On the interval coloring impropriety of graphs

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ABSTRACT

An *improper interval (edge) coloring* of a graph G is an assignment of integer colors to the edges of G satisfying the condition that, for every vertex $v \in V(G)$, the set of colors assigned to the edges incident with v forms an integral interval. An interval coloring is *k-improper* if at most k edges with the same color all share a common endpoint. The minimum integer k such that there exists a k-improper interval coloring of the graph G is the *interval coloring impropriety* of G, denoted by $\mu_{int}(G)$. In this paper, we provide a construction of an interval coloring of a subclass of complete multipartite graphs. Additionally, we determine improved upper bounds on the interval coloring impropriety of several classes of graphs, namely 2-trees, iterated triangulations, and outerplanar graphs. Finally, we investigate the interval coloring impropriety of the corona product of two graphs, $G \odot H$.

Keywords: improper coloring, interval coloring, interval coloring impropriety

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1. Introduction

Let G be a finite graph with vertex set $V(G)$, and edge set $E(G)$. An *edge coloring* of a graph G is a function $\phi : E(G) \to \mathcal{C}$ assigning labels, or colors, from a set $\mathcal{C}$, normally $\mathbb{Z}$, to the edges of G. An edge coloring is *proper* if the colors are assigned in a way so that adjacent edges (those sharing an endpoint) in G receive different colors. Otherwise, the edge coloring is *improper*.

Asratian and Kamalian [4, 5] introduced the problem of interval k-coloring a graph

as a model for creating timetables without gaps for teachers and students. A proper edge coloring of a graph G is said to be an *interval k-coloring* (also called a consecutive coloring, for example in [12]) if exactly k colors are used to color the edges of G and the colors assigned to the edges incident to each vertex $v \in V(G)$ are distinct and form a set of consecutive integers. Note that there exist graphs that do not have an interval coloring, for example complete graphs K_n for odd $n \geq 3$. This leads to the question of whether a given graph has an interval coloring, which has been shown to be NP-complete to determine for bipartite graphs [28]. This problem has been investigated for several classes of graphs, including trees, regular graphs and complete bipartite graphs [15, 19], subclasses of bipartite graphs [13, 15, 24], outerplanar graphs with maximum degree 3 [23], Cartesian, strong, lexicographic and tensor products and compositions of graphs [12, 22, 25, 29], complete tripartite graphs $K_{1,m,n}$ [14], and biregular graphs [1, 3, 7, 9, 10, 16, 18, 27, 30]. However, there remain many classes of graphs for which it is not known whether an interval coloring exists for all graphs in the class, including $(3,4)$-biregular graphs.

Hudák et al. [17] introduced *improper interval (edge) colorings*, motivated by scheduling problems without gaps in the schedule, but with some degree of conflict permitted. An improper edge coloring of a graph G is said to be an *improper interval (edge) coloring* if the colors assigned to the edges incident with each vertex $v \in V(G)$ form an integral interval. Subsequently, Casselgren and Petrosyan [8] introduced the *interval coloring impropriety* of a graph. An improper interval coloring is *k-improper* if at most k edges with the same color all share a common endpoint. The smallest integer k such that a graph G has a k-improper interval coloring is called the *interval coloring impropriety* (or simply *impropriety*) of G, denoted by $\mu_{int}(G)$. Note that assigning every edge of G the same color produces a $\Delta(G)$-improper interval coloring, so the interval coloring impropriety is well-defined for every graph. If G has a proper interval coloring, then $\mu_{int}(G) = 1$.

In this paper, we investigate the impropriety of several classes of graphs. In Section 2, we begin with some preliminary definitions and results. In Section 3, we consider complete multipartite graphs, providing a construction of a coloring that shows that a particular subclass of complete multipartite graphs is interval colorable. In Section 4, we provide an upper bound on the impropriety of a 2-tree, and on the impropriety of the n-th iterated triangulation, both in terms of the maximum degree of the graph. In Section 5, we improve upon the upper bound on the impropriety of an outerplanar graph given in [8]. In Section 6, we derive upper bounds on the impropriety of the corona product of two graphs. Finally, in Section 7, we give some directions for further study of the improper interval impropriety of a graph and pose some open problems.

2. Preliminaries

For a positive integer n, we use $[n]$ to denote the set $\{1, 2, \ldots, n\}$. All graphs are assumed to be simple and finite, unless otherwise indicated. For a vertex v in a graph G, we use $N_G(v)$ to denote the neighborhood of v in G, and $d_G(v)$ to denote the degree of v in G, or $N(v)$ and $d(v)$, respectively, when the graph is obvious from context. For a positive

integer k, if $v \in V(G)$ satisfies $d_G(v) = k$, then we say that v is a *k-vertex* in G. The maximum degree of a vertex in G is denoted by $\Delta(G)$ and the minimum degree of a vertex in G by $\delta(G)$. A vertex v for which $N(v) \cup \{v\} = V(G)$ is called a *universal vertex*. A *clique* is a set of pairwise adjacent vertices in a graph. A complete graph, denoted by K_n, is a graph of order n in which $V(K_n)$ forms a clique. A *path* in a graph G is denoted by $v_1 v_2 \ldots v_n$, where $v_i v_{i+1} \in E(G)$ for $i \in [n-1]$. We say that v_1 and v_n are the *endpoints* of the path and denote the path graph with n vertices by P_n. For vertices $u, v \in V(G)$, the *distance* from u to v, denoted by $d_G(u, v)$ is the minimum length of a path with endpoints u and v. If no such path exists, then $d_G(u, v) = \infty$. Similarly, a *cycle* in a graph G is denoted by $v_1 v_2 \ldots v_n v_1$, where $v_i v_{i+1} \in E(G)$ for $i \in [n-1]$ and $v_n v_1 \in E(G)$. The cycle graph with n vertices is denoted by C_n. For a positive integer $k \geq 3$, we refer to C_k as a *k-cycle*. A *star* S_n is a graph that consists of a center vertex v and leaves $u_1, u_2, \ldots, u_n$ with $vu_i \in E(S_n)$ for $i \in [n]$. Note that $S_n = K_{1,n}$. A *spider graph* is a tree with a center vertex of degree at least 3, and all other vertices with degree at most 2. The family of spider graphs whose center vertex has degree exactly k is denoted by SP_k. A *caterpillar graph* is a tree for which the removal of all leaves produces a path. A *wheel graph* is a graph formed by connecting a single universal vertex to all vertices of a cycle. For a positive integer n, we use W_n to denote a wheel graph with n vertices.

A *planar graph* is a graph which can be drawn in the plane without any edges crossing. A *plane graph* is a planar graph with a specific planar embedding. Thus, a plane graph divides the plane into a set of regions, called *faces*. Each face is bounded by a closed walk called the *boundary of the face*. Each plane graph has exactly one unbounded face, and we call such a face the *outer face*. All the other bounded faces of a plane graph are called *inner faces*. An *outerplanar graph* is a planar graph that has an embedding in which all vertices are on the boundary of the outer face. For all graph theory terminology not defined here, refer to [6].

The *chromatic index* of a graph G, denoted by $\chi'(G)$, is the minimum number of colors required to construct a proper edge coloring of G. Next, we state some existing results about the chromatic index and its relationship with impropriety.

Theorem 2.1 (Vizing's Theorem). *For any graph G, either $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$.*

Graphs with $\chi'(G) = \Delta(G)$ are called *Class 1* and those with $\chi'(G) = \Delta(G) + 1$ are called *Class 2*. There are some families of graphs that are known to be Class 1.

Theorem 2.2 (König's Edge Coloring Theorem). *If a graph G is bipartite, then G is Class 1.*

Knowing whether a graph is Class 1 or Class 2 can help with determining whether the graph has an interval coloring or, if not, what the impropriety of the graph is.

Theorem 2.3 ([5]). *If a graph G is interval colorable (i.e. $\mu_{int}(G) = 1$), then G is Class 1.*

Equivalently, a graph that is Class 2 does not admit an interval coloring, and thus has impropriety at least two.

Theorem 2.4 ([8]). *If G is a regular graph of Class i, then $\mu_{int}(G) = i$ for $i \in \{1, 2\}$.*

As a corollary, this result implies that complete graphs and cycles of even order are interval colorable, while those of odd order have impropriety 2.

For general graphs, it is known that the impropriety is at most two if the degree of every vertex is small.

Theorem 2.5 ([8]). *If G is a graph with $\Delta(G) \leq 5$, then $\mu_{int}(G) \leq 2$.*

Finally, we make use of the following results that determine the impropriety of some well-known classes of graphs, whose maximum degree is unbounded.

Theorem 2.6 ([19]). *If T is a forest, then T is interval colorable (i.e. $\mu_{int}(T) = 1$).*

Theorem 2.7 ([8]). *For a wheel graph W_n, $n \geq 4$, $\mu_{int}(W_n) = 1$ if and only if $n \in \{4, 7, 10\}$. Otherwise, $\mu_{int}(W_n) = 2$.*

3. A subclass of complete multipartite graphs

The impropriety of complete multipartite graphs was first studied in [8]. We recall the following result.

Theorem 3.1 ([8]). *If $n_1, \ldots, n_r \geq 1$, then $\mu_{\text{int}}(K_{n_1,\ldots,n_r}) \leq \left\lceil \frac{r}{2} \right\rceil$.*

Casselgren and Petrosyan [8] then conjectured that, for a complete multipartite graph G, $\mu_{\text{int}}(G) \leq 2$. This conjecture was recently solved by Mkrtchyan [21]. Note that this upper bound is best possible, since there exist exist complete multipartite graphs of Class 2, thus with impropriety 2. However, this raises the question of which complete multipartite graphs are interval colorable and, given that the upper bound in Theorem 3.1 is in terms of the number of parts of the complete multipartite graph, whether the number of parts plays a role in this classification. In this section, we provide a family of complete multipartite graphs with an arbitrarily large number of parts that are interval colorable.

