Primitive star decompositions of graphs

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ABSTRACT

A decomposition $\mathcal{C}$ of a graph G is primitive if no proper, nontrivial subset of $\mathcal{C}$ is a decomposition of an induced subgraph of G. The existence of primitive decompositions has been studied for several decompositions, including path and cycle decompositions for complete and cocktail party graphs. In this work, we classify the existence of primitive star decompositions for complete graphs.

Keywords: star decomposition, subsystems, primitive (star) decomposition, subdecomposition

2020 Mathematics Subject Classification: 05C20, 05C51, 05C70.

1. Introduction

In this work we investigate the existence of primitive graph decompositions, which was formalized by Asplund et al. [1]. We begin with the following question:

Question 1.1. *Let G be a graph with vertex set V, and let $\mathcal{G}$ be a decomposition of G. Do there exist nonempty subsets $\mathcal{G}' \subseteq \mathcal{G}$ and $V' \subseteq V$ such that $\mathcal{G}'$ is a decomposition of the induced subgraph of G with vertex set V'?*

Such subsets $\mathcal{G}'$ are *subdecompositions* of $\mathcal{G}$. Trivially, the answer to the previous question is yes, as $\mathcal{G}$ is a subdecomposition of itself. So we ask a more interesting question:

Question 1.2. *Let G be a graph with a decomposition $\mathcal{G}$. Does $\mathcal{G}$ contain any proper subdecompositions?*

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If the answer to the previous question is no, we say $\mathcal{G}$ is a *primitive decomposition* of G, and this property has been investigated in several cases. In 2000, Rodger and Spicer [6] classified the existence of primitive $(K_4 - e)$-decompositions of complete graphs. In 2012, Dinavahi and Rodger [4] classified the existence of primitive P_n-decompositions of complete graphs. In 2022, Asplund et al. [1] investigated primitive C_n-decompositions of both complete graphs and cocktail party graphs and, in 2023, Schroeder [7] settled an unresolved question from the previous work involving C_4-decompositions of cocktail party graphs. By another name, 2-primitivity of triple systems (C_3-decompositions of K_n) was introduced in 1969 by Doyen [5], in a paper discussing *non-degenerate triple systems*, and in 2024, Schroeder et al. [8] classified a similar result for C_3-decompositions of $K_n - I$. In this paper, we continue the exploration of this topic with star decompositions.

Given positive integers m and n, we say that (m, n) is *valid* if either $n = 1$ or both $n \geq 2m$ and $n(n - 1)$ is divisible by $2m$. This is a necessary condition for K_n to have an m-star decomposition; the latter condition ensures an integral number of stars in such a decomposition, and the former arises from an m-star decomposition having at least $n - 1$ stars (at most one vertex is not a root for a star in the decomposition). Tarsi [3] demonstrated these conditions are sufficient.

Theorem 1.3. *Let n and m be positive integers. Then K_n has an m-star decomposition if and only if (m, n) is valid.*

In this work, we show the necessary conditions are also sufficient for the existence of primitive m-star decompositions of K_n.

Theorem 1.4. *Let n and m be positive integers. Then K_n has a primitive m-star decomposition if and only if $m = 1$ and $n \in \{1, 2\}$, or $m \neq 1$ and (m, n) is valid.*

2. Preliminaries

For a positive integer m, an m-star is a graph which is isomorphic to $K_{1,m}$. The vertex in the part of size 1 is the *root* of S, denoted $r(S)$, and the vertex set of size m is the *pendant vertex set* of S, denoted as $P(S)$. In addition, we use the notation $(x; D)$ to denote the star with root x and pendant vertex set D.

Let G and H be graphs. The *join* of G and H, denoted $G + H$, is a graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set

$$E(G + H) = E(G) \cup E(H) \cup \{xy : x \in V(G), \ y \in V(H)\}.$$

For a graph G and a vertex set V with cardinality n and disjoint from $V(G)$, the join of G with the n independent vertices in V is $G + \overline{K_n}$, where $V = V(\overline{K_n})$, and we abbreviate this graph as $G + V$.

Let G be a graph, ϕ be a permutation of $V(G)$, $e \in E(G)$, and H a subgraph of G. We define $\phi(e) = \{\phi(i), \phi(j)\}$, where $e = \{i, j\}$ for some $i, j \in V(G)$, and $\phi(H)$ is the graph such that $V(\phi(H)) = \{\phi(v) : v \in V(H)\}$ and $E(\phi(H)) = \{\phi(e) : e \in E(H)\}$. If $\mathcal{G}$ is a set of subgraphs of G, we use $V(\mathcal{G})$ to denote the union of the vertex sets of all graphs in $\mathcal{G}$.

Let $\alpha = (a_1, a_2, \ldots, a_k)$ be a sequence of integers. We let $\Sigma_i(\alpha)$ denote the partial sum $a_1 + a_2 + \cdots + a_i$, with the convention $\Sigma_0(\alpha) = 0$. Furthermore, for positive integers $n_1, \ldots, n_k$, we denote the sequence with n_1 copies of a_1, n_2 copies of a_2, $\ldots$, and n_k copies of a_k with $(a_1 : n_1)(a_2 : n_2) \cdots (a_k : n_k)$.

Let G be a digraph. Then G is a *bi-tournament* if $V(G)$ may be partitioned as $V_0 \cup V_1$ such that for each edge $(x, y) \in E(G)$, if $x \in V_i$, then $y \in V_{1-i}$, and for all $x \in V_0$, $y \in V_1$, either $(x, y) \in E(G)$ or $(y, x) \in E(G)$, but not both. We say that such a bi-tournament has $(|V_0|, |V_1|)$ vertices. Suppose that $V_0 = \{x_1, \ldots, x_n\}$ and $V_1 = \{y_1, \ldots, y_m\}$, and for each $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$, let k_i and ℓ_j be the out-degrees of x_i and y_j in G, respectively. The *score sequences* of G are the sequences $(k_1, \ldots, k_n)$ and $(\ell_1, \ldots, \ell_m)$, and we assume these sequences are non-decreasing. Beineke and Moon [2] classified when a pair of sequences is a set of score sequences for a bi-tournament.

Theorem 2.1. *Let $n, m \geq 1$ and $\alpha = (a_1, \ldots, a_n)$ and $\beta = (b_1, \ldots, b_m)$ be non-decreasing, non-negative integer sequences of lengths n and m, respectively. Then α and β are score sequences for a bi-tournament with (n, m) vertices if and only if $\Sigma_k(\alpha) + \Sigma_\ell(\beta) \geq k\ell$ for each $k \in \{0, \ldots, n\}$ and $\ell \in \{0, \ldots, m\}$, with equality when $k = n$ and $\ell = m$.*

We now demonstrate a special case of Theorem 2.1.

