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Nonsplit graphs with split maximal edge induced subgraphs

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ABSTRACT

A *split graph* is a graph in which the vertices can be partitioned into an independent set and a clique. We show that every nonsplit graph has at most four split maximal proper edge induced subgraphs. The exhaustive list of fifteen classes of nonsplit graphs having a split maximal proper edge induced subgraph is determined in this paper.

Keywords: Split graph, card, deck, Reconstruction

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1. Introduction

All graphs considered in this paper are finite, simple and undirected. We shall denote the set of vertices by $V(G)$ and the set of edges by $E(G)$. Terms not defined here are taken as in [12]. The subgraph of G induced by a vertex subset S is denoted by $\langle S\rangle$. The set of all vertices adjacent to v in G is denoted by $N_G(v)$ and it is called the *neighborhood* of v in G and $N_G[v] = N_G(v) \cup \{v\}$.

The one edge deleted unlabeled subgraph $G - e$ is called an *edge card* of G. The *edge deck* of G, denoted $ED(G)$, is the multiset of edge deleted subgraphs (edge cards) of G, and we refer to the elements of $ED(G)$ as *maximal proper edge induced subgraphs* (or simply *edge cards* or *ecards*). The *edge reconstruction number* of G, denoted $ERN(G)$, is the size of a smallest subset of $ED(G)$ that determines G up to isomorphism, if such a subset exists. More precisely, if $ERN(G) = k$, then there are edges $e_1, e_2, \ldots, e_k$ in G such that if H is a graph with edges $e'_1, e'_2, \ldots, e'_k$ where $G - e_i \cong H - e'_i$ for $i = 1, 2, \ldots, k$, then $G \cong H$. Graphs for which $ERN(G)$ is defined, that is, those graphs that are determined by their

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edge deck, are said to be *edge reconstructible*. Edge reconstruction numbers are known for only few classes of graphs [9, 10]. The famous Edge Reconstruction Conjecture, proposed by Harary [5], asserts that every graph with at least four edges is edge reconstructible. The vertex reconstructible graphs (or simply reconstructible graphs) and the vertex reconstruction number (or simply reconstruction number, denoted by $RN(G)$) of a graph are defined similarly with vertex deletions instead of edge deletions. The $RN(G)$ was first defined by Harary and Plantholt [6]. Bollobas [2] has shown that almost all graphs G (in the probabilistic sense) have $RN(G) = 3$. Monikandan and Anu [11] have recently shown that the reconstruction number of most of the separable graphs (connected graphs with a cut vertex) with pendant vertices is three if the reconstruction number of all other connected graphs without pendant vertices is three, which strengthen the above result of Bollobas [2]. For a survey of early results on the reconstruction number problems, see [1]. The reconstruction number (edge reconstruction number) of G is expected to serve as a measure of the level of difficulty of reconstructing G from the deck (edge deck) of G.

A *clique* of a graph G is a vertex subset inducing a complete subgraph of G. A vertex subset I of G is called an *independent set* if no pair of distinct vertices of I are adjacent in G. A graph is *split* if the vertex set can be partitioned into an independent set and a clique. Hammer and Foldes [4] characterized split graphs as graphs G such that both G and the complement graph $\bar{G}$ are chordal graphs (graphs in which every cycle of length at least four contains a chord). Many characterizations and properties of split graphs were obtained ([8]; Ch. 8 & 9). A graph is distance-hereditary if for all connected induced subgraphs F of G, $d_F(u, v) = d_G(u, v)$ for every pair of vertices $u, v \in V(F)$. Distance-hereditary graphs were introduced by Howorka [7], who was also the first to characterize these graphs. Devi Priya and Monikandan [3] have proved a reduction that all graphs are reconstructible if the family $\mathscr{F}$ of all non-distance hereditary 2-connected graphs G with $diam(G) = 2$ or $diam(G) = diam(\overline{G}) = 3$ is reconstructible. So, any result proving or determining the possibility of the (edge) reconstructibility of a subclass of $\mathscr{F}$ assumes importance. Many family of connected split graphs without end vertices belong to $\mathscr{F}$.

We show that every nonsplit graph has at most four split ecards and so if at least five ecards of a graph are split, then the graph is split. The exhaustive list of fifteen classes of nonsplit graphs having a split ecard is given, which is the main result of this paper.

2. Nonsplit graphs with split edge cards

The following characterization of split graphs was obtained by Foldes and Hammer[4].

Theorem 2.1. *A graph is split if and only if it has no induced subgraph isomorphic to C_5, C_4 or $2K_2$.*

Lemma 2.2. *Every edge card of a nonsplit graph containing an induced subgraph isomorphic to C_5 is nonsplit.*

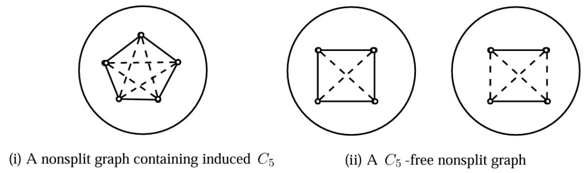


Fig. 1. Nonsplit graphs

Proof. Let G be a nonsplit graph containing an induced subgraph T isomorphic to C_5. For an edge $e \in E(G) - E(T)$, the ecard $G - e$ contains T as an induced subgraph and hence each ecard $G-e$ is nonsplit. For an edge $e' = ab \in E(T)$, $\langle N_{T-e'}[\{a, b\}]\rangle$ isomorphic to $2K_2$ and hence every ecard of G is nonsplit. □

As we want to list out all nonsplit graphs having split ecards, in view of the above lemma, we assume that all nonsplit graphs considered hereafter have no induced subgraph isomorphic to C_5 (i.e, C_5 – free nonsplit graphs).