Parts of a complete multipartite graph are its maximal independent sets. Let G be a graph $K_{s,t,s,t,\ldots,s,t}$ with k parts of size s and k parts of size t. We say that G has $m = 2k$ parts. Then, we denote the parts of G by $A_1, B_1, \ldots, A_k, B_k$, where $|A_i| = s$ and $|B_i| = t$ for every $i \in [k]$. The set of edges between parts A and B is denoted by $E(AB)$ to simplify the expressions in the rest of the section. If A and B are parts of G where $V(A) = \{x_1, \ldots, x_s\}$ and $V(B) = \{y_1, \ldots, y_t\}$, then we say that we color $E(AB)$ *sequentially with shift* α if the edge $x_i y_j$ is assigned color $\alpha + i + j - 1$ for every $i \in [s]$, $j \in [t]$. Notice that, in this case, colors on edges incident to x_i are $\alpha + i, \ldots, \alpha + i + t - 1$, and colors on edges incident to y_j are $\alpha + j, \ldots, \alpha + j + s - 1$. In matrix form, the sequential coloring with shift α between

parts A and B is presented in Table 1.

Suppose that vertices of A and B are incident with some already colored edges. We will say for short that we *extend the coloring sequentially* to color $E(AB)$ in the following way. Let $\beta + i - 1$ be the maximum color incident with x_i ($i \in [s]$), and let $\beta + j - 1$ be the maximum color incident with y_j ($j \in [t]$). Then color $E(AB)$ sequentially with shift β.

Table 1. The sequential coloring with shift α in matrix form

	x_1	x_2	$\cdots$	x_s
y_1	$\alpha + 1$	$\alpha + 2$	$\cdots$	$\alpha + s$
y_2	$\alpha + 2$	$\alpha + 3$	$\cdots$	$\alpha + s + 1$
$\vdots$	$\vdots$			$\vdots$
y_t	$\alpha + t$	$\alpha + t + 1$	$\cdots$	$\alpha + s + t - 1$

Additionally, in the following proofs, we will assume that the vertices are labelled (since this labelling is needed for the sequential coloring to be well-defined). More precisely, if the parts of the graph $K_{s,t,s,t,\ldots,s,t}$ are A_1, B_1, $\ldots$, A_k, B_k, where $|A_i| = s$, $|B_i| = t$, then let $V(A_i) = \{x_{i,1}, \ldots, x_{i,s}\}$ and $V(B_i) = \{y_{i,1}, \ldots, y_{i,t}\}$ for all $i \in [k]$.

It follows from Theorem 3.1 that $\mu_{\text{int}}(K_{s,t}) = 1$ for $s, t \geq 1$. To present the basic case of the coloring used later in the section, we recall the proof of this result. Let (A, B) be the bipartition of G and vertices labelled as described above. Color $E(AB)$ sequentially with shift 0, and observe that this is a proper interval coloring of G. Now, we extend this approach to complete multipartite graphs with a larger number of parts.

Table 2. The edge-coloring of $K_{4,3,4,3}$

	$x_{1,1}$	$x_{1,2}$	$x_{1,3}$	$x_{1,4}$	$y_{1,1}$	$y_{1,2}$	$y_{1,3}$	$x_{2,1}$	$x_{2,2}$	$x_{2,3}$	$x_{2,4}$	$y_{2,1}$	$y_{2,2}$	$y_{2,3}$
$x_{1,1}$					1	2	3	4	5	6	7	8	9	10
$x_{1,2}$					2	3	4	5	6	7	8	9	10	11
$x_{1,3}$					3	4	5	6	7	8	9	10	11	12
$x_{1,4}$					4	5	6	7	8	9	10	11	12	13
$y_{1,1}$	1	2	3	4				8	9	10	11	5	6	7
$y_{1,2}$	2	3	4	5				9	10	11	12	6	7	8
$y_{1,3}$	3	4	5	6				10	11	12	13	7	8	9
$x_{2,1}$	4	5	6	7	8	9	10					1	2	3
$x_{2,2}$	5	6	7	8	9	10	11					2	3	4
$x_{2,3}$	6	7	8	9	10	11	12					3	4	5
$x_{2,4}$	7	8	9	10	11	12	13					4	5	6
$y_{2,1}$	8	9	10	11	5	6	7	1	2	3	4			
$y_{2,2}$	9	10	11	12	6	7	8	2	3	4	5			
$y_{2,3}$	10	11	12	13	7	8	9	3	4	5	6			

Lemma 3.2. *For $s, t \geq 1$, the graph $K_{s,t,s,t}$ is interval colorable.*

Proof. Using the notation defined above, for $i \in [2]$, color $E(A_i B_i)$ sequentially with shift 0, color $E(A_1 A_2)$ sequentially with shift t, color $E(A_1 B_2)$ sequentially with shift $s + t$, color $E(B_1 B_2)$ sequentially with shift s, and color $E(A_2 B_1)$ sequentially with shift $s + t$. For an example with $s = 4$ and $t = 3$, see Table 2.

Using the above coloring, the colors on edges incident with $x_{i,1}$ are $i, \ldots, i + t - 1$, $i + t, \ldots, i + s + t - 1$, $i + s + t, \ldots, i + s + 2t - 1$, which is an interval with no color occurring more than once. The interval property can be similarly verified at each of the other vertices. Thus, $\mu_{\text{int}}(K_{s,t,s,t}) = 1$. $\hfill\square$

Theorem 3.3. *If $m = 2^\ell$ for some $\ell \geq 1$, then the complete multipartite graph $K_{s,t,s,t,\ldots,s,t}$ with m parts is interval colorable.*

Proof. For the cases where $m = 2$ or $m = 4$, see Lemma 3.2 and the example before it. We give an algorithmic proof for $m = 2^{\ell+1}$ when $\ell \geq 2$ that the graph $K_{s,t,\ldots,s,t}$ with m parts is interval colorable. First, we describe some general conventions used in the rest of the proof.

For the sake of notation, we use matrices to describe the set of colors assigned to edges between parts. We will define an $n \times n$ matrix (with n being the number of vertices in the graph) with rows and columns indexed by the vertices in the graph. We will call this matrix the coloring matrix. Note that, by symmetry, we only need to describe the entries of the coloring matrix above the main diagonal, since the entries below the main diagonal are obtained by transposing the coloring matrix. For $i \in [\frac{m}{2}]$, let A_i and B_i be the parts of size s and t, respectively. The rows and columns of the coloring matrix will follow the ordering $A_1, B_1, A_2, B_2, \ldots, A_{\frac{m}{2}}, B_{\frac{m}{2}}$. Denote by 0_s and 0_t the zero matrices of size $s \times s$ and $t \times t$, respectively. Let $X_{s,k}$ be the $s \times s$ matrix recording the sequential coloring with shift $(k - 1)s + kt$, for $k \geq 1$, on the edges between two parts of size s. Let $X_{t,k}$ be the $t \times t$ matrix recording the sequential coloring with shift $ks + (k - 1)t$, for $k \geq 1$, on the edges between two parts of size t. Let $Y_{s,k}$ (similarly, $Y_{t,k}$) be the $s \times t$ (resp. $t \times s$) matrix recording the sequential coloring with shift $ks + kt$, for $k \geq 0$, on the edges between a part of size s and a part of size t. For example, see Table 3. The rows and columns of each of these matrices are indexed by the vertices in the corresponding parts. Note that a particular row or column of the coloring matrix gives the colors assigned to all of the edges incident with the corresponding vertex.

Table 3. The edge-coloring of $K_{s,t,s,t}$

	A_1	B_1	A_2	B_2
A_1	0_s	$Y_{s,0}$	$X_{s,1}$	$Y_{s,1}$
B_1	$Y_{t,0}$	0_t	$Y_{t,1}$	$X_{t,1}$
A_2	$X_{s,1}$	$Y_{s,1}$	0_s	$Y_{s,0}$
B_2	$Y_{t,1}$	$X_{t,1}$	$Y_{t,0}$	0_t

Now, in a particular row of $Y_{s,k}$, the colors that occur in that row are $ks + kt + i, ks + kt + i + 1, \ldots, ks + kt + i + t - 1 = ks + (k + 1)t + i - 1$, as in the definition of sequential

coloring. Similarly, those that occur in a particular row of $X_{s,k+1}$ are $ks + (k + 1)t + i, ks + (k + 1)t + i + 1, \ldots, ks + (k + 1)t + i + s - 1 = (k + 1)s + (k + 1)t + i - 1$. From this, it is clear that the colors in any row of the coloring matrix formed from submatrices $Y_{s,0}, X_{s,1}, Y_{s,1}, \ldots, X_{s,r}, Y_{s,r}$, for some $r \geq 1$, form a set of consecutive integers.

Additionally, in a particular row of $Y_{t,k}$, the colors that occur in that row are $ks + kt + j, ks + kt + j + 1, \ldots, ks + kt + j + s - 1 = (k + 1)s + kt + j - 1$, as in the definition of sequential coloring. Similarly, those that occur in a particular row of $X_{t,k+1}$ are $(k+1)s+kt+j, (k+1)s+kt+j+1, \ldots, (k+1)s+kt+j+t-1 = (k+1)s+(k+1)t+j-1$. From this, it is clear that the colors in any row of the coloring matrix formed from submatrices $Y_{t,0}, X_{t,1}, Y_{t,1}, \ldots, X_{t,r}, Y_{t,r}$, for some $r \geq 1$, form a set of consecutive integers.