Lemma 2.2. *Let $n, m \geq 1$ and $\alpha = (a_1, \ldots, a_n)$ and $\beta = (b_1, \ldots, b_m)$ be non-decreasing, non-negative integer sequences of lengths n and m, respectively. Then α and β are score sequences for a bi-tournament with (n, m) vertices if $\Sigma_n(\alpha) + \Sigma_m(\beta) = nm$ and there exist integers p and t such that $1 \leq p \leq n$, $0 \leq t \leq m$, $a_{p+1} = a_{p+2} = \cdots = a_n = t$, and $p \leq b_1$.*

Proof. For $u \in \{0, \ldots, n\}$ and $v \in \{0, \ldots, m\}$, define $g(u, v) = \Sigma_u(\alpha) + \Sigma_v(\beta)$. Since $g(n, m) = nm$, it is sufficient to show that $g(u, v) \geq uv$ over the domain of g, as the result then follows from Theorem 2.1.

Clearly $g(u, 0) \geq 0$ for each $u \in \{1, \ldots, n\}$. Now suppose that $g(u, v) \geq uv$ for some $u \in \{0, \ldots, p\}$ and $v \in \{0, \ldots, m-1\}$. Then

$$g(u, v+1) = g(u, v) + b_{v+1} \geq uv + b_{v+1} \geq uv + b_1 \geq uv + p \geq uv + u = u(v+1),$$

so $g(u, v+1) \geq u(v+1)$. Now let $v \in \{0, \ldots, m\}$. Then

$$g(n, v) = \Sigma_n(\alpha) + \Sigma_v(\beta) = nm - \left(\sum_{j=v+1}^{m} b_j \right) \geq nm - (m-v)n = nv.$$

Finally, suppose $u \in \{p+1, \ldots, n-1\}$ and $v \in \{1, \ldots, m\}$. Then

$$g(u, v) = g(p, v) + t(u - p) \geq pv + t(u - p) = uv + (t - v)(u - p). \tag{1}$$

Similarly we have that

$$g(u, v) = g(n, v) - t(n - u) \geq nv - t(n - u) = uv + (v - t)(n - u). \tag{2}$$

Since both $u - p \geq 0$ and $n - u \geq 0$, we have that $g(u,v) \geq uv$ by (1) if $t \geq v$ or by (2) if $t \leq v$. □

Again, let G be a graph. A set $\mathcal{D}$ of subgraphs of G is a *decomposition* of G if $\{E(H) : H \in \mathcal{D}\}$ is a partition of $E(G)$. A subset $\mathcal{S}$ of $\mathcal{D}$ is a *subdecomposition* of $\mathcal{D}$ if there exists an induced subgraph H of G such that $\mathcal{S}$ is a decomposition of H. We say H is the induced subgraph of G *associated* to $\mathcal{S}$. Recall that if $\mathcal{D}$ has no proper subdecompositions, then $\mathcal{D}$ is primitive.

Example 2.3. Let $G = K_5$ with $V(G) = \mathbb{Z}_5$, and let

$$\mathcal{D} = \{(1; \{0,2\}), (2; \{0,3\}), (3; \{0,1\}), (4; \{0,1\}), (4; \{2,3\}), \text{ and}$$
$$\mathcal{E} = \{(0; \{1,2\}), (1; \{2,3\}), (2; \{3,4\}), (3; \{4,0\}), (4; \{0,1\})\}.$$

Then $\mathcal{D}' = \{(1; \{0,2\}), (2; \{0,3\}), (3; \{0,1\})\}$ is a proper subdecomposition of $\mathcal{D}$, and hence $\mathcal{D}$ is not primitive. However, $\mathcal{E}$ has no proper subdecompositions, so $\mathcal{E}$ is primitive. Furthermore, observe that if we define $S = (0; \{1,2\})$ and $\phi(x) = x + 1$ for each $x \in \mathbb{Z}_5$, then $\mathcal{E} = \{\phi^i(S) : i \in \{0,1,2,3,4\}\}$. See Figure 1.
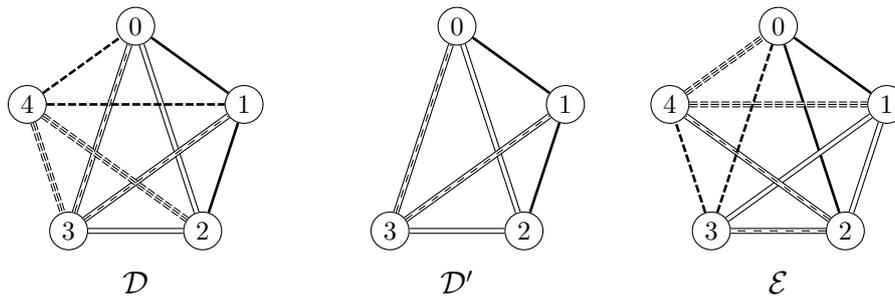

Fig. 1. The decompositions $\mathcal{D}$ and $\mathcal{E}$ of K_5, along with the proper subdecomposition $\mathcal{D}'$ of $\mathcal{D}$, as outlined in Example 2.3

The following lemma outlines a fairly obvious construction for an m-star decomposition, which is used in several subsequent arguments.

Lemma 2.4. *Let a and m be positive integers with $m \geq 3$. Then $K_{a,m}$ has an m-star decomposition.*

Proof. Let V and W be the parts of the vertex set of $K_{a,m}$ with cardinalities a and m, respectively. Then $\mathcal{D} = \{(x; W) : x \in V\}$ is an m-star decomposition of $K_{a,m}$. Observe that any subset of $\mathcal{D}$ is a subdecomposition, so $\mathcal{D}$ is not primitive for any $a \geq 2$. □

We now present two lemmas, originally introduced and proved by Asplund et al. [1], which lends itself better to proving primitivity, as well as providing a useful property involving the intersection of subdecompositions.

Lemma 2.5 (Lemma 2.1 in [1]). *Let G be a graph with a decomposition $\mathcal{G}$ and $\mathcal{S} \subseteq \mathcal{G}$ with $\mathcal{S} \neq \varnothing$.*

(a) Then $\mathcal{S}$ is a subdecomposition of $\mathcal{G}$ if and only if for each $v, w \in V(\mathcal{S})$, if $vw \in E(G)$ and $C \in \mathcal{G}$ such that $vw \in E(C)$, then $C \in \mathcal{S}$.

(b) If $V(\mathcal{S}) = V(G)$, then $\mathcal{S} = \mathcal{G}$.

(c) If $\mathcal{G}$ is primitive and $\mathcal{S}$ is a subdecomposition of $\mathcal{G}$, then $\mathcal{S} = \mathcal{G}$.

(d) If $\mathcal{S}$ being a subdecomposition of $\mathcal{G}$ implies $\mathcal{S} = \mathcal{G}$, then $\mathcal{G}$ is primitive.

Lemma 2.6 (Lemma 2.3 in [1]). *Let G be a graph with a decomposition $\mathcal{G}$ and let $\mathcal{H}$ be a nonempty subset of $\mathcal{G}$ associated to a subgraph H of G. If $\mathcal{S}$ is a subdecomposition of $\mathcal{G}$, then $\mathcal{S} \cap \mathcal{H}$ is either empty or a subdecomposition of $\mathcal{H}$. Furthermore, if $\mathcal{H}$ is a primitive decomposition of H, then $\mathcal{S} \cap \mathcal{H}$ is empty or $\mathcal{H} \subseteq \mathcal{S}$.*

We close this section with a pair of similar lemmas that provides a method for building a primitive decomposition for a graph from primitive decompositions of its subgraphs, which is leveraged in later arguments.