Lemma 2.3. *Every nonsplit graph has at most four split edge cards.*

Proof. Let G be a nonsplit graph containing an induced subgraph T isomorphic to C_4 or $2K_2$. If an ecard, obtained by deleting an edge from a copy of C_4 (or $2K_2$) in G, may possibly contain no induced subgraph isomorphic to C_4 (or $2K_2$). Consequently, any nonsplit graph can have at most four edges such that the corresponding ecards may not contain C_4(or $2K_2$) as an induced subgraph. Since the other type have two edges of the same type, the graph G has at most four split ecards. □

Corollary 2.4. *If an edge deck of a graph G contains five split edge cards, then the unknown parent graph G is split.*

We now proceed to find all nonsplit graphs having precisely k split ecards, where $k = 1, 2, 3$, or 4. As a prelude, we define few notation for the sake of clarity.

Notation 2.5. Let U and W be disjoint subsets of $V(G)$.
(i) By $U \sim W$ means that there is a vertex in U adjacent to at least one vertex in W;
(ii) By $U \sim\sim W$, we mean that every vertex in U is adjacent to every vertex in W;
(iii) By $U \nsim W$, we mean that there is a vertex in U not adjacent to at least one vertex in W;
(iv) By $U \nsim\nsim W$ means that no vertex in U is adjacent to a vertex in W.

For $U = \{u\}$, we just write $u \sim W$ and $u \nsim W$ instead of $U \sim W$ and $U \nsim W$, respectively.

Let $\mathscr{G}$ be the collection of all nonsplit graphs containing an induced subgraph isomorphic to C_4 or $2K_2$.. Let $G \in \mathscr{G}$. Let $T(G)$ be a subset of $V(G)$ such that the subgraph induced by $T(G)$ contains all induced subgraph isomorphic to C_4 or $2K_2$. Then $G - e$ is nonsplit for all $e \in E(G) - E(\langle T(G) \rangle)$ and if $G - e$ is a split ecard , then $e \in E(\langle T(G) \rangle)$. Therefore, all the vertices (if any) of G that are not in $T(G)$ can be partitioned into a clique $C(G)$ and an independent set $I(G)$, where $C(G)$ and $I(G)$ may be empty. That is, $V(G) = C(G) \cup I(G) \cup T(G)$. If no confusion arises, we simply use T, C, I instead of $T(G)$, $C(G)$, $I(G)$ respectively.

In view of Theorem 2.1, we have the following two properties $P(C_4)$ and $P(2K_2)$.

$P(C_4)$: In a nonsplit graph G containing an induced subgraph $\langle K \rangle \cong C_4$, if $G - v_1v_2$ is a split ecard, where $v_1, v_2 \in K$, then vertices in $K - \{v_1, v_2\}$ must lie in the clique partition of $G - v_1v_2$ and v_1 and v_2 must lie in the independent partition of $G - v_1v_2$.

$P(2K_2)$: In a nonsplit graph G containing an induced subgraph $\langle K \rangle \cong 2K_2$, if $G - v_1v_2$ is a split ecard, where $v_1, v_2 \in K$, then v_1 and v_2 must lie in the independent partition of $G - v_1v_2$ and one vertex in $K - \{v_1, v_2\}$ must lie in the clique partition of $G - v_1v_2$.

2.1. Nonsplit graphs with one split maximal edge induced subgraph

Here we want to find a nonsplit graph H with exactly one split edge card, say $H - e$. Being a split edge card of a nonsplit graph, H must contain an induced subgraph T isomorphic to C_4 or $2K_2$. Every ecard $H - e'$, obtained by deleting an edge except from a copy of C_4 (or $2K_2$) in H, must contain an induced subgraph isomorphic to C_4 (or $2K_2$) and hence $H - e'$ is nonsplit and the required edge e of H must lie in all induced subgraphs that are isomorphic to C_4 or $2K_2$. Thus, we can partite the collection of nonsplit graphs with exactly one split edge card as below:

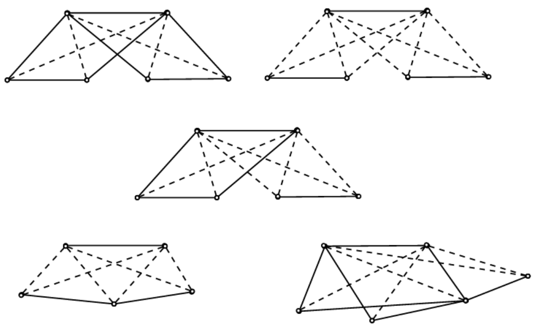


Fig. 2. Forbidden subgraphs for nonsplit graphs with one split edge card

In all the figures in this paper, a single line denotes the existence of an edge, a double line denotes the existence of all possible edges, dashed single line denotes nonexistence of an edge, and dashed double line denotes the nonexistence of any edge.