A similar argument shows that any column formed by submatrices $Y_{s,0}, X_{t,1}, Y_{s,1}, \ldots, X_{t,r}, Y_{s,r}$ and by $Y_{t,0}, X_{s,1}, Y_{t,1}, \ldots, X_{s,r}, Y_{t,r}$, for some $r \geq 1$, also forms a set of consecutive integers.

Therefore, we can use the matrices $X_{s,k}, X_{t,k}, Y_{s,k}$ and $Y_{t,k}$ to generate an interval coloring of the complete multipartite graph with the described number of parts as follows.

Let $m = 2^{\ell+1}$ for $\ell \geq 2$. We partition the parts into two sub-parts $M_1 = \{A_1, B_1, \ldots, A_{\frac{m}{4}}, B_{\frac{m}{4}}\}$ and $M_2 = \{A_{\frac{m}{4}+1}, B_{\frac{m}{4}+1}, \ldots, A_{\frac{m}{2}}, B_{\frac{m}{2}}\}$. By the algorithm described above, we color the edges with both ends in M_i, $i \in [2]$. Thus, all that remains is to color the edges with one endpoint in M_1 and the other in M_2. To do this, we extend the coloring in the rows corresponding to A_1 and B_1 to M_2 sequentially using $X_{s,k}, X_{t,k}, Y_{s,k}$, and $Y_{t,k}$ with $k \in \{\frac{m}{4}, \ldots, \frac{m}{2} - 1\}$. See Table 4 for the case when $\ell = 2$.

Table 4. Initial coloring step of $m = 8$

	A_1	B_1	A_2	B_2	A_3	B_3	A_4	B_4
A_1	0_s	$Y_{s,0}$	$X_{s,1}$	$Y_{s,1}$	$X_{s,2}$	$Y_{s,2}$	$X_{s,3}$	$Y_{s,3}$
B_1	$Y_{t,0}$	0_t	$Y_{t,1}$	$X_{t,1}$	$Y_{t,2}$	$X_{t,2}$	$Y_{t,3}$	$X_{t,3}$
A_2	$X_{s,1}$	$Y_{s,1}$	0_s	$Y_{s,0}$				
B_2	$Y_{t,1}$	$X_{t,1}$	$Y_{t,0}$	0_t				
A_3	$X_{s,2}$	$Y_{s,2}$			0_s	$Y_{s,0}$	$X_{s,1}$	$Y_{s,1}$
B_3	$Y_{t,2}$	$X_{t,2}$			$Y_{t,0}$	0_t	$Y_{t,1}$	$X_{t,1}$
A_4	$X_{s,3}$	$Y_{s,3}$			$X_{s,1}$	$Y_{s,1}$	0_s	$Y_{s,0}$
B_4	$Y_{t,3}$	$X_{t,3}$			$Y_{t,1}$	$X_{t,1}$	$Y_{s,0}$	0_t

Take each 2×2 submatrix with rows corresponding to A_1 and B_1 and copy it to color the diagonal and off-diagonal 2×2 blocks in the unfilled submatrix corresponding to the rows corresponding to A_2 and B_2. See Table 5 for the case $\ell = 2$ and Table 6 for the case $\ell = 3$.

Repeat the same method for the 4×4 submatrices corresponding to the rows A_1, B_1, A_2, B_2 to complete the submatrices corresponding to the rows A_3, B_3, A_4, B_4. See Table 7 for the case when $\ell = 4$. We repeat this process for $2^j \times 2^j$ submatrices for $j \in [\frac{m}{8}]$ corresponding to $A_1, B_1, \ldots, A_{j-1}, B_{j-1}$ until we fill in the whole matrix.

Now, each row of the coloring matrix has one occurrence of each of the matrices in either the sequence $Y_{s,0}, X_{s,1}, Y_{s,1}, \ldots, X_{s,r}, Y_{s,r}$ or the sequence $Y_{t,0}, X_{t,1}, Y_{t,1}, \ldots, X_{t,r}, Y_{t,r}$

for $r = \frac{m}{2} - 1$ and, as described above, this corresponds to an integral interval of colors at the edges incident to each vertex. Thus, we have constructed an interval coloring of $K_{s,t,s,t,\ldots,s,t}$ with m parts. This completes the proof. $\square$

Given that the impropriety of any complete multipartite graph is either one or two, the following question naturally arises.

Question 3.4. *Is there a characterization of the complete multipartite graphs that are interval colorable?*

Note that Theorem 3.3 provides a step towards such a characterization.

Table 5. Matrix showing the edge colors of $K_{s,t,s,t,s,t,s,t}$

	A_1	B_1	A_2	B_2	A_3	B_3	A_4	B_4
A_1	0_s	$Y_{s,0}$	$X_{s,1}$	$Y_{s,1}$	$X_{s,2}$	$Y_{s,2}$	$X_{s,3}$	$Y_{s,3}$
B_1	$Y_{t,0}$	0_t	$Y_{t,1}$	$X_{t,1}$	$Y_{t,2}$	$X_{t,2}$	$Y_{t,3}$	$X_{t,3}$
A_2	$X_{s,1}$	$Y_{s,1}$	0_s	$Y_{s,0}$	$X_{s,3}$	$Y_{s,3}$	$X_{s,2}$	$Y_{s,2}$
B_2	$Y_{t,1}$	$X_{t,1}$	$Y_{t,0}$	0_t	$Y_{t,3}$	$X_{t,3}$	$Y_{t,2}$	$X_{t,2}$
A_3	$X_{s,2}$	$Y_{s,2}$	$X_{s,3}$	$Y_{s,3}$	0_s	$Y_{s,0}$	$X_{s,1}$	$Y_{s,1}$
B_3	$Y_{t,2}$	$X_{t,2}$	$Y_{t,3}$	$X_{t,3}$	$Y_{t,0}$	0_t	$Y_{t,1}$	$X_{t,1}$
A_4	$X_{s,3}$	$Y_{s,3}$	$X_{s,2}$	$Y_{s,2}$	$X_{s,1}$	$Y_{s,1}$	0_s	$Y_{s,0}$
B_4	$Y_{t,3}$	$X_{t,3}$	$Y_{t,2}$	$X_{t,2}$	$Y_{t,1}$	$X_{t,1}$	$Y_{s,0}$	0_t

Table 6. Initial coloring for $m = 16$

	A_1	B_1	A_2	B_2	A_3	B_3	A_4	B_4	A_5	B_5	A_6	B_6	A_7	B_7	A_8	B_8
A_1	0_s	$Y_{s,0}$	$X_{s,1}$	$Y_{s,1}$	$X_{s,2}$	$Y_{s,2}$	$X_{s,3}$	$Y_{s,3}$	$X_{s,4}$	$Y_{s,4}$	$X_{s,5}$	$Y_{s,5}$	$X_{s,6}$	$Y_{s,6}$	$X_{s,7}$	$Y_{s,7}$
B_1	$Y_{t,0}$	0_t	$Y_{t,1}$	$X_{t,1}$	$Y_{t,2}$	$X_{t,2}$	$Y_{t,3}$	$X_{t,3}$	$Y_{t,4}$	$X_{t,4}$	$Y_{t,5}$	$X_{t,5}$	$Y_{t,6}$	$X_{t,6}$	$Y_{t,7}$	$X_{t,7}$
A_2	$X_{s,1}$	$Y_{s,1}$	0_s	$Y_{s,0}$	$X_{s,3}$	$Y_{s,3}$	$X_{s,2}$	$Y_{s,2}$	$X_{s,5}$	$Y_{s,5}$	$X_{s,4}$	$Y_{s,4}$	$X_{s,7}$	$Y_{s,7}$	$X_{s,6}$	$Y_{s,6}$
B_2	$Y_{t,1}$	$X_{t,1}$	$Y_{t,0}$	0_t	$Y_{t,3}$	$X_{t,3}$	$Y_{t,2}$	$X_{t,2}$	$Y_{t,5}$	$X_{t,5}$	$Y_{t,4}$	$X_{t,4}$	$Y_{t,7}$	$X_{t,7}$	$Y_{t,6}$	$X_{t,6}$
A_3	$X_{s,2}$	$Y_{s,2}$	$X_{s,3}$	$Y_{s,3}$	0_s	$Y_{s,0}$	$X_{s,1}$	$Y_{s,1}$								
B_3	$Y_{t,2}$	$X_{t,2}$	$Y_{t,3}$	$X_{t,3}$	$Y_{t,0}$	0_t	$Y_{t,1}$	$X_{t,1}$								
A_4	$X_{s,3}$	$Y_{s,3}$	$X_{s,2}$	$Y_{s,2}$	$X_{s,1}$	$Y_{s,1}$	0_s	$Y_{s,0}$								
B_4	$Y_{t,3}$	$X_{t,3}$	$Y_{t,2}$	$X_{t,2}$	$Y_{t,1}$	$X_{t,1}$	$Y_{t,0}$	0_t								
A_5	$X_{s,4}$	$Y_{s,4}$	$X_{s,5}$	$Y_{s,5}$					0_s	$Y_{s,0}$	$X_{s,1}$	$Y_{s,1}$	$X_{s,2}$	$Y_{s,2}$	$X_{s,3}$	$Y_{s,3}$
B_5	$Y_{t,4}$	$X_{t,4}$	$Y_{t,5}$	$X_{t,5}$					$Y_{t,0}$	0_t	$Y_{t,1}$	$X_{t,1}$	$Y_{t,2}$	$X_{t,2}$	$Y_{t,3}$	$X_{t,3}$
A_6	$X_{s,5}$	$Y_{s,5}$	$X_{s,4}$	$Y_{s,4}$					$X_{s,1}$	$Y_{s,1}$	0_s	$Y_{s,0}$	$X_{s,3}$	$Y_{s,3}$	$X_{s,2}$	$Y_{s,2}$
B_6	$Y_{t,5}$	$X_{t,5}$	$Y_{t,4}$	$X_{t,4}$					$Y_{t,1}$	$X_{t,1}$	$Y_{t,0}$	0_t	$Y_{t,3}$	$X_{t,3}$	$Y_{t,2}$	$X_{t,2}$
A_7	$X_{s,6}$	$Y_{s,6}$	$X_{s,7}$	$Y_{s,7}$					$X_{s,2}$	$Y_{s,2}$	$X_{s,3}$	$Y_{s,3}$	0_s	$Y_{s,0}$	$X_{s,1}$	$Y_{s,1}$
B_7	$Y_{t,6}$	$X_{t,6}$	$Y_{t,7}$	$X_{t,7}$					$Y_{t,2}$	$X_{t,2}$	$Y_{t,3}$	$X_{t,3}$	$Y_{t,0}$	0_t	$Y_{t,1}$	$X_{t,1}$
A_8	$X_{s,7}$	$Y_{s,7}$	$X_{s,6}$	$Y_{s,6}$					$X_{s,3}$	$Y_{s,3}$	$X_{s,2}$	$Y_{s,2}$	$X_{s,1}$	$Y_{s,1}$	0_s	$Y_{s,0}$
B_8	$Y_{t,7}$	$X_{t,7}$	$Y_{t,6}$	$X_{t,6}$					$Y_{t,3}$	$X_{t,3}$	$Y_{t,2}$	$X_{t,2}$	$Y_{t,1}$	$X_{t,1}$	$Y_{t,0}$	0_t