Lemma 2.7. *Let G be a graph with subgraphs H, and K with $V(G) = V(H) \cup V(K)$. Let $\mathcal{G}$ and $\mathcal{H}$ be decompositions of G and H, respectively, and both $\mathcal{K}$ and $\mathcal{K}'$ be decompositions of K such that*

(a) $\mathcal{H}$ and $\mathcal{K}$ are subsets of $\mathcal{G}$,

(b) $\mathcal{H}$ is primitive,

(c) $\mathcal{H} \cap \mathcal{K}$ is nonempty,

(d) any subdecomposition of $\mathcal{G}$ intersects $\mathcal{H}$ nontrivially, and

(e) $\mathcal{K}'$ is contained in all subdecompositions of $(\mathcal{G} \backslash \mathcal{K}) \cup \mathcal{K}'$.

Then $(\mathcal{G} \backslash \mathcal{K}) \cup \mathcal{K}'$ is a primitive decomposition of G.

Proof. Let $\mathcal{G}' = (\mathcal{G} \backslash \mathcal{K}) \cup \mathcal{K}'$ and $\mathcal{S}'$ be a subdecomposition of $\mathcal{G}'$ which associates to an induced subgraph I in G. Let $\mathcal{S} = (\mathcal{S}' \backslash \mathcal{K}') \cup \mathcal{K}$. Then $\mathcal{S} \subseteq \mathcal{G}$, and since $\mathcal{K}' \subseteq \mathcal{S}'$ and $\mathcal{K}$ and $\mathcal{K}'$ are both decompositions of K, we have that $\mathcal{S}$ also associates to I, so $\mathcal{S}$ is a subdecomposition of $\mathcal{G}$. Since $\mathcal{H} \cap \mathcal{S}$ is nonempty and $\mathcal{H}$ is primitive, it follows that $\mathcal{H} \subseteq \mathcal{S}$, by Lemma 2.6. Hence $V(\mathcal{S}) = V(G)$, so $\mathcal{S} = \mathcal{G}$. Therefore $\mathcal{S}' = (\mathcal{G} \backslash \mathcal{K}) \cup \mathcal{K}' = \mathcal{G}'$. Hence $\mathcal{G}'$ is primitive by Lemma 2.5(d). $\qquad\square$

Lemma 2.8. *Let G be a graph with subgraphs H, J, K with $V(G) = V(H) \cup V(J)$. Let $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{J}$ be decompositions of G, H, and J, respectively, and both $\mathcal{K}$ and $\mathcal{K}'$ be decompositions of K such that*

(a) $\mathcal{H}$, $\mathcal{J}$, and $\mathcal{K}$ are subsets of $\mathcal{G}$,

(b) $\mathcal{H}$ and $\mathcal{J}$ are primitive,

(c) both $\mathcal{H} \cap \mathcal{K}$ and $\mathcal{J} \cap \mathcal{K}$ are nonempty,

(d) any subdecomposition of $\mathcal{G}$ intersects $\mathcal{H} \cup \mathcal{J}$ nontrivially, and

(e) $\mathcal{K}'$ is contained in all subdecompositions of $(\mathcal{G} \backslash \mathcal{K}) \cup \mathcal{K}'$.

Then $(\mathcal{G} \backslash \mathcal{K}) \cup \mathcal{K}'$ is a primitive decomposition of G.

Proof. Let $\mathcal{G}' = (\mathcal{G}\backslash\mathcal{K}) \cup \mathcal{K}'$ and $\mathcal{S}'$ be a subdecomposition of $\mathcal{G}'$ which associates to an induced subgraph I in G. Let $\mathcal{S} = (\mathcal{S}'\backslash\mathcal{K}') \cup \mathcal{K}$. Then $\mathcal{S} \subseteq \mathcal{G}$, and since $\mathcal{K}' \subseteq \mathcal{S}'$ and $\mathcal{K}$ and $\mathcal{K}'$ are both decompositions of K, we have that $\mathcal{S}$ also associates to I, meaning $\mathcal{S}$ is a subdecomposition of $\mathcal{G}$. Since $\mathcal{K} \subseteq \mathcal{S}$ and both $\mathcal{H} \cap \mathcal{K}$ and $\mathcal{J} \cap \mathcal{K}$ are nonempty, it follows that $\mathcal{H} \cap \mathcal{S}$ and $\mathcal{J} \cap \mathcal{S}$ are nonempty. Furthermore, since $\mathcal{H}$ and $\mathcal{J}$ are primitive, it follows that $\mathcal{H} \subseteq \mathcal{S}$ and $\mathcal{J} \subseteq \mathcal{S}$, by Lemma 2.6. Hence $V(G) = V(H) \cup V(J) \subseteq V(\mathcal{S})$, so $\mathcal{S} = \mathcal{G}$. Therefore $\mathcal{S}' = (\mathcal{G}\backslash\mathcal{K}) \cup \mathcal{K}' = \mathcal{G}'$. Hence $\mathcal{G}'$ is primitive by Lemma 2.5(d). $\square$

3.   Star decompositions with odd star size

In this section, we prove Theorem 1.4 for odd $m \geq 3$. First, we directly show the existence of primitive m-star decompositions of K_n when (m, n) is valid and $n < 3m$, and then we provide an inductive construction to handle cases with larger n.

Lemma 3.1. *Let $m \geq 2$ be a positive integer. Then K_{2m+1} has a primitive m-star decomposition.*

Proof. Let G be isomorphic to K_{2m+1} with vertex set $V(G) = \mathbb{Z}_{2m+1}$. Define S as the subgraph of G given by $S = (0; \{1, \ldots, m\})$, and let ϕ be the permutation of $V(G)$ given by $\phi(x) = x + 1$. Let $\mathcal{D} = \{\phi^i(S) : i \in \mathbb{Z}_{2m+1}\}$. Then $\mathcal{D}$ is an m-star decomposition of G, which we now show is primitive.

Let $\mathcal{S}$ be a subdecomposition of $\mathcal{D}$. Then $\phi^i(S) \in \mathcal{S}$ for some $i \in \mathbb{Z}_{2m+1}$. Since $m \geq 2$, we have $i+1, i+2 \in V(S)$, and hence $i+1, i+2 \in V(\mathcal{S})$. Since $(i+1)(i+2) \in E(\phi^{i+1}(S))$, it follows that $\phi^{i+1}(S) \in \mathcal{S}$. An iterative application of this gives that $\mathcal{S} = \mathcal{D}$, and hence $\mathcal{S}$ is not proper. So $\mathcal{D}$ is primitive by Lemma 2.5(d). See Example 2.3 for an illustration of this decomposition when $m = 2$. $\square$

Lemma 3.2. *Let m and n be positive integers such that $m \geq 2$, (m, n) is valid, and $n < 3m$. Then K_n has a primitive m-star decomposition.*

Proof. By Theorem 1.3, K_n has an m-star decomposition $\mathcal{D}$, and assume that $\mathcal{D}$ is not primitive. Then $n \neq 2m+1$ by Lemma 3.1. Let $\mathcal{S}$ be a proper subdecomposition of $\mathcal{D}$ with associated induced subgraph H of K_n. Then H is isomorphic to K_ℓ, where $2m \leq \ell < n$; hence $n \neq 2m$. So $n \geq 2m + 2$, which implies that $m \geq 3$.

