{i-1-\ell}^\ell, v_{i-1+\ell}^{\ell-1})$ or $(v_{i-1+\ell}^{\ell-1}, v_{i-1-\ell}^\ell)$ for some $\ell \in \{1, 2, \ldots, k-1\}$. Since $A^0 \cap (C_{i-1} \cup C_i) = \emptyset$, $(v_{i-\ell}^0, v_{i+\ell}^0) \in C_i$ and $(v_{i-1-\ell}^0, v_{i-1+\ell}^0) \in C_{i-1}$, we have that any chord of C' is of the form $(v_{i+\ell}^\ell, v_{i-\ell}^{\ell-1})$ or $(v_{i-1+\ell}^{\ell-1}, v_{i-1-\ell}^\ell)$. Let p be the least value of ℓ for which $(v_{i+p}^p, v_{i-p}^{p-1})$ or $(v_{i-1+p}^{p-1}, v_{i-1-p}^p)$ is a chord of C'. If $(v_{i+p}^p, v_{i-p}^{p-1})$ is a chord of C', then $v_i^0 v_{i+1}^1 v_{i+2}^2 \ldots v_{i+p-1}^{p-1} v_{i+p}^p v_{i-p}^{p-1} v_{i-(p-1)}^{p-2} \ldots v_{i-1}^0 v_i^0$ is an induced odd

cycle. If $(v_{i-1+p}^{p-1}, v_{i-1-p}^{p})$ is a chord of C', then $v_i^0 v_{i+1}^1 v_{i+2}^2 \ldots v_{i+p-1}^{p-1} v_{i-p-1}^{p} v_{i-p}^{p-1} \ldots v_{i-1}^0 v_i^0$ is an induced odd cycle. In both the cases, $\mu_m(D)$ has an induced odd cycle, which is a kernel-less subdigraph, a contradiction. $\qquad\square$

5. Kernel-perfectness of $D_1 \times_Y D_2$

Throughout this section, assume that D_1 is a digraph, D_2 is a symmetric digraph and the subset $Y \subseteq V(D_2)$ is nonempty.

As D_1 is an induced subdigraph of $D_1 \times_Y D_2$, if D_1 is kernel-imperfect, then so is $D_1 \times_Y D_2$. So, in what follows, we assume that D_1 is kernel-perfect.

Even if D_1 is kernel-perfect, $D_1 \times_Y D_2$ may be kernel-imperfect. For example, let C : $v_0 v_1 \ldots v_{2n} v_0$, $n \geq 2$, be an odd cycle. Take $D_1 = C + v_0 v_{2i}$ for some $i \in \{1, 2, \ldots, n-1\}$, a symmetric digraph D_2, and Y, a proper subset of $V(D_2)$. Clearly, D_1 is kernel-perfect. Let $y \in Y$, $x \in V(D_2)\backslash Y$ and $xy, yx \in A(D_2)$. From the definition of $D_1 \times_Y D_2$, we have an induced odd cycle $(v_0, x)(v_1, y)(v_2, y) \ldots (v_{2i-1}, y)(v_{2i}, x)(v_{2i+1}, y)(v_{2i+2}, y) \ldots (v_{2n}, y)(v_0, x)$ in $D_1 \times_Y D_2$. Hence $D_1 \times_Y D_2$ is kernel-imperfect.

Our intention is to search for kernel-perfect digraphs D_1 so that $D_1 \times_Y D_2$ is kernel-perfect.

Theorem 5.1. *Let D_1 be a digraph, D_2 be a symmetric digraph and Y, a proper subset of $V(D_2)$. Then $D_1 \times_Y D_2$ is kernel-perfect if D_1 satisfies at least one of the following properties:*
  (1) *every odd cycle in D_1 possesses at least two symmetric arcs;*
  (2) *every cycle in D_1 has a symmetric arc;*
  (3) *transitive or right-pretransitive or left-pretransitive.*

Proof. We prove it for D_1 satisfing (1) by showing that every odd cycle in $D_1 \times_Y D_2$ possesses at least two symmetric arcs. Let C be an odd cycle in $D_1 \times_Y D_2$. By the definition of $D_1 \times_Y D_2$, there is a closed odd walk C' in D_1, which corresponds to the sequence of the first entries of the vertices of C. As every closed odd walk contains an odd cycle, C' contains an odd cycle C'' in D_1. By hypothesis, C'' has at least two symmetric arcs in D_1. The arcs in C corresponding to the two symmetric arcs of C'' are symmetric in $D_1 \times_Y D_2$ and hence C has at least two symmetric arcs in $D_1 \times_Y D_2$.

For D_1 satisfing (2), proof is similar to that of (1).