Table 7. Matrix showing the edge colors of $K_{s,t,\ldots,s,t}$ with $2^4 = 16$ parts

	A_1	B_1	A_2	B_2	A_3	B_3	A_4	B_4	A_5	B_5	A_6	B_6	A_7	B_7	A_8	B_8
A_1	0_s	$Y_{s,0}$	$X_{s,1}$	$Y_{s,1}$	$X_{s,2}$	$Y_{s,2}$	$X_{s,3}$	$Y_{s,3}$	$X_{s,4}$	$Y_{s,4}$	$X_{s,5}$	$Y_{s,5}$	$X_{s,6}$	$Y_{s,6}$	$X_{s,7}$	$Y_{s,7}$
B_1	$Y_{t,0}$	0_t	$Y_{t,1}$	$X_{t,1}$	$Y_{t,2}$	$X_{t,2}$	$Y_{t,3}$	$X_{t,3}$	$Y_{t,4}$	$X_{t,4}$	$Y_{t,5}$	$X_{t,5}$	$Y_{t,6}$	$X_{t,6}$	$Y_{t,7}$	$X_{t,7}$
A_2	$X_{s,1}$	$Y_{s,1}$	0_s	$Y_{s,0}$	$X_{s,3}$	$Y_{s,3}$	$X_{s,2}$	$Y_{s,2}$	$X_{s,5}$	$Y_{s,5}$	$X_{s,4}$	$Y_{s,4}$	$X_{s,7}$	$Y_{s,7}$	$X_{s,6}$	$Y_{s,6}$
B_2	$Y_{t,1}$	$X_{t,1}$	$Y_{t,0}$	0_t	$Y_{t,3}$	$X_{t,3}$	$Y_{t,2}$	$X_{t,2}$	$Y_{t,5}$	$X_{t,5}$	$Y_{t,4}$	$X_{t,4}$	$Y_{t,7}$	$X_{t,7}$	$Y_{t,6}$	$X_{t,6}$
A_3	$X_{s,2}$	$Y_{s,2}$	$X_{s,3}$	$Y_{s,3}$	0_s	$Y_{s,0}$	$X_{s,1}$	$Y_{s,1}$	$X_{s,6}$	$Y_{s,6}$	$X_{s,7}$	$Y_{s,7}$	$X_{s,4}$	$Y_{s,4}$	$X_{s,5}$	$Y_{s,5}$
B_3	$Y_{t,2}$	$X_{t,2}$	$Y_{t,3}$	$X_{t,3}$	$Y_{t,0}$	0_t	$Y_{t,1}$	$X_{t,1}$	$Y_{t,6}$	$X_{t,6}$	$Y_{t,7}$	$X_{t,7}$	$Y_{t,4}$	$X_{t,4}$	$Y_{t,5}$	$X_{t,5}$
A_4	$X_{s,3}$	$Y_{s,3}$	$X_{s,2}$	$Y_{s,2}$	$X_{s,1}$	$Y_{s,1}$	0_s	$Y_{s,0}$	$X_{s,7}$	$Y_{s,7}$	$X_{s,6}$	$Y_{s,6}$	$X_{s,5}$	$Y_{s,5}$	$X_{s,4}$	$Y_{s,4}$
B_4	$Y_{t,3}$	$X_{t,3}$	$Y_{t,2}$	$X_{t,2}$	$Y_{t,1}$	$X_{t,1}$	$Y_{t,0}$	0_t	$Y_{t,7}$	$X_{t,7}$	$Y_{t,6}$	$X_{t,6}$	$Y_{t,5}$	$X_{t,5}$	$Y_{t,4}$	$X_{t,4}$
A_5	$X_{s,4}$	$Y_{s,4}$	$X_{s,5}$	$Y_{s,5}$	$X_{s,6}$	$Y_{s,6}$	$X_{s,7}$	$Y_{s,7}$	0_s	$Y_{s,0}$	$X_{s,1}$	$Y_{s,1}$	$X_{s,2}$	$Y_{s,2}$	$X_{s,3}$	$Y_{s,3}$
B_5	$Y_{t,4}$	$X_{t,4}$	$Y_{t,5}$	$X_{t,5}$	$Y_{t,6}$	$X_{t,6}$	$Y_{t,7}$	$X_{t,7}$	$Y_{t,0}$	0_t	$Y_{t,1}$	$X_{t,1}$	$Y_{t,2}$	$X_{t,2}$	$Y_{t,3}$	$X_{t,3}$
A_6	$X_{s,5}$	$Y_{s,5}$	$X_{s,4}$	$Y_{s,4}$	$X_{s,7}$	$Y_{s,7}$	$X_{s,6}$	$Y_{s,6}$	$X_{s,1}$	$Y_{s,1}$	0_s	$Y_{s,0}$	$X_{s,3}$	$Y_{s,3}$	$X_{s,2}$	$Y_{s,2}$
B_6	$Y_{t,5}$	$X_{t,5}$	$Y_{t,4}$	$X_{t,4}$	$Y_{t,7}$	$X_{t,7}$	$Y_{t,6}$	$X_{t,6}$	$Y_{t,1}$	$X_{t,1}$	$Y_{t,0}$	0_t	$Y_{t,3}$	$X_{t,3}$	$Y_{t,2}$	$X_{t,2}$
A_7	$X_{s,6}$	$Y_{s,6}$	$X_{s,7}$	$Y_{s,7}$	$X_{s,4}$	$Y_{s,4}$	$X_{s,5}$	$Y_{s,5}$	$X_{s,2}$	$Y_{s,2}$	$X_{s,3}$	$Y_{s,3}$	0_s	$Y_{s,0}$	$X_{s,1}$	$Y_{s,1}$
B_7	$Y_{t,6}$	$X_{t,6}$	$Y_{t,7}$	$X_{t,7}$	$Y_{t,4}$	$X_{t,4}$	$Y_{t,5}$	$X_{t,5}$	$Y_{t,2}$	$X_{t,2}$	$Y_{t,3}$	$X_{t,3}$	$Y_{t,0}$	0_t	$Y_{t,1}$	$X_{t,1}$
A_8	$X_{s,7}$	$Y_{s,7}$	$X_{s,6}$	$Y_{s,6}$	$X_{s,5}$	$Y_{s,5}$	$X_{s,4}$	$Y_{s,4}$	$X_{s,3}$	$Y_{s,3}$	$X_{s,2}$	$Y_{s,2}$	$X_{s,1}$	$Y_{s,1}$	0_s	$Y_{s,0}$
B_8	$Y_{t,7}$	$X_{t,7}$	$Y_{t,6}$	$X_{t,6}$	$Y_{t,5}$	$X_{t,5}$	$Y_{t,4}$	$X_{t,4}$	$Y_{t,3}$	$X_{t,3}$	$Y_{t,2}$	$X_{t,2}$	$Y_{t,1}$	$X_{t,1}$	$Y_{t,0}$	0_t

4. k-trees

A k-*tree* is a graph obtained from K_{k+1} by repeatedly adding vertices in such a way that each added vertex v has exactly k neighbors U, where U is a clique. For example, 1-trees are exactly trees, 2-trees are maximal series-parallel graphs, and include the maximal outerplanar graphs. Planar 3-trees are also known as Apollonian networks.

Every tree has impropriety 1, by Theorem 2.6. In this section, we study the impropriety of 2-trees and of a subfamily of planar 3-trees.

4.1. 2-trees

We begin by determining an upper bound on the impropriety of a general 2-tree, before restricting our focus to two particular subclasses of 2-trees.

Theorem 4.1. *If G is a 2-tree with $\Delta(G) \geq 3$, then $\mu_{\mathrm{int}}(G) \leq \left\lceil \frac{\Delta(G)}{3} \right\rceil$.*

Proof. Let G be a minimal counterexample to the statement with respect to the number of vertices. So G is a 2-tree with $\Delta(G) \geq 3$ and $\mu_{\mathrm{int}}(G) > \left\lceil \frac{\Delta(G)}{3} \right\rceil$. If $\Delta(G) = 3$, then G is a diamond graph which satisfies $\mu_{\mathrm{int}}(G) = 1 = \left\lceil \frac{3}{3} \right\rceil$, so it is not a counterexample; see Figure 1.
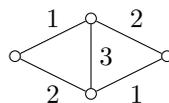
Fig. 1. A diamond graph and its interval coloring with impropriety 1

If $4 \leq \Delta(G) \leq 5$, then by Theorem 2.5, $\mu_{\mathrm{int}}(G) \leq 2 = \left\lceil \frac{\Delta(G)}{3} \right\rceil$. Therefore, $\Delta(G) \geq 6$

for the minimal counterexample G.