Let $J = \overline{H}$ in K_n and $W = V(G)\backslash V(H)$; then $\mathcal{D}\backslash\mathcal{S}$ is a decomposition of J and $|W| = n - \ell$. Let $x \in V(H)$. Then $\deg_J(x) = n - \ell$, and hence $1 \leq \deg_J(x) \leq m - 1$. So $r(S) \notin V(H)$ for each $S \in \mathcal{D}\backslash\mathcal{S}$. For each $x \in W$, define $f(x) = |\{S \in \mathcal{J} : r(S) = x\}|$. Then the average value of f over W is

$$\frac{1}{n-\ell} \sum_{x \in W} f(x) = \frac{|E(J)|}{m(n-\ell)} = \frac{n+\ell-1}{2m} \geq \frac{4m+1}{2m} > 2.$$

Hence $f(x) \geq 3$ for some $x \in W$, and therefore $\deg_J(x) \geq 3m$, which is not possible. Hence $\mathcal{D}$ is a primitive m-star decomposition of K_n. $\square$

Notably, it follows from this argument that all m-star decompositions of K_n for which $n = 2m$ or $2m + 2 \leq n < 3m$ are primitive. Interestingly, K_{2m+1} does have an m-star decomposition that is not primitive, which arises by first decomposing K_{2m+1} as the union of K_{2m} and $K_{1,2m}$.

The next two lemmas, along with Lemma 3.2, are sufficient to show that Theorem 1.4 holds for odd $m \geq 3$ and $n \geq 3m$.

Lemma 3.3. *Let m be a positive odd integer, $m \geq 3$, K_m be a complete graph on a vertex set W, and V be a vertex set of cardinality $(m + 1)/2$ disjoint from W. Then $K_m + V$ has an m-star decomposition which contains two stars Q and R such that $r(Q) \in W$, $r(R) \in (P(Q) \cap W)$, and $|V \cap P(Q) \cap P(R)| \geq 2$.*

Proof. Identify W with the ring of integers modulo m, $\mathbb{Z}_m$. For each $i \in W$, define the m-star S_i as $S_i = (i; \{i + 1, \ldots, i + (m - 1)/2\} \cup V)$. Define $\mathcal{G} = \{S_i : i \in W\}$, and as such, $\mathcal{G}$ is an m-star decomposition of G.

Observe that by defining $Q = S_0$ and $R = S_1$, the conditions on Q and R outlined in the lemma statement are satisfied. $\square$

Lemma 3.4. *Let a and m be positive integers such that $m \geq 3$ and K_a has a primitive m-star decomposition. Then K_{a+m} has a primitive m-star decomposition if and only if m is odd.*

Proof. First suppose that K_{a+m} has a primitive m-star decomposition. Then $2m$ divides both $a(a - 1)$ and $(a + m)(a + m - 1)$. Hence $2m$ divides their difference, which is $m(m - 2a - 1)$, and therefore $m - 2a - 1$ must be even. So m is odd.

Now, suppose m is odd. Let U, V, and W be disjoint sets with cardinalities $a - (m+1)/2$, $(m+1)/2$, and m, respectively; note that $a \geq 2m$ by Theorem 1.3, since K_a has an m-star decomposition, so U is well-defined. Then $\{H, F, J\}$ is a decomposition of K_{a+m}, where $H = K_a$ with vertex set $U \cup V$, $J = K_m + V$ with vertex set $W \cup V$, and $F = K_{a-v,m}$ with U and W as the parts of its vertex set.

Let $\mathcal{H}$, $\mathcal{J}$, and $\mathcal{F}$ be the m-star decompositions of H, J, and F given by the hypothesis and Lemmas 3.3 and 2.4. Then $\mathcal{H}$ is primitive and $\mathcal{H} \cup \mathcal{J} \cup \mathcal{F}$, which we denote as $\mathcal{G}$, is an m-star decomposition of K_{a+m}.

Let $Q \in \mathcal{J}$ and $R \in \mathcal{J}$ such that $r(R) \in (P(Q) \cap W)$ and $|V \cap P(Q) \cap P(R)| \geq 2$; let $a = r(Q)$, $b = r(R)$, and $\{c, d\} \subseteq V \cap P(Q) \cap P(R)$ where $c \neq d$. Without loss of generality, we may assume there exists a star $S \in \mathcal{H}$ such that $r(S) = c$, $d \in P(S)$, and $P(S) \cap U$ is nonempty; let $e \in P(S) \cap U$. Observe that $(e; W) \in \mathcal{F}$, and let $T = (e; W)$. See Figure 2(a).

Let $\mathcal{K} = \{Q, S, T\}$, and define $\mathcal{K}'$ as the set of subgraphs $\{Q', S', T'\}$, where

$$E(Q') = (E(Q) \backslash \{ac\}) \cup \{ae\},$$
$$E(S') = (E(S) \backslash \{ce\}) \cup \{ca\}, \text{ and}$$
$$E(T') = (E(T) \backslash \{ea\}) \cup \{ec\}. \tag{3}$$

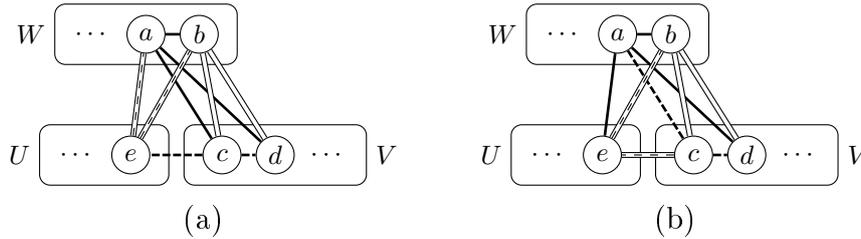(a)   (b)

Fig. 2.   (a) Edges of Q (——), R (═══), S (------), and T (====), stars from the decomposition $\mathcal{G}$ defined in the proof of Lemma 3.4

(b) Edges of Q' (——), R (═══), S' (------), and T' (====), stars from the decomposition $\mathcal{G}'$ defined in the proof of Lemma 3.4

See Figure 2(b). We seek to show that $(\mathcal{G}\backslash\mathcal{K})\cup\mathcal{K}'$, which we henceforth denote as $\mathcal{G}'$, is a primitive m-star decomposition of K_{a+m}. Observe that $V(G) = V(H) \cup V(K)$, so it is sufficient to show that conditions (a) through (e) of Lemma 2.7 are satisfied, as the result then follows. The first three follow immediately by construction, so we now demonstrate that (d) and (e) are met.