Let $G_1 \in \mathscr{G}$ with $T(G_1) = \{a_1, a_2, a_3, a_4, a_5\}$, $a_1a_2, a_3a_4, a_4a_5 \in E(G_1)$ and $a_1a_3, a_1a_4, a_2a_3, a_2a_4, a_1a_5, a_2a_5 \notin E(G_1)$ and let $G_2 \in \mathscr{G}$ with $T(G_2) = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, $a_1a_2, a_2a_3, a_3a_4, a_1a_4, a_1a_5, a_3a_5, a_3a_6 \in E(G_2)$ and $a_1a_3, a_2a_4, a_2a_5, a_1a_6, a_2a_6 \notin E(G_2)$.

A nonsplit graph G has exactly one split ecard only if G belongs to any one of the following five subfamilies $\mathscr{F}1i$ of graphs in $\mathscr{G}$, for $i = 1, 2, 3, 4, 5$.

$\mathscr{F}11$: Graphs containing two induced C_4 with exactly one common edge.

$\mathscr{F}12$: Graphs containing two induced $2K_2$ with exactly one common edge.

$\mathscr{F}13$: Graphs containing an induced C_4 and an induced $2K_2$ with exactly one common edge.

$\mathscr{F}14$: Graphs containing G_1 as an induced subgraph.

$\mathscr{F}15$: Graphs containing G_2 as an induced subgraph.

2.1.1. The family $\mathscr{F}11$. Let $G_3 \in \mathscr{G}$ with $T(G_3) = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, $a_1a_2, a_2a_3, a_3a_4, a_4a_5, a_5a_6, a_6a_1, a_1a_4$ $\in E(G_3)$ and $a_1a_3, a_2a_4, a_4a_6, a_1a_5 \notin E(G_3)$. We shall now construct G_3 such that $G_3 - a_1a_4$ to be split.

$\underline{a_1a_4\text{-ecard}}$:

If a_1 was lying in a clique partition of the maximal edge induced subgraph $G_3 - a_1a_4$, then, since $a_1a_3, a_1a_4 \notin E(G_3 - a_1a_4)$ and $a_3a_4 \in E(G_3 - a_1a_4)$, both a_3 and a_4 would not lie in an independent set of the maximal edge induced subgraph $G_3 - a_1a_4$, giving a contradiction by $P(C_4)$. Therefore a_1 lies in the independent set partition of the maximal edge induced subgraph $G_3 - a_1a_4$ and $a_1 \nsim\nsim I(G_3)$. Similarly, a_4 lies in the independent set partition of the maximal edge induced subgraph $G_3 - a_1a_4$ and $a_4 \nsim\nsim I(G_3)$.

If a_2 was lying in an independent partition of the maximal edge induced subgraph $G_3 - a_1a_4$, then, since $a_1a_2, a_2a_3 \in E(G_3 - a_1a_4)$ and $a_1a_3 \notin E(G_3 - a_1a_4)$, both a_1 and a_3 would not lie in a clique of the maximal edge induced subgraph $G_3 - a_1a_4$, giving a contradiction by $P(C_4)$. Therefore a_2 lies in the clique partition of the maximal edge induced subgraph $G_3 - a_1a_4$ and $a_2 \sim\sim C(G_3)$. Similarly, a_3, a_5 and a_6 lie in the clique partition of the maximal edge induced subgraph $G_3 - a_1a_4$ and $\{a_3, a_5, a_6\} \sim\sim C(G_3)$. Therefore, a necessary condition for $G_3 - a_1a_4$ to be a split ecard of G_3 is $\{a_2, a_3, a_5, a_6\} \sim\sim C(G_3)$ & $\{a_1, a_4\} \nsim\nsim I(G_3)$.

A nonsplit graph G_3 has only one split ecard $G_3 - a_1a_4$ if and only if it satisfies

1C.1: $\{a_2, a_3, a_5, a_6\} \sim\sim C(G_3)$, $\{a_1, a_4\} \nsim\nsim I(G_3)$ and $\{a_2, a_3\} \sim\sim \{a_5, a_6\}$ (Figure 3(i)) (The label 1C we use here to mean a condition under one split ecard case).

2.1.2. The family $\mathscr{F}12$. Let $G_4 \in \mathscr{G}$ with $T(G_4) = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, a_1a_2, a_3a_4, a_5a_6 $\in E(G_4)$ and $a_1a_3, a_1a_4, a_2a_3, a_2a_4, a_3a_5, a_3a_6, a_4a_5, a_4a_6 \notin E(G_4)$. We shall now construct G_4 such that $G_4 - a_3a_4$ to be split.

$\underline{a_3a_4\text{-ecard}}$:

If a_3 was lying in a clique of the maximal edge induced subgraph $G_4 - a_3a_4$, then, since $a_1a_3, a_2a_3 \notin E(G_4 - a_3a_4)$ and $a_1a_2 \in E(G_4 - a_3a_4)$, both a_1 and a_2 would not lie in an independent set of the maximal edge induced subgraph $G_4 - a_3a_4$, giving a contradiction by $P(2K_2)$. Therefore a_3 lies in the independent set partition of the maximal edge induced subgraph $G_4 - a_3a_4$ and $a_3 \nsim\nsim I(G_4)$. Similarly, a_4 lies in the independent set partition of the maximal edge induced subgraph $G_4 - a_3a_4$ and $a_4 \nsim\nsim I(G_4)$.