Let v be a vertex of degree 2 in G. Note that v can be considered as the last vertex added to G in the process of obtaining a 2-tree, so $N(v) \cong K_2$. Let $N(v) = \{u, w\}$ and $uw \in E(G)$. Since v is the last vertex added to G, $G - v$ is also a 2-tree. Since $\Delta(G) \geq 6$, we have $\Delta(G - v) \geq 5$ and $\mu_{\mathrm{int}}(G - v) \leq \left\lceil \frac{\Delta(G-v)}{3} \right\rceil$ by the minimality of G. Let φ be a k-improper interval coloring of $G - v$, where $k \leq \left\lceil \frac{\Delta(G-v)}{3} \right\rceil$.

Now, we can construct the coloring φ' of $E(G)$ in the following way. First, for $e \in E(G - v)$, we let $\varphi'(e) = \varphi(e)$. Let $x = \varphi(uw) = \varphi'(uw)$. Then at least one of the colors in $\{x - 1, x, x + 1\}$ appears at most $\left\lceil \frac{\Delta(G-v)}{3} \right\rceil - 1 \leq \left\lceil \frac{\Delta(G)}{3} \right\rceil - 1$ times on the edges incident to u (and by symmetry, to w). Let x_u (resp. x_w) be a color in $\{x - 1, x, x + 1\}$ that appears at most $\left\lceil \frac{\Delta(G)}{3} \right\rceil - 1$ times on the edges incident to u (resp. w).

If $x_u = x$, then assigning $\varphi'(uv) = x_u$ and $\varphi'(wv) = x_w$ provides us a $\left\lceil \frac{\Delta(G)}{3} \right\rceil$-improper interval coloring of G, which is a contradiction.

Suppose $x_u = x - 1$ (resp. $x_u = x + 1$). If $x_w = x - 1$ (resp. $x_w = x + 1$), then assigning $\varphi'(uv) = x_u$ and $\varphi'(wv) = x_w$ provides us a $\left\lceil \frac{\Delta(G)}{3} \right\rceil$-improper interval coloring of G, which is a contradiction. If $x_w \neq x - 1$ (resp. $x_w \neq x + 1$), then that means $x - 1$ (resp. $x + 1$) appears on an edge incident with w. Thus, there is at least one color in $\{x - 2, x - 1, x\}$ (resp. $\{x + 2, x + 1, x\}$) appearing at most $\left\lceil \frac{\Delta(G)}{3} \right\rceil - 1$ times on the edges incident to w. Let this color be x'_w. Then assigning $\varphi'(uv) = x_u$ and $\varphi'(wv) = x'_w$ provides us a $\left\lceil \frac{\Delta(G)}{3} \right\rceil$-improper interval coloring of G, which is a contradiction. This concludes the proof. $\qquad\square$

Note that the smallest 2-tree and the only one with $\Delta(G) = 2$ is K_3, and $\mu_{\mathrm{int}}(G) = 2 > 1 = \left\lceil \frac{2}{3} \right\rceil$. Thus the condition that $\Delta(G) \geq 3$ in the preceding theorem is justified. Moreover, we were not able to find any 2-tree with impropriety strictly greater than 2. On the other hand, there are clearly 2-trees with impropriety 2, for example K_3. Thus we pose the following conjecture.

Conjecture 4.2. *If G is a 2-tree, then $\mu_{int}(G) \leq 2$.*

We note, however, that such a bound does not exist for the larger class of 2-degenerate graphs, graphs for which every subgraph has a vertex of degree at most 2 [8, Theorem 3.8]. In the following we provide a better bound for two subfamilies of 2-trees, supporting Conjecture 4.2.

For $n \geq 1$, the *square of a path* P_n is the graph G with vertices $V(G) = V(P_n)$ and edges $E(G) = \{uv \colon d_{P_n}(u, v) \leq 2\}$. Observe that the unique 2-tree on 4 vertices (a diamond) is the square of P_4. Notice also that if $n \geq 5$, then G contains a vertex v of degree 2, such that $N_G(v) = \{x, y\}$, xy is an edge of G, $d_{G-v}(x) = 2$ and $d_{G-v}(y) = 3$.

Proposition 4.3. *If G is the square of a path P_n, $n \geq 4$, then $\mu_{\mathrm{int}}(G) = 1$.*

Proof. Let G be the minimal counterexample with respect to the number of vertices.

Then $|V(G)| \geq 5$ since the diamond graph is interval colorable. Let v be a vertex of degree 2, such that $N_G(v) = \{x, y\}$, $xy \in E(G)$, $d_{G-v}(x) = 2$ and $d_{G-v}(y) = 3$. Notice that $G - v$ is the square of P_{n-1}. Then, by the minimality of G, $\mu_{\text{int}}(G - v) = 1$. Let φ be an interval coloring of $G - v$, and let $\varphi(xy) = a$. By a short case analysis, the coloring of $G - v$ can be extended by appropriately coloring edges vx and vy to give $\mu_{\text{int}}(G) = 1$. See Figure 2 for the analysis of different cases.
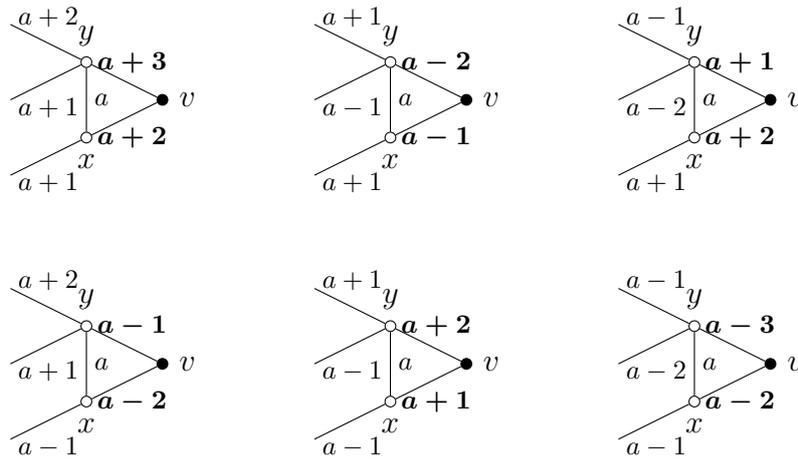
Fig. 2. Case analysis for Proposition 4.3

$\square$

Note that squares of paths are a special case of 2-paths [20, 26]. A *2-path* is an alternating sequence of distinct edges and triangles $(e_0, t_1, e_1, t_2, \ldots, e_n)$, starting and ending with an edge, such that t_i contains exactly two of the distinct edges in the sequence, e_{i-1} and e_i. For example, see Figure 3.
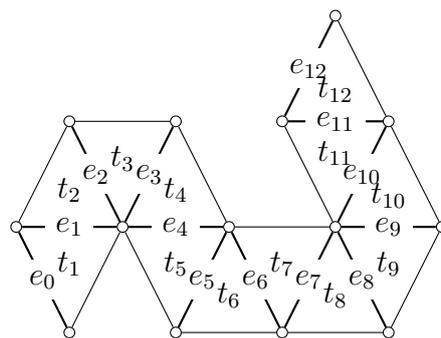
Fig. 3. An example of a 2-path

Proposition 4.4. *If G is a 2-path, then $\mu_{\text{int}}(G) \leq 2$.*

Proof. Let $(e_0, t_1, e_1, t_2, \ldots, e_n)$ be the sequence defining G. Define an edge-coloring $\varphi \colon E(G) \to \mathbb{N}$ in the following way. For every $i \in \{0\} \cup [n]$, let $\varphi(e_i) = i$, and for every $i \in [n]$, color the remaining edge of t_i with color i.

Let $v \in V(G)$. By definition of a 2-path, there exist indices $0 \leq a \leq b \leq n$ such that v is incident to e_i if and only if $a \leq i \leq b$. Thus v is incident to a triangle t_j if and only if $\max\{a, 1\} \leq j \leq \min\{b + 1, n\}$. Hence, the colors on edges incident to v are $a, \dots, b$ and $\max\{a, 1\}, \dots, \min\{b + 1, n\}$. This is an interval and each color occurs at most twice. Therefore, $\mu_{\mathrm{int}}(G) \leq 2$. $\qquad\square$

4.2.   Iterated triangulations

In this section we consider a subfamily of 3-trees. The n-th *iterated triangulation* (n) is the plane graph defined as follows. The graph K_3 is (0). For each $i \geq 0$, obtain $(i+1)$ from (i) by adding a new vertex v_f to the interior of each inner face f of (i) and connecting v_f with each of the three vertices on f. For example, see Figure 4. Note that (n) is a planar graph, and the boundary of each face of (n) is a 3-cycle. In addition, (n) is a special case of the Apollonian network, sometimes called the *complete Apollonian network*. Recall that Apollonian networks are precisely the planar 3-trees.
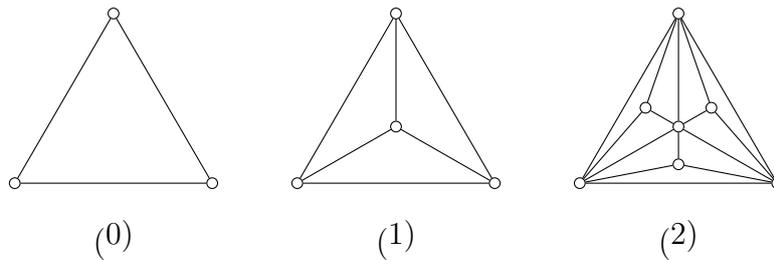
Fig. 4. Illustrations for $Tr(0)$, $Tr(1)$, and $Tr(2)$

Theorem 4.5. *For every $n \geq 1$, $\mu_{\mathrm{int}}(Tr(n)) \leq \left\lceil \frac{\Delta(Tr(n))}{3} \right\rceil$.*

Proof. By observing a construction of $Tr(n)$, we see that $Tr(n)$ contains $Tr(i)$ for all $i \in \{0\} \cup [n]$. For $i \in [n]$, let us call the vertices in $V(Tr(i)) \setminus V((i-1)) \subseteq V((n))$, the *$i$-th vertices* in $Tr(n)$.