Let $\mathcal{S}$ be a subdecomposition of $\mathcal{G}$, and assume that $\mathcal{S}\cap\mathcal{H}$ is empty. Then either $\mathcal{S}\cap\mathcal{J}$ or $\mathcal{S}\cap\mathcal{F}$ is nonempty. If $\mathcal{S}\cap\mathcal{J}$ is nonempty, then $V \subseteq V(\mathcal{S})$, and since $|V|\geq 2$, it follows that $\mathcal{S}\cap\mathcal{H}$ is nonempty, which is a contradiction. So $\mathcal{S}\cap\mathcal{J}$ is empty and $\mathcal{S}\cap\mathcal{F}$ is nonempty. Then $(x;W) \in \mathcal{S}$ for some $x \in U$. So $W \in V(\mathcal{S})$, which implies $\mathcal{S}\cap\mathcal{J}$ is nonempty, which contradicts. So $\mathcal{S}\cap\mathcal{H}$ is nonempty, and hence (d) is satisfied.

Now, let $\mathcal{S}'$ be a subdecomposition of $\mathcal{G}'$. First, we show that $\mathcal{S}'\cap\mathcal{K}'$ is nonempty. Suppose to the contrary. Then $\mathcal{S}'$ is also a subdecomposition of $\mathcal{G}$ and $\{Q,S,T\}\cap\mathcal{S}$ is empty. However, since $\mathcal{S}'$ is nontrivial, $\mathcal{S}'$ must intersect nontrivially with either $\mathcal{H}$, $\mathcal{J}$, or $\mathcal{F}$. If $\mathcal{S}'\cap\mathcal{H}$ is nonempty, then $\mathcal{H}\subseteq\mathcal{S}'$ by Lemma 2.6, meaning $S \in \mathcal{S}'$, which is a contradiction. If $\mathcal{S}\cap\mathcal{J}$ is nonempty, then $V \subseteq V(\mathcal{S})$, which implies that either $S \in \mathcal{S}$ or $\mathcal{S}'\cap\mathcal{H}$ is nonempty, which is another contradiction. If $\mathcal{S}'\cap\mathcal{F}$ is nonempty, then $W \subseteq V(\mathcal{S}')$ and hence $\mathcal{S}'\cap\mathcal{J}$ is nonempty, which is, again, a contradiction. Therefore $\mathcal{S}'\cap\mathcal{K}'$ is nonempty.

We conclude by showing that $\mathcal{K}'\subseteq\mathcal{S}$. We consider three cases.

- Suppose $Q' \in \mathcal{S}'$. Then $e,b \in V(\mathcal{S}')$, and hence $T' \in \mathcal{S}'$.

- Suppose $T' \in \mathcal{S}'$. Then $c,b \in V(\mathcal{S}')$, and hence $R \in \mathcal{S}$. So $d \in V(\mathcal{S}')$, implying that $S' \in \mathcal{S}$.

- Suppose $S' \in \mathcal{S}'$. Then $a,d \in V(\mathcal{S}')$, and hence $Q' \in \mathcal{S}'$.

The above implications show that since $\mathcal{K}'\cap\mathcal{S}'$ is nonempty, then $\mathcal{K}'\subseteq\mathcal{S}'$, which satisfies condition (e) in Lemma 2.7. Hence $\mathcal{G}'$ is primitive, and therefore K_{a+m} has a primitive m-star decomposition.   □

Combining the results of Theorem 1.3 with Lemmas 3.2 and 3.4 admits the following classification.

Theorem 3.5. *Let $m \geq 3$ be an odd integer. Then K_n has a primitive m-star decomposition if and only if (m,n) is valid.*

4. Star decompositions with even star size

Lemma 3.2 proved Theorem 1.4 when $m \geq 2$ and $2m \leq n < 3m$, regardless of the parity of m. However, Lemma 3.4 relies on m being odd, meaning another method is needed when m is even. First, we directly prove the existence of primitive m-star decompositions of K_n when $n < 4m$ with two constructions, dependent on the parity of n. Then, we prove an inductive result weaker than Lemma 3.4 to resolve the remaining cases of Theorem 1.4.

Lemma 4.1. *Let $m \geq 3$ and n be a positive, even integer. Suppose there exist integers p and q such that $p \geq 2$, $1 \leq q \leq m - 2$, $m = p + 2q + 1$, and $n = 4p + 6q + 4$. Then there exists a primitive m-star decomposition of K_n.*

Proof. Since p and q are nonnegative, it follows that

$$m + q \leq 2p + 2q + 2 + \frac{1}{m}\binom{2q+1}{2} \leq 2m + 2q + 1.$$

Then we may find integers s and t such that $0 \leq s \leq 2q + 1$, $m + q \leq t \leq 2m$, and $s + t = 2p + 2q + 2 + \frac{1}{m}\binom{2q+1}{2}$. Observe that $n - 2m = 2p + 2q + 2$. Let the sequences α and β, each with length $2m$ and $n - 2m$, respectively, be defined as

$$\alpha = (0 : 2m - t)(m : t) \text{ and } \beta = (m - q : 2q + 1 - s)(m : 2p + 1)(2m - q : s).$$

Since $\Sigma_{2m}(\alpha) + \Sigma_{n-2m}(\beta) = 2m(n - 2m)$ and $2m - t \leq m - q$ (which is equivalent to $t \geq m + q$), then by Lemma 2.2, there exists a bi-tournament T on $(2m, n - 2m)$ vertices for which α and β are score sequences. Let V_a, V_b, and V_c be the vertex sets $V_a = \{a_i : i \in \mathbb{Z}_{2m}\}$, $V_b = \{b_j : j \in \mathbb{Z}_{2p+1}\}$, and $V_c = \{c_k : k \in \mathbb{Z}_{2q+1}\}$. We may take V_a and $V_b \cup V_c$ to be the vertex sets of T, and hence there exist a t-subset $X \subseteq \mathbb{Z}_{2m}$ and an s-subset $Y \subseteq \mathbb{Z}_{2q+1}$ such that

$$|N_T(a_i)| = m \text{ if } i \in X \text{ and } 0 \text{ if } i \notin X,$$
$$|N_T(b_j)| = m, \text{ and}$$
$$|N_T(c_k)| = 2m - q \text{ if } k \in Y \text{ and } m - q \text{ if } k \notin Y.$$

Without loss of generality, we may assume that $N_T(b_0) = \{a_0, \ldots, a_{m-1}\}$.