By $P(2K_2)$, either a_1 or a_2 lies in the clique partition of the maximal edge induced subgraph $G_4 - a_3a_4$, since $\langle\{a_1, a_2, a_3, a_4\}\rangle \cong 2K_2$. Similarly, by $P(2K_2)$, either a_5 or a_6 lies in the clique partition of the maximal edge induced subgraph $G_4 - a_3a_4$. Therefore, one of the following nine conditions (X1-X9) must be a necessary condition for the maximal edge induced subgraph $G_4 - a_3a_4$ of G_4 to be a split.

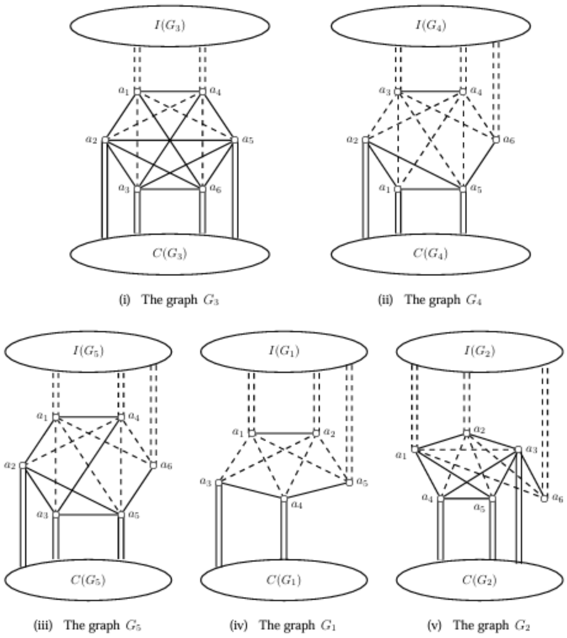


Fig. 3. Nonsplit graphs with one split ecard

X1 : $\{a_1, a_2, a_5, a_6\} \sim\sim C(G_4)$ & $\{a_3, a_4\} \nsim\nsim I(G_4)$
X2 : $\{a_1, a_2, a_5\} \sim\sim C(G_4)$ & $\{a_3, a_4, a_6\} \nsim\nsim I(G_4)$
X3 : $\{a_1, a_2, a_6\} \sim\sim C(G_4)$ & $\{a_3, a_4, a_5\} \nsim\nsim I(G_4)$
X4 : $\{a_1, a_5, a_6\} \sim\sim C(G_4)$ & $\{a_2, a_3, a_4\} \nsim\nsim I(G_4)$
X5 : $\{a_2, a_5, a_6\} \sim\sim C(G_4)$ & $\{a_1, a_3, a_4\} \nsim\nsim I(G_4)$
X6 : $\{a_1, a_5\} \sim\sim C(G_4)$ & $\{a_2, a_3, a_4, a_6\} \nsim\nsim I(G_4)$
X7 : $\{a_1, a_6\} \sim\sim C(G_4)$ & $\{a_2, a_3, a_4, a_5\} \nsim\nsim I(G_4)$
X8 : $\{a_2, a_5\} \sim\sim C(G_4)$ & $\{a_1, a_3, a_4, a_6\} \nsim\nsim I(G_4)$
X9 : $\{a_2, a_6\} \sim\sim C(G_4)$ & $\{a_1, a_3, a_4, a_5\} \nsim\nsim I(G_4)$

A nonsplit graph G_4 has only one split ecard $G_4 - a_3a_4$ if and only if it satisfies one of the following adjacency conditions (1C.2) to (1C.4).

(1C.2:) $\{a_1, a_2, a_5, a_6\} \sim\sim C(G_4), \{a_3, a_4\} \nsim\nsim I(G_4)$ and $\{a_1, a_2\} \sim\sim \{a_5, a_6\}$.

(1C.3:) $\{a_1, a_2, a_5\} \sim\sim C(G_4), \{a_3, a_4, a_6\} \nsim\nsim I(G_4)$ and $\{a_1, a_2\} \sim\sim \{a_5\}$ (Figure 3(ii)).

(1C.4:) $\{a_1, a_5\} \sim\sim C(G_4)$, $\{a_2, a_3, a_4, a_6\} \nsim\nsim I(G_4)$, $a_1 \sim a_5$ and $a_2 \nsim a_6$.