Let $C = v_1 v_2 v_3 v_1$ be a 3-cycle that is the boundary of the outer face of $Tr(n)$. Then C is a $Tr(0)$. Let φ be the edge coloring of C defined by $\varphi(v_1 v_2) = 1$, $\varphi(v_2 v_3) = 2$, and $\varphi(v_3 v_1) = 3$.

For each $i \in [n]$, given the edge coloring φ of $Tr(i - 1)$, we color the edges of $Tr(i)$ by giving colors as follows. Let v be a i-th vertex in $Tr(n)$. Then v is located inside the face f of $Tr(i - 1)$. Let $C_f = u_1 u_2 u_3 u_1$ be the boundary of the face f. Then for each $j \in [3]$, we let $\varphi(v u_j) = h$, where $h \in [3] \setminus \{\varphi(u_j u_{j+1}), \varphi(u_{j-1} u_j)\}$ and the addition in the subscript is modulo 3.

Then for each $v \in V(Tr(n))$, edges incident with v will receive all three colors in $[3]$ appearing periodically, in the counterclockwise direction. For example, the edges incident with v will receive x, y, z, x, y, z, $\dots$, where $\{x, y, z\} = [3]$. Therefore, φ is a k-improper interval coloring of $Tr(n)$, with $k \leq \left\lceil \frac{\Delta(Tr(n))}{3} \right\rceil$. $\qquad\square$

Note that the theorem does not hold for $Tr(0)$, since $Tr(0) = K_3$ and thus has impropriety $2 > \left\lceil \frac{2}{3} \right\rceil$.

5. Outerplanar graphs

An outerplanar graph is Class 1 unless it is an odd cycle. In [8], Casselgren and Petrosyan proved the following theorem on the impropriety of an outerplanar graph with respect to the maximum degree.

Theorem 5.1 ([8]). *If G is an outerplanar graph, then $\mu_{int}(G) \leq \left\lceil \frac{\Delta(G)}{4} \right\rceil + 1$.*

They also made the following conjecture, and proved that the Conjecture 5.2 is true when $\Delta(G) \leq 8$.

Conjecture 5.2 ([8]). *If G is an outerplanar graph, then $\mu_{int}(G) \leq 2$.*

This conjecture is sharp if true since every odd cycle has impropriety exactly 2. In this section, we improve Theorem 5.1. Note that this result proves that Conjecture 5.2 is true when $\Delta(G) \leq 10$. We will make use of the following lemma.

Lemma 5.3. *Suppose G is a 2-connected outerplanar graph such that every 2-vertex in G is on a 3-cycle. Then there is a 2-vertex in G that has a neighbor of degree at most 4.*

Proof. Assume that G is an outerplanar graph not satisfying Lemma 5.3. Then each neighbour of a 2-vertex has degree at least 5. Denote by S the set of all 2-vertices. Then, every other vertex in the graph may have at most two neighbors from S. Thus, the graph $G \setminus S$ is an outerplanar graph of minimum degree 3, a contradiction. $\square$

Theorem 5.4. *For any outerplanar graph G with $\Delta(G) \geq 6$, $\mu_{int}(G) \leq \left\lceil \frac{\Delta(G)}{5} \right\rceil$.*

Proof. Let G be a minimal counterexample to the statement with respect to the number of vertices. So G is an outerplanar graph with $\Delta(G) \geq 6$ and $\mu_{int}(G) > \left\lceil \frac{\Delta(G)}{5} \right\rceil$.

Claim 1. G is 2-connected.

Proof. Suppose on the contrary that G not 2-connected.

If G is disconnected, and G is a disjoint union of two subgraphs G_1 and G_2, then by the minimality of G, $\mu_{int}(G_1) \leq \left\lceil \frac{\Delta(G)}{5} \right\rceil$ and $\mu_{int}(G_2) \leq \left\lceil \frac{\Delta(G)}{5} \right\rceil$. This implies $\mu_{int}(G) \leq \left\lceil \frac{\Delta(G)}{5} \right\rceil$, which is a contradiction. Thus, G is connected.

Let v be a cut vertex of G, and let G_1 and G_2 be two subgraphs of G such that $V(G_1) \cap V(G_2) = \{v\}$. By the minimality of G, $\mu_{int}(G_1) \leq \left\lceil \frac{\Delta(G)}{5} \right\rceil$ and $\mu_{int}(G_2) \leq \left\lceil \frac{\Delta(G)}{5} \right\rceil$. We shift the colors of G_2 appropriately so that the colors on the edges incident with v form an interval. This implies that $\mu_{int}(G) \leq \left\lceil \frac{\Delta(G)}{5} \right\rceil$, which is a contradiction. $\square$

Since G is a 2-connected outerplanar graph, by Lemma 5.3, G has a 2-vertex v that is either

(i) on a 3-cycle vuw such that $d_G(u) \leq 4$, or;

(ii) not on a 3-cycle.

Case 1. Suppose (i) holds. Then G has a 2-vertex that is on a 3-cycle vuw such that $d_G(u) \leq 4$. By the minimality of G, $G - v$ has an $\left\lceil \frac{\Delta(G)}{5} \right\rceil$-improper interval coloring φ. Since $\Delta(G) \geq 6$, $\mu_{\text{int}}(G) > \left\lceil \frac{\Delta(G)}{5} \right\rceil \geq 2$. Let $\varphi(uw) = x$.

Since $d_{G-v}(u) \leq 3$, we may assign $\varphi(vu) \in \{x, x+1\}$ so that the interval property at u is preserved and the impropriety at u is still at most $\left\lceil \frac{\Delta(G)}{5} \right\rceil$. If assigning $\varphi(vw) = x$ does not make the impropriety at w greater than $\left\lceil \frac{\Delta(G)}{5} \right\rceil$, then it is a contradiction that $\mu_{\text{int}}(G) > \left\lceil \frac{\Delta(G)}{5} \right\rceil$. Thus, assigning $\varphi(vw) = x$ makes the impropriety at w greater than $\left\lceil \frac{\Delta(G)}{5} \right\rceil$, and this means that w is incident with at least $\left\lceil \frac{\Delta(G)}{5} \right\rceil$ edges in $G - v$ that are colored x by φ. Similarly, w is incident with at least $\left\lceil \frac{\Delta(G)}{5} \right\rceil$ edges in $G - v$ that are colored $x + 1$ by φ.

Since $d_{G-v}(u) \leq 3$, we may assign $\varphi(vu) \in \{x+1, x+2\}$ so that the interval property at u is preserved and the impropriety at u is still at most $\left\lceil \frac{\Delta(G)}{5} \right\rceil$. If assigning $\varphi(vw) = x+2$ does not make the impropriety at w greater than $\left\lceil \frac{\Delta(G)}{5} \right\rceil$, then it is a contradiction that $\mu_{\text{int}}(G) > \left\lceil \frac{\Delta(G)}{5} \right\rceil$. Thus, assigning $\varphi(vw) = x + 2$ makes the impropriety at w greater than $\left\lceil \frac{\Delta(G)}{5} \right\rceil$, and this means that w is incident with at least $\left\lceil \frac{\Delta(G)}{5} \right\rceil$ edges in $G - v$ that are colored $x + 2$ by φ.

Since $d_{G-v}(u) \leq 3$, we may assign $\varphi(vu) \in \{x, x-1\}$ so that the interval property at u is preserved and the impropriety at u is still at most $\left\lceil \frac{\Delta(G)}{5} \right\rceil$. If assigning $\varphi(vw) = x-1$ does not make the impropriety at w greater than $\left\lceil \frac{\Delta(G)}{5} \right\rceil$, then it is a contradiction that $\mu_{\text{int}}(G) > \left\lceil \frac{\Delta(G)}{5} \right\rceil$. Thus, assigning $\varphi(vw) = x - 1$ makes the impropriety at w greater than $\left\lceil \frac{\Delta(G)}{5} \right\rceil$, and this means that w is incident with at least $\left\lceil \frac{\Delta(G)}{5} \right\rceil$ edges in $G - v$ that are colored $x - 1$ by φ.

By the above arguments, w is incident with at least $4 \cdot \left\lceil \frac{\Delta(G)}{5} \right\rceil$ edges in $G - v$ that are colored by a color in $\{x - 1, x, x + 1, x + 2\}$. Since there are at most $\Delta(G) - 1$ edges in $G - v$ that are incident with w, w is incident with less than $\left\lceil \frac{\Delta(G)}{5} \right\rceil$ edges in $G - v$ colored $x - 2$ by φ. Thus, we may assign $\varphi(vu) \in \{x - 1, x - 2\}$ and $\varphi(vw) = x - 2$, and obtain an $\left\lceil \frac{\Delta(G)}{5} \right\rceil$-improper interval coloring φ of G, which is a contradiction.