Let G be a graph isomorphic to K_n with vertex set $V_a \cup V_b \cup V_c$. For each $k \in \mathbb{Z}_{2q+1}$, let $N'_T(c_k)$ be a $(m - q)$-subset of $N_T(c_k)$; note that $|N_T(c_k)\backslash N'_T(c_k)| = m$ when $k \in Y$ and $N'_T(c_k) = N_T(c_k)$ when $k \notin Y$. For each $i \in \mathbb{Z}_{2m}$, define $x_i = b_0$ if $i \in \{0, \ldots, m - 1\}$ and $x_0 = a_{i+m}$ if $i \in \{m, \ldots, 2m - 1\}$. For $i \in \mathbb{Z}_{2m}$, $j \in \mathbb{Z}_{2p+1}$, and $k \in \mathbb{Z}_{2q+1}$, define the subgraphs A_i, B_j, and C_k of G as the stars

$$A_i = (a_i; \{a_{i+1}, \ldots, a_{i+m-1}, x_i\}),$$
$$B_j = (b_j; \{b_{j+1}, \ldots, b_{j+p}\} \cup V_c), \text{ and}$$
$$C_k = (c_k; \{c_{k+1}, \ldots, c_{k+q}\} \cup N'_T(c_k)).$$

For each $i \in X$, $j \in \mathbb{Z}_{2p+1}\backslash\{0\}$, and $k \in Y$, define

$$A'_i = (a_i; N_T(a_i)),$$
$$B'_j = (b_j; N_T(b_j)), \text{ and}$$
$$C'_k = (c_k; N_T(c_k)\backslash N'_T(c_k)).$$

Next, define $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{A}'$, $\mathcal{B}'$, and $\mathcal{C}'$ as the sets

$$\mathcal{A} = \{A_i : i \in \mathbb{Z}_{2m}\}, \qquad \mathcal{A}' = \{A'_i : i \in X\},$$
$$\mathcal{B} = \{B_j : j \in \mathbb{Z}_{2p+1}\}, \qquad \mathcal{B}' = \{B'_j : j \in \mathbb{Z}_{2p+1}\backslash\{0\}\},$$
$$\mathcal{C} = \{C_k : k \in \mathbb{Z}_{2q+1}\}, \text{ and} \qquad \mathcal{C}' = \{C'_k : k \in Y\}.$$

Finally, define $\mathcal{D} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{A}' \cup \mathcal{B}' \cup \mathcal{C}'$. We claim that $\mathcal{D}$ is a primitive m-star decomposition of K_n. By its construction, we have that $\mathcal{D}$ is an m-star decomposition, so we only need to show $\mathcal{D}$ is primitive. To that end, let $\mathcal{S}$ be a nontrivial subdecomposition of $\mathcal{D}$. First, we begin with two useful facts about $\mathcal{S}$.

(a) Suppose $\mathcal{A} \cap \mathcal{S} \neq \varnothing$. Then there exists $A_i \in \mathcal{S}$ for some $i \in \mathbb{Z}_{2m}$. Since $m \geq 3$, we have that $a_{i+1}, a_{i+2} \in V(\mathcal{S})$. So $A_{i+1} \in \mathcal{S}$ as well, since $a_{i+1}a_{i+2} \in E(A_{i+1})$. An iterative application of this argument gives that $\mathcal{A} \subseteq \mathcal{S}$, and as such, $V_a \subseteq V(\mathcal{S})$.

(b) Suppose $\mathcal{B} \cap \mathcal{S} \neq \varnothing$. Then there exists $B_j \in \mathcal{S}$ for some $j \in \mathbb{Z}_{2p+1}$. Then $V_c \subseteq V(\mathcal{S})$ and, since $p \geq 2$, $b_{j+1} \in V(\mathcal{S})$ as well. Hence $B_{j+1} \in \mathcal{S}$, since $b_{j+1}x \in E(B_{j+1})$ for each $x \in V_c$. An iterative application of this argument gives that $\mathcal{B} \subseteq \mathcal{S}$, and as such, $V_b \cup V_c \subseteq V(\mathcal{S})$.

Therefore, it is sufficient to show that $\mathcal{A} \cap \mathcal{S}$ and $\mathcal{B} \cap \mathcal{S}$ are nonempty, as (a) and (b) imply $V(\mathcal{S}) = V(G)$, and hence $\mathcal{D}$ is primitive by Lemma 2.5(b). Consider the following five cases.

(1) Suppose that $\mathcal{A} \cap \mathcal{S} \neq \varnothing$. Then from property (a), we have that $\mathcal{A} \subseteq \mathcal{S}$, and hence $V_a \subseteq V(\mathcal{S})$ and $b_0 \in V(\mathcal{S})$ as well. Since $b_0a_m \in E(A'_m)$, we have that $A'_m \in \mathcal{S}$, and since $p \geq 2$, there exists some $b_j \in V(A'_m)$, where $j \in \mathbb{Z}_{2p+1}$ and $j \neq 0$. Since either $b_0b_j \in E(B_0)$ or $E(B_j)$, we have that $\mathcal{B} \cap \mathcal{S}$ is nonempty.

(2) Suppose that $\mathcal{B} \cap \mathcal{S} \neq \varnothing$. Then $V_c \subseteq V(\mathcal{S})$, and since $q \geq 1$, $C_k \in \mathcal{S}$ for each $k \in \mathbb{Z}_{2q+1}$. Hence $\mathcal{C} \cap \mathcal{S}$ is nonempty.

(3) Suppose that $\mathcal{C} \cap \mathcal{S} \neq \varnothing$. Then $C_k \in \mathcal{S}$ for some $k \in \mathbb{Z}_{2q+1}$. Since $m - q \geq 2$, there exist distinct $a_h, a_i \in V(\mathcal{S})$; without loss of generality, suppose $h = i + \ell$ for some $\ell \in \{1, \ldots, m\}$. Hence $A_i \in \mathcal{S}$, and therefore $\mathcal{A} \cap \mathcal{S}$ is nonempty.

(4) Suppose $\mathcal{A}' \cap \mathcal{S} \neq \varnothing$. Then $|V_b \cap V(\mathcal{S})| \geq p \geq 2$, and so $\mathcal{B} \cap \mathcal{S}$ is nonempty.

(5) Suppose $(\mathcal{B}' \cup \mathcal{C}') \cap \mathcal{S}$ is nonempty. Then $|V_a \cap V(\mathcal{S})| \geq m \geq 3$, and hence $\mathcal{A} \cap \mathcal{S}$ is nonempty.

The previous five facts imply that, since $\mathcal{S}$ is nonempty, $\mathcal{A} \cap \mathcal{S}$ and $\mathcal{B} \cap \mathcal{S}$ are both nonempty. Hence $\mathcal{D}$ is primitive. $\square$

The next lemma and proof is very similar to that of Lemma 4.1, however, modifications to the construction are needed due to the parity of n.