2.1.3. The family $\mathscr{F}13$. Let $G_5 \in \mathscr{G}$ with $T(G_5) = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, $a_1a_2, a_2a_3, a_3a_4, a_4a_1, a_5a_6 \in E(G_5)$ and $a_1a_3, a_2a_4, a_1a_5, a_1a_6, a_4a_5, a_4a_6 \notin E(G_5)$. We shall now construct G_5 such that $G_5 - a_1a_4$ to be split.

a_1a_4-ecard:

If a_1 was lying in a clique of the maximal edge induced subgraph $G_5 - a_1a_4$, then, since $a_1a_3, a_1a_4 \notin E(G_5 - a_1a_4)$ and $a_3a_4 \in E(G_5 - a_1a_4)$, both a_3 and a_4 would not lie in an independent set of the maximal edge induced subgraph $G_5 - a_1a_4$, giving a contradiction by $P(C_4)$. Therefore a_1 lies in the independent set partition of the maximal edge induced subgraph $G_5 - a_1a_4$ and $a_1 \nsim\nsim I(G_5)$. Similarly, a_4 lies in the independent set partition of the maximal edge induced subgraph $G_5 - a_1a_4$ and $a_4 \nsim\nsim I(G_5)$. Similarly, by $P(C_4)$, both a_2 and a_3 lie in the clique partition of the maximal edge induced subgraph $G_5 - a_1a_4$ and $\{a_2, a_3\} \sim\sim C(G_5)$ and by $P(2K_2)$, either a_5 or a_6 lies in the clique partition of the ecard $G_5 - a_1a_4$. Therefore, one of the following three conditions (Y1-Y3) must be a necessary condition for the maximal edge induced subgraph $G_5 - a_1a_4$ of G_5 to be a split.

Y1 : $\{a_2, a_3, a_5, a_6\} \sim\sim C(G_5)$ & $\{a_1, a_4\} \nsim\nsim I(G_5)$
Y2 : $\{a_2, a_3, a_5\} \sim\sim C(G_5)$ & $\{a_1, a_4, a_6\} \nsim\nsim I(G_5)$
Y3 : $\{a_2, a_3, a_6\} \sim\sim C(G_5)$ & $\{a_1, a_4, a_5\} \nsim\nsim I(G_5)$

A nonsplit graph G_5 has only one split ecard $G_5 - a_1a_4$ if and only if it satisfies one of the following adjacency conditions (1C.5) to (1C.6).

(1C.5:) $\{a_2, a_3, a_5, a_6\} \sim\sim C(G_5), \{a_1, a_4\} \nsim\nsim I(G_5)$ and $\{a_2, a_3\} \sim\sim \{a_5, a_6\}$.

(1C.6:) $\{a_2, a_3, a_5\} \sim\sim C(G_5), \{a_1, a_4, a_6\} \nsim\nsim I(G_5)$ and $\{a_2, a_3\} \sim\sim \{a_5\}$ (Figure 3(iii)).

2.1.4. The family $\mathscr{F}14$. Let $G_1 \in \mathscr{G}$ with $T(G_1) = \{a_1, a_2, a_3, a_4, a_5\}$, $a_1a_2, a_3a_4, a_4a_5 \in E(G_1)$ and $a_1a_3, a_1a_4, a_2a_3, a_2a_4, a_1a_5, a_2a_5 \notin E(G_1)$. We shall now construct G_1 such that $G_1 - a_1a_2$ to be split.

<u>a_1a_2-ecard</u>:

If a_1 was lying in a clique of the maximal edge induced subgraph $G_1 - a_1a_2$, then, since $a_1a_3, a_1a_4 \notin E(G_1 - a_1a_2)$ and $a_3a_4 \in E(G_1 - a_1a_2)$, both a_3 and a_4 would not lie in an independent set of the maximal edge induced subgraph $G_1 - a_1a_2$, giving a contradiction by $P(2K_2)$. Therefore a_1 lies in the independent set partition of the maximal edge induced subgraph $G_1 - a_1a_2$ and $a_1 \nsim\nsim I(G_1)$. Similarly, a_2 lies in the independent set partition of the maximal edge induced subgraph $G_1 - a_1a_2$ and $a_2 \nsim\nsim I(G_1)$. By $P(2K_2)$, either a_3 or a_4 lies in the clique partition of the maximal edge induced subgraph $G_1 - a_1a_2$, since $\langle\{a_1, a_2, a_3, a_4\}\rangle \cong 2K_2$. Similarly, by $P(2K_2)$, either a_4 or a_5 lies in the clique partition of the ecard $G_1 - a_1a_2$. Therefore, one of the following five conditions (Z1-Z5) must be a necessary condition for maximal edge induced subgraph $G_1 - a_1a_2$ of G_1 to be a split.

Z1 : $\{a_3, a_4, a_5\} \sim\sim C(G_1)$ & $\{a_1, a_2\} \nsim\nsim I(G_1)$
Z2 : $\{a_3, a_4\} \sim\sim C(G_1)$ & $\{a_1, a_2, a_5\} \nsim\nsim I(G_1)$
Z3 : $\{a_4, a_5\} \sim\sim C(G_1)$ & $\{a_1, a_2, a_3\} \nsim\nsim I(G_1)$
Z4 : $\{a_3, a_5\} \sim\sim C(G_1)$ & $\{a_1, a_2, a_4\} \nsim\nsim I(G_1)$
Z5 : $\{a_4\} \sim\sim C(G_1)$ & $\{a_1, a_2, a_3, a_5\} \nsim\nsim I(G_1)$

A nonsplit graph G_1 has only one split ecards $G_1 - a_1a_2$ if and only if it satisfies one of the following adjacency conditions (1C.7) to (1C.10).