Case 2. Suppose (ii) holds. Then G has a 2-vertex v that is not on a 3-cycle. Let $N_G(v) = \{u, w\}$. Since v is not on a 3-cycle, $uw \notin E(G)$. By the minimality of G, $G - v + uw$ has an $\left\lceil \frac{\Delta(G)}{5} \right\rceil$-improper interval coloring φ. Let $\varphi(uw) = x$. Then by assigning $\varphi(vu) = \varphi(vw) = x$, φ becomes an $\left\lceil \frac{\Delta(G)}{5} \right\rceil$-improper interval coloring of G, which is a contradiction. □

6. Corona products

Let G and H be graphs, with $V(G) = \{v_1, v_2, \ldots, v_n\}$. First introduced by Frucht and Harary [11], the *corona product* of G and H, denoted by $G \odot H$, is the graph obtained by taking G and n disjoint copies of H, say $H_1, H_2, \ldots, H_n$, and for each $i \in [n]$, adding the edge $v_i x$ for every $x \in V(H_i)$. Note that $K_1 \odot H$ is obtained from H by adding a universal vertex to H. See Figure 5 for an example of the corona product $C_4 \odot P_3$.
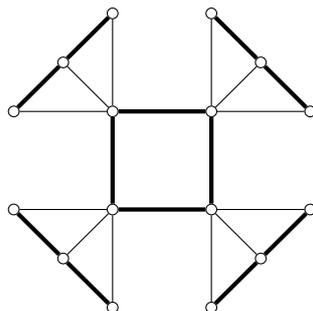
Fig. 5. The corona product $C_4 \odot P_3$

We first determine an upper bound on the impropriety of some corona products of the form $K_1 \odot H$, where the graph H belongs to one of several well-known classes of graphs. After establishing this result, we expand it to an upper bound on the impropriety of $G \odot H$, where G is any arbitrary graph.

Theorem 6.1. *Let H be a graph that is one of the following:*
  (1) *a path P_k,*
  (2) *a cycle C_k,*
  (3) *a star S_k,*
  (4) *a spider SP_k ($k \leq 4$).*
Then $\mu_{\text{int}}(K_1 \odot H) \leq 2$.
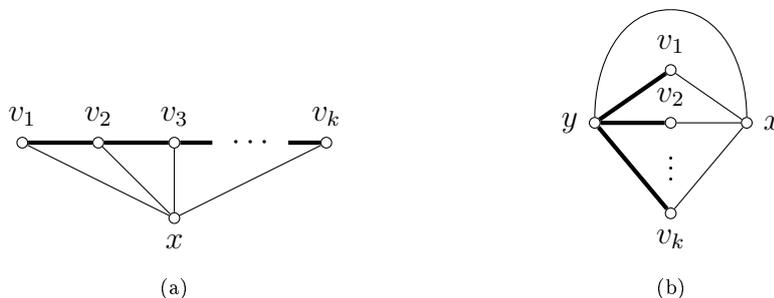
Fig. 6. Illustrations of $G = K_1 \odot H$, where H is path on k vertices or H is a star with k leaves

Proof. (1) Let H be a path $v_1 v_2 \ldots v_k$. Let $G = K_1 \odot H$ be obtained by attaching a universal vertex x to every vertex of H. See Figure 6a. Let us define $\varphi : E(G) \to \mathbb{N}$ by the following rule. For each $i \in [k-1]$, let $\varphi(v_i x) = \varphi(v_i v_{i+1}) = i$. Let $\varphi(v_k x) = k$. Then φ is a 2-improper interval coloring of G. Thus, $\mu_{\text{int}}(G) \leq 2$.

(2) If H is a cycle, then $K_1 \odot H$ is a wheel graph, which has impropriety at most 2 by Theorem 2.7. Thus, $\mu_{\text{int}}(G) \leq 2$.

(3) Let H be a star with its center vertex y and leaves $v_1, v_2, \ldots, v_k$. Let $G = K_1 \odot H$ be obtained by attaching a universal vertex x to every vertex of H. See Figure 6b. Let us define $\varphi : E(G) \to \mathbb{N}$ by the following rule. For each $i \in [k]$, let $\varphi(yv_i) = \varphi(xv_i) = i$. Let $\varphi(xy) = k + 1$. Then φ is a 2-improper interval coloring of G. Thus, $\mu_{\text{int}}(G) \leq 2$.

(4) Let H be a spider graph with center v and four legs $(u_1, u_2, \ldots, u_k)$, $(w_1, w_2, \ldots, w_\ell)$, $(y_1, y_2, \ldots, y_s)$, and $(z_1, z_2, \ldots, z_t)$. Let $G = K_1 \odot H$ be obtained by attaching a universal vertex x to every vertex of H. See Figure 7. Let us define $\varphi : E(G) \to \mathbb{Z}$ by the following rule. Let $\varphi(xv) = 0$, $\varphi(vu_1) = \varphi(xu_1) = \varphi(vw_1) = \varphi(xw_1) = 1$, and $\varphi(vy_1) = \varphi(xy_1) = \varphi(vz_1) = \varphi(xz_1) = -1$. For each $i \in [k-1]$, let $\varphi(u_{i+1}u_i) = \varphi(u_{i+1}x) = i+1$. For each $i \in [\ell-1]$, let $\varphi(w_{i+1}w_i) = \varphi(w_{i+1}x) = i+1$. For each $i \in [s-1]$, let $\varphi(y_{i+1}y_i) = \varphi(y_{i+1}x) = -(i+1)$. For each $i \in [t-1]$, let $\varphi(z_{i+1}z_i) = \varphi(z_{i+1}x) = -(i+1)$. Then φ is a 2-improper interval coloring of G. Thus, $\mu_{\text{int}}(G) \leq 2$.
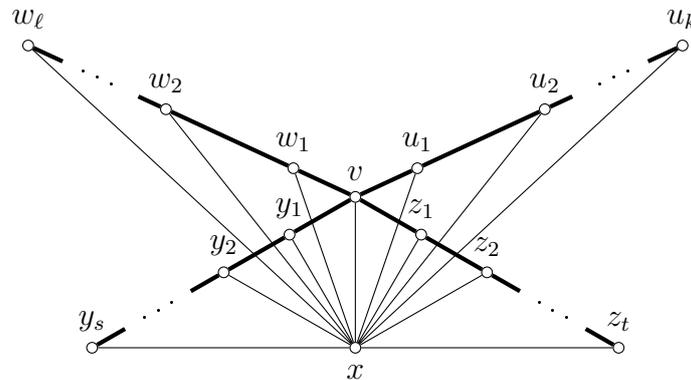
Fig. 7. Illustration of $G = K_1 \odot H$, where H is a spider graph with four legs

The case for a spider graph with less than four legs follows analogously from the above proof.   □

As a corollary to Theorem 6.1, we obtain the following result.

Corollary 6.2. *Let H be a graph that is one of the following:*
   (1) a path P_k,
   (2) a cycle C_k,
   (3) a star S_k,
   (4) a spider SP_k $(k \leq 4)$.
Then for any graph G, $\mu_{\text{int}}(G \odot H) \leq \max\{2, \mu_{\text{int}}(G)\}$.

Proof. Note that $G \odot H$ is obtained by identifying each vertex of G with a universal vertex of a copy of $K_1 \odot H$. Let φ be a $\mu_{int}(G)$-improper interval coloring of G. By Theorem 6.1, for each $v \in V(G)$, we can extend φ into a copy of $K_1 \odot H$ with impropriety at most 2. This can be done by increasing the colors used in the improper interval coloring of each copy of $K_1 \odot H$ so that the minimum color assigned to an edge incident with the

universal vertex is one greater than the maximum color assigned by φ to an edge in $E(G)$ incident with the corresponding vertex in G. Thus, $\mu_{\text{int}}(G \odot H) \leq \max\{2, \mu_{\text{int}}(G)\}$. $\square$

We present an example of a corona product of two graphs that attains this upper bound.

Example 6.3. Consider the graph $G = C_3 \odot P_2$. Let u, v, w be the vertices of C_3 and a_i, b_i the vertices of each copy of P_2, for $i \in \{u, v, w\}$. See Figure 8. We will show that there does not exist an interval coloring of G, and then construct a 2-improper interval coloring. Observe that $2 = \max\{2, \mu_{\text{int}}(C_3)\}$.
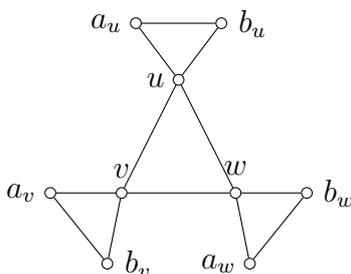
Fig. 8. The corona product $C_3 \odot P_2$ in Example 6.3

Suppose, without loss of generality, that the edge $a_u b_u$ receives color x. Then the edges $u a_u$ and $u b_u$ must receive the colors $x-1$ and $x+1$ in an interval coloring of G. Moreover, one of uv and uw must also receive color x. Without loss of generality, assume that uv receives color x. This means that the edge uw must receive either color $x - 2$ or $x + 2$.

Now, the edges $v a_v$ and $v b_v$ must receive colors different from x, that differ from each other by exactly 2 (in order to properly color $a_v b_v$) and such that the addition of exactly one color will form an interval with x. The available options are $\{x-3, x-1\}$, $\{x-1, x+1\}$ and $\{x + 1, x + 3\}$.

First suppose that $v a_v$ and $v b_v$ are colored using $x-3$ and $x-1$. Then, in order to form an interval at the vertex v, the edge vw must receive color $x - 2$ and thus uw receives color $x+2$, in order to form a proper coloring. However, this means that at vertex w, the colors $x - 2$ and $x + 2$ appear with only two other incident edges left to form an interval. Thus, an interval coloring is not possible in this case. A similar argument shows that either of the other two color options for $v v_a$ and $v b_v$ cannot lead to an interval coloring of G. Therefore, $\mu_{int}(G) \geq 2$.