Lemma 4.2. *Let $m \geq 3$ and n be a positive, odd integer. Suppose there exist integers p and q such that $p \geq 2$, $1 \leq q \leq m - 2$, $m = p + 2q$, and $n = 4p + 6q + 1$. Then there exists a primitive m-star decomposition of K_n.*

Proof. Since p and q are nonnegative, it follows that

$$m + q \leq 2p + 2q + \frac{1}{m}\binom{2q}{2} \leq 2m + 2q.$$

Then we may find integers s and t such that $0 \leq s \leq 2q$, $m + q \leq t \leq 2m$, and $s + t = 2p + 2q + \frac{1}{m}\binom{2q}{2}$. Observe that $n - 2m = 2p + 2q + 1$. Let the sequences α and β, each with length $2m$ and $n - 2m$, respectively, be defined as

$$\alpha = (0 : 2m - t)(m : t) \text{ and } \beta = (m - q : 2q - 1 - s)(m : 2p + 1)(2m - q : s)(2m : 1).$$

Since $\Sigma_{2m}(\alpha) + \Sigma_{n-2m}(\beta) = 2m(n - 2m)$ and $2m - t \leq m - q$ (which is equivalent to $t \geq m + q$), then by Lemma 2.2, there exists a bi-tournament T on $(2m, n - 2m)$ vertices for which α and β are score sequences. Let V_a, V_b, and V_c be the vertex sets $V_a = \{a_i : i \in \mathbb{Z}_{2m}\}$, $V_b = \{b_j : j \in \mathbb{Z}_{2p+1}\}$, and $V_c = \{c_k : k \in \mathbb{Z}_{2q-1}\} \cup \{\infty\}$. We may take V_a and $V_b \cup V_c$ to be the vertex sets of T, and hence there exist a t-subset $X \subseteq \mathbb{Z}_{2m}$ and an s-subset $Y \subseteq \mathbb{Z}_{2q-1}$ such that

$$
\begin{aligned}
&|N_T(a_i)| = m \text{ if } i \in X \text{ and } 0 \text{ if } i \notin X, \\
&|N_T(b_j)| = m, \\
&|N_T(\infty)| = 2m, \text{ and} \\
&|N_T(c_k)| = 2m - q \text{ if } k \in Y \text{ and } m - q \text{ if } k \notin Y.
\end{aligned}
$$

Without loss of generality, we may assume that $N_T(b_0) = \{a_0, \ldots, a_{m-1}\}$.

Let G be a graph isomorphic to K_n with vertex set $V_a \cup V_b \cup V_c$. For each $k \in \mathbb{Z}_{2q+1}$, let $N_T'(c_k)$ be a $(m - q)$-subset of $N_T(c_k)$; note that $|N_T(c_k) \backslash N_T'(c_k)| = m$ when $k \in Y$ and $N_T'(c_k) = N_T(c_k)$ when $k \notin Y$. Let $\{N_1, N_2\}$ partition $N_T(\infty)$ such that $|N_1| = |N_2| = m$. For each $i \in \mathbb{Z}_{2m}$, define $x_i = b_0$ if $i \in \{0, \ldots, m-1\}$ and $x_0 = a_{i+m}$ if $i \in \{m, \ldots, 2m-1\}$. For $i \in \mathbb{Z}_{2m}$, $j \in \mathbb{Z}_{2p+1}$, and $k \in \mathbb{Z}_{2q-1}$, define the subgraphs A_i, B_j, and C_k of G as the stars

$$
\begin{aligned}
A_i &= (a_i; \{a_{i+1}, \ldots, a_{i+m-1}, x_i\}), \\
B_j &= (b_j; \{b_{j+1}, \ldots, b_{j+p}\} \cup V_c), \text{ and} \\
C_k &= (c_k; \{c_{k+1}, \ldots, c_{k+q-1}\} \cup \{\infty\} \cup N_T'(c_k)).
\end{aligned}
$$

For each $i \in X$, $j \in \mathbb{Z}_{2p+1} \backslash \{0\}$, and $k \in Y$, define

$$
\begin{aligned}
A_i' &= (a_i; N_T(a_i)), \\
B_j' &= (b_j; N_T(b_j)), \text{ and} \\
C_k' &= (c_k; N_T(c_k) \backslash N_T'(c_k)),
\end{aligned}
$$

and define C'_∞ and C''_∞ as $(\infty; N_1)$ and $(\infty; N_2)$, respectively. Next, define $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{A}'$, $\mathcal{B}'$, and $\mathcal{C}'$ as the sets

$$\begin{aligned}
\mathcal{A} &= \{A_i : i \in \mathbb{Z}_{2m}\}, & \mathcal{A}' &= \{A'_i : i \in X\}, \\
\mathcal{B} &= \{B_j : j \in \mathbb{Z}_{2p+1}\}, & \mathcal{B}' &= \{B'_j : j \in \mathbb{Z}_{2p+1}\backslash\{0\}\}, \\
\mathcal{C} &= \{C_k : k \in \mathbb{Z}_{2q+1}\}, \text{ and} & \mathcal{C}' &= \{C'_k : k \in Y\} \cup \{C'_\infty, C''_\infty\}.
\end{aligned}$$

Finally, define $\mathcal{D} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{A}' \cup \mathcal{B}' \cup \mathcal{C}'$. Using an argument identical to that which is used in the proof of Lemma 4.1 (by replacing $\mathbb{Z}_{2q+1}$ with $\mathbb{Z}_{2q-1}$ where appropriate), we have that $\mathcal{D}$ is primitive. $\qquad\square$

Lemma 4.3. *Let $m \geq 4$, m even, and n be an integer such that $3m \leq n < 4m$ and (m, n) is valid. Then K_n has a primitive m-star decomposition.*

Proof. First, observe that if m is even, then $(m, 3m)$ and $(m, 3m + 1)$ are not valid. Furthermore, since $m \geq 3$, we have that $(m, 3m + 2)$ and $(m, 4m - 1)$ are not valid. Hence, we may assume that $3m + 3 \leq n \leq 4m - 2$.

Suppose that n is even. Let $p = n - 1 - 3m$ and $q = 2m - \frac{n}{2}$. Then $p \geq 2$, $1 \leq q \leq m - 2$, $m = p + 2q + 1$ and $n = 4p + 6q + 4$. Therefore, by Lemma 4.1, K_n has a primitive m-star decomposition.

Suppose that n is odd. Let $p = n - 1 - 3m$ and $q = 2m - (n - 1)/2$. Then $p \geq 2$ and $1 \leq q \leq m - 2$, $m = p + 2q$, and $n = 4p + 6q + 1$. Therefore, by Lemma 4.2, K_n has a primitive m-star decomposition. $\qquad\square$

Observe that the proof of Lemma 4.3 only uses the hypothesis that m is even to discount $(m, 3m)$ and $(m, 3m + 1)$ from being valid. One could simply produce a primitive m-star decomposition of K_{3m} and K_{3m+1} when m is odd using the earlier constructions, and then Lemma 4.3 would hold for all $m \geq 3$. However, the next two lemmas have constructions which depend heavily on m being even, and thus it is necessary to have distinct constructions, based on the parity of m.

Lemma 4.4. *Let m be a positive even integer, $m \geq 4$, K_{2m-1} be a complete graph on a vertex set W, and V be a vertex set of cardinality $m + 1$ disjoint from W. Then $K_{2m-1} + V$ has a primitive m-star decomposition which contains two stars Q and R such that $r(Q) \in W$, $r(R) \in (P(Q) \cap W)$, and $|V \cap P(Q) \cap P(R)| \geq 2$.*

Proof. Let $\{V_1, V_2\}$ be a partition of V where $|V_1| = (m + 2)/2$ and $|V_2| = m/2$, and identify W with the ring of integers modulo $2m - 1$, $\mathbb{Z}_{2m-1}$. Let $x \in V_1$. For each $i \in W$, define the m-stars S_i and T_i as

$$\begin{aligned}
S_i &= (i; V_1 \cup \{i + 1, \ldots, i + (m/2) - 1\}) \text{ and} \\
T_i &= (i; V_2 \cup \{i + (m/2), \ldots, i + m - 1\}).
\end{aligned}$$

Define $\mathcal{G} = \{S_i, T_i : i \in W\}$, and as such, $\mathcal{G}$ is a decomposition of G.