(1C.7:) $\{a_3, a_4, a_5\} \sim\sim C(G_1), \{a_1, a_2\} \nsim\nsim I(G_1)$ and $a_3 \sim a_5$.

(1C.8:) $\{a_3, a_4\} \sim\sim C(G_1)$ and $\{a_1, a_2, a_5\} \nsim\nsim I(G_1)$(Figure 3(iv)).

(1C.9:) $\{a_3, a_5\} \sim\sim C(G_1)$, $\{a_1, a_2, a_4\} \nsim\nsim I(G_1)$ and $a_3 \sim a_5$.

(1C.10:) $\{a_4\} \sim\sim C(G_1)$, $\{a_1, a_2, a_3, a_5\} \nsim\nsim I(G_1)$ and $a_3 \nsim a_5$.

2.1.5. The family $\mathscr{F}15$. Let $G_2 \in \mathscr{G}$ with $T(G_2) = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, $a_1a_2, a_2a_3, a_3a_4, a_1a_4, a_1a_5, a_3a_5, a_3a_6 \in E(G_2)$ and $a_1a_3, a_2a_4, a_2a_5, a_1a_6, a_2a_6 \notin E(G_2)$.

We shall now construct G_2 such that $G_2 - a_1a_2$ to be split.

a_1a_2-ecard:

If a_1 was lying in a clique of the maximal edge induced subgraph $G_2 - a_1a_2$, then, since $a_1a_2, a_1a_3 \notin E(G_2 - a_1a_2)$ and $a_2a_3 \in E(G_2 - a_1a_2)$, both a_2 and a_3 would not lie in an independent set of the maximal edge induced subgraph $G_2 - a_1a_2$, giving a contradiction by $P(C_4)$. Therefore a_1 lies in the independent set partition of the maximal edge induced subgraph $G_2 - a_1a_2$ and $a_1 \nsim\nsim I(G_2)$. Similarly, a_2 lies in the independent set partition of the maximal edge induced subgraph $G_2 - a_1a_2$ and $a_2 \nsim\nsim I(G_2)$. Similarly, by $P(C_4)$, a_3, a_4 and a_5 lie in the clique partition of the maximal edge induced subgraph $G_2 - a_1a_2$ and $\{a_3, a_4, a_5\} \sim\sim C(G_2)$.

Therefore, one of the following two conditions (W1-W2) must be a necessary condition for maximal edge induced subgraph $G_2 - a_1a_2$ of G_2 to be a split.

W1 : $\{a_3, a_4, a_5, a_6\} \sim\sim C(G_2)$ & $\{a_1, a_2\} \nsim\nsim I(G_2)$

W2 : $\{a_3, a_4, a_5\} \sim\sim C(G_2)$ & $\{a_1, a_2, a_6\} \nsim\nsim I(G_2)$

A nonsplit graph G_2 has only one split ecards $G_2 - a_1a_2$ if and only if it satisfies one of the following two adjacency conditions (1C.11) to (1C.12).

(1C.11:) $\{a_3, a_4, a_5, a_6\} \sim\sim C(G_2), \{a_1, a_2\} \nsim\nsim I(G_2)$, $a_4 \sim\sim \{a_5, a_6\}$ and $a_5 \sim a_6$.

(1C.12:) $\{a_3, a_4, a_5\} \sim\sim C(G_2)$, $\{a_1, a_2, a_6\} \nsim\nsim I(G_2)$and $a_4 \sim a_5$(Figure 3(v)).

From Subsections 2.1.1 to 2.1.5, we have twelve classes of graphs obtained by applying conditions 1C.1 to 1C.12. Clearly, nonsplit graphs in these twelve classes only have exactly one split ecards. Therefore, we have the following theorem.

Theorem 2.6. A nonsplit graph G has exactly one split ecard if and only if G lies in the class of graphs satisfying the conditions 1C.1 to 1C.12.

2.2. *Nonsplit graphs with two split maximal edge induced subgraphs*

A nonsplit graph G with such T has exactly two split ecards only if G belongs to any one of the following two families of graphs.

$\mathscr{F}21$: Graphs, in $\mathscr{G}$, containing two induced C_4 with exactly three common vertices (two common edges).

$\mathscr{F}22$: Graphs, in $\mathscr{G}$, containing an induced union of two complete graph on two vertices.

2.2.1. The family $\mathscr{F}21$. Let $G_6 \in \mathscr{G}$ with $T(G_6) = \{a_1, a_2, a_3, a_4, a_5\}$, $a_1a_2, a_2a_3, a_3a_4, a_4a_1, a_2a_5, a_4a_5 \in E(G_6)$ and $a_1a_3, a_2a_4, a_3a_5 \notin E(G_6)$. We shall now construct G_6 such that $G_6 - a_2a_3$ and $G_6 - a_3a_4$ are to be split.

a_2a_3-ecard:

By $P(C_4)$, a_2 and a_3 lie in an independent set of the maximal edge induced subgraph $G_6 - a_2a_3$ and $\{a_2, a_3\} \nsim\nsim I(G_6)$ and a_1, a_4 and a_5 lie in the clique partition of the maximal edge induced subgraph $G_6 - a_2a_3$ and $\{a_1, a_4, a_5\} \sim\sim C(G_6)$.