A 2-improper interval coloring φ of G can be constructed as follows. Let $\varphi(i a_i) = 1$, $\varphi(i b_i) = 3$, and $\varphi(a_i b_i) = 2$ for $i \in \{u, v, w\}$. Finally, let $\varphi(uv) = \varphi(vw) = \varphi(uw) = 2$. Since only the color 2 appears twice at the vertices u, v and w, this is a 2-improper interval coloring of the graph. Therefore, $\mu_{int}(G) = 2$.

It remains an open question whether there exists an infinite family of graphs $G \odot H$ with $G \neq K_1$ whose impropriety attains the upper bound of Corollary 6.2.

As a step towards determining the impropriety of the graph $G \odot T$, where T is a tree, we begin by investigating the special case where T is a caterpillar. Similar to paths, stars

and spiders, we construct a 2-improper interval coloring of $K_1 \odot C$, where C is a caterpillar and use this to determine an upper bound on $\mu_{int}(G \odot C)$ for any graph G.

Theorem 6.4. *Let C be a caterpillar. Then $\mu_{int}(G \odot C) \leq \max\{2, \mu_{int}(G)\}$.*

Proof. Let $v_1, v_2, \ldots, v_k$ be the vertices of the path obtained by removing the leaves of the caterpillar C, with $v_i v_{i+1} \in E(C)$ for $i \in [k-1]$.

We construct a $\max\{2, \mu_{int}(G)\}$-improper interval coloring of $G \odot C$, $\varphi : E(G \odot C) \to \mathbb{N}$, as follows. Let $\hat{\varphi}$ be a $\mu_{int}(G)$-improper interval coloring of G. For each edge $e \in E(G)$, let $\varphi(e) = \hat{\varphi}(e)$.

Let C_x be the copy of C corresponding to the vertex $x \in V(G)$. Let $m(x)$ be the maximum value of $\varphi(e)$ over all edges $e \in E(G)$ that are incident with x. Now we color each C_x as follows. Let $\ell_1, \ell_2, \ldots, \ell_{c_1}$ be the leaves in C_x adjacent to the vertex v_1. Then, let $\varphi(v_1 \ell_i) = m(x) + i$ for $1 \leq i \leq c_1$. Let $\varphi(v_1 v_2) = m(x) + c_1 + 1$. For each vertex v_i, $2 \leq i \leq k$, let $\ell_{c_{i-1}+2}, \ell_{c_{i-1}+3}, \ldots, \ell_{c_i}$ be the leaves in C_x adjacent to v_i. Let $\varphi(v_i \ell_j) = m(x) + j$, for $c_{i-1} + 2 \leq j \leq c_i$. Finally, for $2 \leq i < k$, let $\varphi(v_i v_{i+1}) = m(x) + c_i + 1$.

To complete the coloring of $G \odot C$, a color must be assigned to each edge containing one vertex $x \in V(G)$ and one vertex in C_x, the corresponding copy of C. Let $\varphi(x v_1) = m(x) + 1$ and $\varphi(x v_i) = m(x) + c_{i-1} + 1$, for $1 < i \leq k$. Finally, let $\varphi(xw) = \varphi(v_i w)$ for every leaf vertex in $w \in V(C)$, where v_i is the vertex adjacent to w in C_x. Note that there is no edge $v_i w$ for which $\varphi(v_i w) = m(x) + c_{i-1} + 1$, so only the color $m(x) + 1$ is assigned to two distinct edges incident with x.

The coloring φ defines an improper interval coloring. Each vertex in G has at most $\max\{2, \mu_{int}(G)\}$ incident edges that share the same color, two containing vertices in the corresponding copy of C and at most $\mu_{int}(G)$ containing other vertices in G. Each vertex in a copy of C has exactly two incident edges that share the same color. Therefore, φ is a $\max\{2, \mu_{int}(G)\}$-improper interval coloring of $G \odot C$. $\square$

So far, we have determined upper bounds on the impropriety of $G \odot H$ only when H is a cycle or belongs to a specific subclass of trees. For an arbitrary graph H, we have the following upper bound on the impropriety of $G \odot H$ in terms of the impropriety of the graph G and the order of the graph H.

Theorem 6.5. *For any graphs G and H, $\mu_{int}(G \odot H) \leq \max\{\mu_{int}(G), |V(H)|\}$.*

Proof. We construct a $\max\{\mu_{int}(G), |V(H)|\}$-improper interval coloring of $G \odot H$, $\varphi : E(G \odot H) \to \mathbb{N}$, as follows. Let $\hat{\varphi}$ be a $\mu_{int}(G)$-improper interval coloring of G. For every edge $e \in E(G)$, let $\varphi(e) = \hat{\varphi}(e)$.

For each vertex $v \in V(G)$, let H_v be the copy of H corresponding to v and let $m(v)$ be the maximum value of φ over all edges in $E(G)$ that are incident with v. Then, for each vertex $w \in V(H_v)$, let $\varphi(vw) = m(v) + 1$. Finally, for every $u, w \in V(H_v)$, let $\varphi(uw) = m(v) + 2$.

This is an improper interval coloring of $G \odot H$. Each vertex in $V(G)$ has at most $\max\{\mu_{int}(G), |V(H)|\}$ incident edges that are assigned the same color. Each vertex v in

each copy of H has at most $\max\{1, d_H(v)\}$ incident edges that are assigned the same color. Since $d_H(v) < |V(H)|$ for every vertex $v \in V(H)$, it follows that the coloring φ is a $\max\{\mu_{int}(G), |V(H)|\}$-improper interval coloring. $\square$

We note that this upper bound on $\mu_{int}(G \odot H)$ is sharp in the case when G is interval colorable and $H \cong K_1$. These are not the only graphs attaining this bound, as the corona product $C_3 \odot P_2$ in Example 6.3 has impropriety $\max\{\mu_{int}(C_3), |V(P_2)|\} = 2$.

Note that the known sharp examples for 6.5 satisfy $|V(H)| \leq 2$. If we add the condition that $|V(H)| \geq 3$ to 6.5, then we obtain a better upper bound on $\mu_{int}(G \odot H)$ as follows.

Theorem 6.6. *For any graphs G and H with $|V(H)| \geq 3$,*

$$\mu_{\text{int}}(G \odot H) \leq \max\left\{\mu_{\text{int}}(G), \left\lceil \frac{|V(H)|}{3} \right\rceil, \Delta(H) + 1\right\}.$$

Proof. Let $k = \max\{\mu_{\text{int}}(G), \lceil \frac{|V(H)|}{3} \rceil, \Delta(H) + 1\}$. We construct a k-improper interval coloring of $G \odot H$, $\varphi : E(G \odot H) \to \mathbb{N}$, as follows. Let $\hat{\varphi}$ be a $\mu_{\text{int}}(G)$-improper interval coloring of G. For every edge $e \in E(G)$, let $\varphi(e) = \hat{\varphi}(e)$.

For each vertex $v \in V(G)$, let H_v be the copy of H corresponding to v and let $m(v)$ be the maximum value of φ over all edges in $E(G)$ that are incident with v. Since $|V(H)| \geq 3$, we can partition $V(H_v)$ into three sets $S_{v,1}, S_{v,2}, S_{v,3}$, where $1 \leq |S_{v,i}| \leq \left\lceil \frac{|V(H_v)|}{3} \right\rceil$ for each $i \in [3]$. For each $i \in [3]$ and $w \in V(S_{v,i})$, let $\varphi(vw) = m(v) + i$, and for every $u, w \in V(H_v)$, let $\varphi(uw) = m(v) + 2$.

This is an improper interval coloring of $G \odot H$. Each vertex in $V(G)$ has at most $\max\{\mu_{\text{int}}(G), \lceil \frac{|V(H)|}{3} \rceil\}$ incident edges that are assigned the same color. Each vertex v in each copy of H has at most $1 + d_H(v)$ incident edges that are assigned the same color. Since $d_H(v) \leq \Delta(H)$ for every vertex $v \in V(H)$, it follows that the coloring φ is a k-improper interval coloring. $\square$

Related to Theorems 6.5 and 6.6, we raise a question supported by 6.2 and 6.4.

Question 6.7. *For any graphs G and H, is it true that $\mu_{\text{int}}(G \odot H) \leq \max\{\mu_{\text{int}}(G), \mu_{\text{int}}(H) + 1\}$?*

7. Conclusion and further directions

In this paper, we have improved the upper bounds on the interval coloring impropriety of several classes of graphs. However, in many cases there remains a significant difference between the known upper bounds presented here and those that have been conjectured.

In addition to Conjecture 4.2, the following is of interest.

Question 7.1. *Does there exist a function f depending on k, but not on G, such that for every k-tree G, $\mu_{\text{int}}(G) \leq f(k)$?*

In this article we have continued the study of the impropriety of products of graphs

initiated by [8] where the Cartesian products of graphs were studied. Considering the strong products of graphs is a natural choice as well. Note that using the same coloring as for the Cartesian product in [8, Proposition 4.17] and coloring the rest of the edges $(u_1, v_1)(u_2, v_2)$ using $\alpha(u_1 u_2) + \beta(v_1 v_2)$, where α (resp. β) is the $\mu_{\text{int}}(G)$-improper interval coloring of G (resp. $\mu_{\text{int}}(H)$-improper interval coloring of H), we get intervals on the edges incident to each vertex, and the obtained bound is

$$\mu_{int}(G \boxtimes H) \leq \max\{\mu_{int}(G), \mu_{int}(H)\} + \Delta(G)\mu_{int}(H) + \Delta(H)\mu_{int}(G). \tag{1}$$

Question 7.2. *Can the bound in* (1) *be improved?*

Note that the impropriety is not the only way to attack the difficulty of interval colorability. The interval coloring thickness has also been introduced and studied (see for example [2]), and it would be interesting to see if there is any relation between the interval coloring impropriety and the interval coloring thickness of a graph.

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