For D_1 satisfing (3), proof follows from Lemma 2.2 and (2). $\qquad\square$

Observation 5.2. *$D_1 \times_Y D_2$ need not be quasi-transitive, even if D_1 is quasi-transitive (For, consider any path $u_1 \to u_2 \to u_3$ of length 2 in D_1 and any arc $v_1 v_2$ of the symmetric digraph D_2 with $v_1 \notin Y$. In $D_1 \times_Y D_2$, $(u_1, v_1) \to (u_2, v_2) \to (u_3, v_1)$. Here, neither $(u_1, v_1) \to (u_3, v_1)$ nor $(u_3, v_1) \to (u_1, v_1)$.). Similarly, $D_1 \times_Y D_2$ need not be left-pretransitive (resp. right-pretransitive), even if D_1 is left-pretransitive (resp. right-pretransitive).*

From Theorems 2.3 and 5.1 (2), we have:

Theorem 5.3. *Let H_1 and H_2 be digraphs $(A(H_1) \cap A(H_2)$ may be nonempty). If $D_1 = H_1 \cup H_2$ with H_1 and H_2 as its induced subdigraphs, D_2 is a symmetric digraph, and Y, a proper subset of $V(D_2)$, then $D_1 \times_Y D_2$ is kernel-perfect in any of the following cases:*
    (1) H_1 is right-pretransitive and H_2 is left-pretransitive;
    (2) both H_1 and H_2 are left-pretransitive or right-pretransitive;
    (3) every cycle of H_1 has a symmetric arc and H_2 is left-pretransitive or right- pretransitive.

Theorem 5.4. *If D_1 is a kernel-perfect quasi-transitive digraph, D_2 is a symmetric digraph, and Y, a proper subset of $V(D_2)$, then $D_1 \times_Y D_2$ is kernel-perfect.*

Proof. Let H be an induced subdigraph of $D_1 \times_Y D_2$. We show that H has a nonempty semikernel. If $\delta^+(H) = 0$, then $\{(u,v) \in V(H) : d_H^+((u,v)) = 0\}$ is a nonempty semikernel of H. So assume $\delta^+(H) \geq 1$. Let D_1' denote the subdigraph of D_1 induced by $\{u \in V(D_1) : (u,v) \in V(H) \text{ for some } v \in V(D_2)\}$, which is clearly a nonempty set. Observe that $\delta^+(D_1') \geq 1$ and D_1' is quasi-transitive. As D_1 is kernel-perfect, D_1' has a kernel, say, J'. Let $x \in J'$. From the proof of Lemma 2.1, $N_{D_1'}^+(x) \subseteq N_{D_1'}^-(x)$. As $x \in V(D_1')$, we have $(x,v) \in V(H)$ for some $v \in Y$. And so $N_H^+((x,v)) \subseteq N_H^-((x,v))$. Therefore, $\{(x,v)\}$ is a semikernel of H. By Theorem 1.6, $D_1 \times_Y D_2$ is kernel-perfect. $\square$

Acknowledgements

References

[1] S. D. Andres and W. Hochstättler. Perfect digraphs. *Journal of Graph Theory*, 79(1):21–29, 2015. https://doi.org/10.1002/jgt.21811.

[2] C. Balbuena, H. Galeana-Sanchez, and M.-k. Guevara. A sufficient condition for kernel perfectness of a digraph in terms of semikernels modulo f. *Acta Mathematica Sinica, English Series*, 28(2):349–356, 2012. https://doi.org/10.1007/s10114-012-9754-6.

[3] J. Bang-Jensen and G. Gutin. *Digraphs: Theory, Algorithms and Applications*. Springer, 2009.

[4] P. Duchet. Graphes noyau-parfaits. In *Annals of Discrete Mathematics*. Volume 9, pages 93–101. Elsevier, 1980. https://doi.org/10.1016/S0167-5060(08)70041-4.

[5] P. Duchet and H. Meyniel. A note on kernel-critical graphs. *Discrete Mathematics*, 33(1):103–105, 1981. https://doi.org/10.1016/0012-365X(81)90264-8.

[6] S. Francis Raj. Connectivity of the generalised mycielskian of digraphs. *Graphs and Combinatorics*, 29(4):893–900, 2013. https://doi.org/10.1007/s00373-012-1151-5.

[7] H. Galeana-Sánchez and V. Neumann-Lara. On kernels and semikernels of digraphs. *Discrete Mathematics*, 48(1):67–76, 1984. https://doi.org/10.1016/0012-365X(84)90131-6.

[8] H. Galeana-Sánchez and V. Neumann-Lara. On kernel-perfect critical digraphs. *Discrete Mathematics*, 59(3):257–265, 1986. https://doi.org/10.1016/0012-365X(86)90172-X.

[9] A. Ghouilà-Houri. Characterization of nonoriented graphs whose edges can be oriented in such a way as to obtain the graph of an order relation. *Comptes rendus de l'Académie des Sciences*, 254:1370–1371, 1962.

[10] M. Harminc. Solutions and kernels of a directed graph. *Mathematica Slovaca*, 32(3):263–267, 1982.

[11] V. Neumann-Lara. Seminúcleos de una digráfica. *Anales del Instituto de Matemáticas*, 2, 1971.

[12] M. Richardson. Solutions of irreflexive relations. *Annals of Mathematics*, 58(3):573–590, 1953. https://doi.org/10.2307/1969755.

[13] J. Topp. Kernels of digraphs formed by some unary operations from other digraphs. *Rostocker Mathematisches Kolloquium*, 21:73–81, 1982.

R. Lakshmi
Department of Mathematics, Annamalai University
Annamalainagar 608 002, India
Department of Mathematics, Dharumapuram Gnanambigai Government Arts College for Women
Mayiladuthurai 609 001, India

D. G. Sindhu
St. Francis de Sales College (Autonomous)
Electronics City Post, Bengaluru 560 100, India