Hence a necessary condition for $G_6 - a_2a_3$ to be a split ecard of G_6 is $\{a_1, a_4, a_5\} \sim\sim C(G_6)$, $\{a_2, a_3\} \nsim\nsim I(G_6)$ and $a_1 \sim a_5$.

a_3a_4-ecard:

By $P(C_4)$, a_3 and a_4 lie in an independent set of the maximal edge induced subgraph

$G_6 - a_3a_4$ and $\{a_3, a_4\} \nsim\nsim I(G_6)$ and a_1, a_2 and a_5 lie in clique of the maximal edge induced subgraph $G_6 - a_3a_4$ and $\{a_1, a_2, a_5\} \sim\sim C(G_6)$. Hence a necessary condition for $G_6 - a_3a_4$ to be a split ecard of G_6 is $\{a_1, a_2, a_5\} \sim\sim C(G_6)$, $\{a_3, a_4\} \nsim\nsim I(G_6)$ and $a_1 \sim a_5$.

In the graph G_6 with split ecards $G_6 - a_2a_3$ and $G_6 - a_3a_4$, a vertex, that is not in both $T(G_6)$ and a clique of $G_6 - a_2a_3$, does not lie in an independent set of $G_6 - a_3a_4$ and similarly a vertex, that is not in both $T(G_6)$ and an independent set of $G_6 - a_2a_3$, does not lie in a clique of $G_6 - a_3a_4$. Therefore, a nonsplit graph G_6 has only two split ecards $G_6 - a_2a_3$ and $G_6 - a_3a_4$ if it satisfies

(2C.1:)$\{a_1, a_2, a_4, a_5\} \sim\sim C(G_6), \{a_2, a_3, a_4\} \nsim\nsim I(G_6)$ and $a_1 \sim a_5$(Figure 4(i)).

2.2.2. The family $\mathscr{F}22$. Let $G_7 \in \mathscr{G}$ with $T(G_7) = \{a_1, a_2, a_3, a_4\}$, $a_1a_2, a_3a_4 \in E(G_7)$ and $a_1a_3, a_1a_4, a_2a_3, a_2a_4 \notin E(G_7)$. We shall now construct G_7 such that $G_7 - a_1a_2$ and $G_7 - a_3a_4$ are to be split.

a_1a_2-ecard:

By $P(2K_2)$, a_1 and a_2 lie in an independent set of the maximal edge induced subgraph $G_7 - a_1a_2$ and $\{a_1, a_2\} \nsim\nsim I(G_7)$ and either a_3 or a_4 lies in the clique partition of the maximal edge induced subgraph $G_7 - a_1a_2$. Therefore, one of the following three conditions (X1-X3) must be a necessary condition for the maximal edge induced subgraph $G_7 - a_1a_2$ of G_7 to be a split.

X1 : $\{a_3, a_4\} \sim\sim C(G_7)$ & $\{a_1, a_2\} \nsim\nsim I(G_7)$
X2 : $\{a_3\} \sim\sim C(G_7)$ & $\{a_1, a_2, a_4\} \nsim\nsim I(G_7)$
X3 : $\{a_4\} \sim\sim C(G_7)$ & $\{a_1, a_2, a_3\} \nsim\nsim I(G_7)$

a_3a_4-ecard:

By $P(2K_2)$, a_3 and a_4 lie in an independent set of the maximal edge induced subgraph $G_7 - a_3a_4$ and $\{a_3, a_4\} \nsim\nsim I(G_7)$ and either a_1 or a_2 lies in the clique of partition of the maximal edge induced subgraph $G_7 - a_3a_4$. Therefore, one of the following three conditions (Y1-Y3) must be a necessary condition for the maximal edge induced subgraph $G_7 - a_3a_4$ of G_7 to be a split.

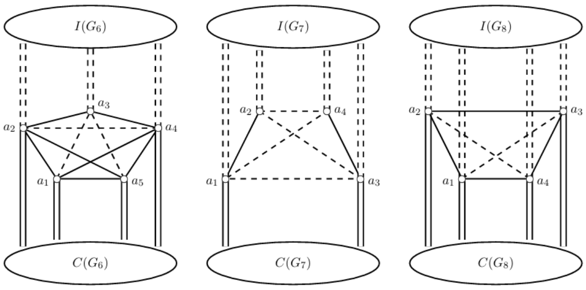
Y1 : $\{a_1, a_2\} \sim\sim C(G_7)$ & $\{a_3, a_4\} \nsim\nsim I(G_7)$
Y2 : $\{a_1\} \sim\sim C(G_7)$ & $\{a_2, a_3, a_4\} \nsim\nsim I(G_7)$
Y3 : $\{a_2\} \sim\sim C(G_7)$ & $\{a_1, a_3, a_4\} \nsim\nsim I(G_7)$

Among all conditions (X1-X3) and (Y1-Y3), a nonsplit graph G_7 has only two maximal edge induced split subgraphs $G_7 - a_1a_2$ and $G_7 - a_3a_4$ if it satisfies

(2C.2:) $\{a_1, a_3\} \sim\sim C(G_7)$ and $\{a_1, a_2, a_3, a_4\} \nsim\nsim I(G_7)$(Figure 4(ii)).

From subsections 2.2.1 to 2.2.2, we have two classes of graphs obtained by applying conditions 2C.1 to 2C.2. Clearly, nonsplit graphs in these two classes only have exactly two split ecards. Therefore, we have the following theorem.

Theorem 2.7. *A nonsplit graph G has exactly two maximal edge induced split subgraphs if and only if G lies in the class of graphs satisfying the conditions 2C.1 and 2C.2.*



(i) The graph G_6 (ii) The graph G_7 (iii) The graph G_8

Fig. 4. Nonsplit graphs with two split ecards and four split ecards

2.3. Nonsplit graphs with three split maximal edge induced subgraphs

This case does not arise, because there is no nonsplit graph H with exactly three split ecards.

2.4. Nonsplit graphs with four split maximal edge induced subgraphs

A nonsplit graph G with such T has exactly four split ecards only if G belongs to the following family of graphs.

$\mathscr{F}41$: graphs in $\mathscr{G}$ containing an induced cycle on four vertices.

The family $\mathscr{F}41$:

Let $G_8 \in \mathscr{G}$ with $T(G_8) = \{a_1, a_2, a_3, a_4\}$, $a_1a_2, a_2a_3, a_3a_4, a_4a_1 \in E(G_8)$ and $a_1a_3, a_2a_4 \notin E(G_8)$. We now construct G_8 such that $G_8 - a_1a_2$, $G_8 - a_2a_3$, $G_8 - a_3a_4$ and $G_8 - a_4a_1$ are to be split.

a_1a_2-ecard:

By $P(C_4)$, a_1 and a_2 lie in an independent set of the maximal edge induced subgraph $G_8 - a_1a_2$ and $\{a_1, a_2\} \nsim\nsim I(G_8)$ and a_3 and a_4 lie in the clique partition of the maximal edge induced subgraph $G_8 - a_1a_2$ and $\{a_3, a_4\} \sim\sim C(G_8)$. Hence a necessary condition for $G_8 - a_1a_2$ to be a split ecard of G_8 is $\{a_3, a_4\} \sim\sim C(G_8)$ and $\{a_1, a_2\} \nsim\nsim I(G_8)$.

a_2a_3-ecard:

By $P(C_4)$, a_2 and a_3 lie in an independent set of the maximal edge induced subgraph $G_8 - a_2a_3$ and $\{a_2, a_3\} \nsim\nsim I(G_8)$ and a_1 and a_4 lie in the clique partition of the maximal edge induced subgraph $G_8 - a_2a_3$ and $\{a_1, a_4\} \sim\sim C(G_8)$. Hence a necessary condition for $G_8 - a_2a_3$ to be a split ecard of G_8 is $\{a_1, a_4\} \sim\sim C(G_8)$ and $\{a_2, a_3\} \nsim\nsim I(G_8)$.

a_3a_4-ecard:

By $P(C_4)$, a_3 and a_4 lie in an independent set of the maximal edge induced subgraph $G_8 - a_3a_4$ and $\{a_3, a_4\} \nsim\nsim I(G_8)$ and a_1 and a_2 lie in the clique partition of the maximal edge induced subgraph $G_8 - a_3a_4$ and $\{a_1, a_2\} \sim\sim C(G_8)$. Hence a necessary condition for $G_8 - a_3a_4$ to be a split ecard of G_8 is $\{a_1, a_2\} \sim\sim C(G_8)$ and $\{a_3, a_4\} \nsim\nsim I(G_8)$.

a_4a_1-ecard:

By $P(C_4)$, a_4 and a_1 lie in an independent set of the maximal edge induced subgraph $G_8 - a_4a_1$ and $\{a_4, a_1\} \nsim\nsim I(G_8)$ and a_2 and a_3 lie in the clique partition of the maximal edge induced subgraph $G_8 - a_4a_1$ and $\{a_2, a_3\} \sim\sim C(G_8)$. Hence a necessary condition for $G_8 - a_4a_1$ to be a split ecard of G_8 is $\{a_2, a_3\} \sim\sim C(G_8)$ and $\{a_1, a_4\} \nsim\nsim I(G_8)$.

Therefore, a nonsplit graph G_8 has only four split ecards $G_8 - a_1a_2$, $G_8 - a_2a_3$, $G_8 - a_3a_4$ and $G_8 - a_4a_1$ if it satisfies

(4C.1:)$\{a_1, a_2, a_3, a_4\} \sim\sim C(G_8)$ and $\{a_1, a_2, a_3, a_4\} \nsim\nsim I(G_8)$(Figure 4(iii)).

Clearly nonsplit graphs in this class only have exactly four split ecards. Therefore, we have the following theorem.

Theorem 2.8. *A nonsplit graph G has exactly four split ecards if and only if G lies in the class of graphs satisfying the conditions 4C.1.*

3. Concluding remarks

In this paper, we have listed

(i) twelve families of nonsplit graphs having exactly one split ecard;

(ii) two families of split graphs having exactly two split ecards; and

(iii) only one family of nonsplit graphs having exactly four split ecards.

Using these fifteen classes of nonsplit graphs, we can check whether any given nonsplit graph has a split ecard or not. This may give us inference to find the edge reconstruction number of nonsplit graphs.

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