Suppose $\mathcal{S}$ is a subdecomposition of $\mathcal{G}$. Then $\mathcal{S}$ is nonempty; first suppose that $S_k \in \mathcal{S}$ for some $k \in W$. It follows that $x, k+1 \in V(\mathcal{S})$ and, since $\{x, k+1\} \in E(S_{k+1})$, we have that $S_{k+1} \in \mathcal{S}$. Applying this iteratively gives that $\{S_i : i \in W\} \subseteq \mathcal{S}$, and hence $W \cup V_1 \subseteq V(\mathcal{S})$. Since $k, k+(m/2) \in V(\mathcal{S})$, it follows that $T_k \in \mathcal{S}$ and hence $V_2 \subseteq V(\mathcal{S})$. Therefore $\mathcal{S} = \mathcal{G}$ by Lemma 2.5(b).

If $T_k \in \mathcal{S}$ for some $k \in W$, then $k+(m/2), k+(m/2)+1 \in V(\mathcal{S})$, and since edge $\{k+(m/2), k+(m/2)+1\} \in E(S_{k+(m/2)})$, it follows that $S_{k+(m/2)} \in \mathcal{S}$. So $\mathcal{S} = \mathcal{G}$ from the above argument. Therefore $\mathcal{G}$ is primitive by Lemma 2.5(d).

Observe that by defining $Q = T_0$ and $R = T_{m/2}$, we have that Q and R satisfy their conditions outlined in the lemma statement. $\qquad\square$

Lemma 4.5. *Let a and m be positive integers such that $a > m \geq 4$ and m is even. If K_a has a primitive m-star decomposition, then K_{a+2m} has a primitive m-star decomposition as well.*

Proof. Let U, V, and W be sets with cardinalities $a - m$, m, and $2m - 1$, respectively, and let ∞ be an element such that $\infty \notin (U \cup V \cup W)$. Let W_1 be a subset of W with cardinality m and $W_2 = (W \backslash W_1) \cup \{\infty\}$. Then $\{H, J, F_1, F_2, E\}$ is a decomoposition of K_{a+2m}, where $H = K_a$ with vertex set $U \cup V$, $J = K_{2m-1} + (V \cup \{\infty\})$ with vertex set $W \cup V \cup \{\infty\}$, $F_1 = K_{a-m,m}$ with U and W_1 as the parts of its vertex set, $F_2 = K_{a-m,m}$ with U and W_2 as the parts of its vertex set, and $E = (\infty; V)$.

Let $\mathcal{H}$, $\mathcal{J}$, $\mathcal{F}_1$, and $\mathcal{F}_2$ be the m-star decompositions of H, J, F_1, and F_2 given by the hypothesis and Lemmas 4.4 and 2.4. Then $\mathcal{H}$ and $\mathcal{J}$ are primitive and $\mathcal{H} \cup \mathcal{J} \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \{E\}$, which we now denote as $\mathcal{G}$, is an m-star decomposition of K_{a+2m}.

Let $Q \in \mathcal{J}$ and $R \in \mathcal{J}$ such that $r(R) \in (P(Q) \cap W)$ and $|V \cap P(Q) \cap P(R)| \geq 2$; let $a = r(Q)$, $b = r(R)$, and $\{c, d\} \subseteq V \cap P(Q) \cap P(R)$ where $c \neq d$. Without loss of generality, we may assume there exists a star $S \in \mathcal{H}$ such that $r(S) = c$, $d \in P(S)$, and $P(S) \cap U$ is nonempty; let $e \in P(S) \cap U$. Also, without loss of generality, we may assume $\{a, b\} \subseteq W_1$. Then $(e; W_1) \in \mathcal{F}_1$, and let $T = (e, W_1)$. See Figure 3(a).
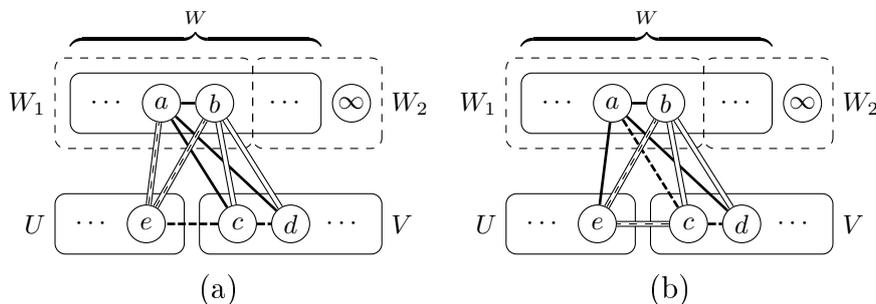

(a)   (b)

Fig. 3. (a) Edges of Q (——), R (══), S (------), and T (════), stars from the decomposition $\mathcal{G}$ defined in the proof of Lemma 4.5
(b) Edges of Q' (——), R (══), S' (------), and T' (════), stars from the decomposition $\mathcal{G}'$ defined in the proof of Lemma 4.5

Similar to the proof of Lemma 3.4, let $\mathcal{K} = \{Q, S, T\}$, and define $\mathcal{K}'$ as the set of

subgraphs $\{Q', S', T'\}$, where Q', S', and T' are defined as given in (3). See Figure 3(b). Again, we define $\mathcal{G}' = (\mathcal{G} \backslash \mathcal{K}) \cup \mathcal{K}'$, and we now show that $\mathcal{G}'$ is a primitive m-star decomposition of K_{a+2m}. Similar to the proof of Lemma 3.4, we need only to demonstrate that (d) and (e) of Lemma 2.8 are satisfied.

Let $\mathcal{S}$ be a subdecomposition of $\mathcal{G}$, and assume that $\mathcal{S} \cap (\mathcal{H} \cup \mathcal{J})$ is empty. Then $\mathcal{S} \subseteq \mathcal{F}_1 \cup \mathcal{F}_2 \cup \{E\}$ and $\mathcal{S}$ is nonempty. Then either $(x; W_1) \in \mathcal{S}$ or $(x; W_2) \in \mathcal{S}$ for some $x \in U$, or $E \in \mathcal{S}$. So either $W_1 \subseteq \mathcal{S}$, $W_2 \subseteq \mathcal{S}$, or $V \subseteq \mathcal{S}$. In the first two cases, it follows that $\mathcal{J} \cap \mathcal{S}$ is nonempty, and $\mathcal{H} \cap \mathcal{S}$ is nonempty in the latter case, and both result in a contradiction. So (d) is satisfied in Lemma 2.8.

Now, let $\mathcal{S}'$ be a subdecomposition of $\mathcal{G}'$. First, we show that $\mathcal{S}' \cap \mathcal{K}'$ is nonempty. Suppose to the contrary. Then $\mathcal{S}'$ is also a subdecomposition of $\mathcal{G}$ and $\{Q, S, T\} \cap \mathcal{S}$ is empty. However, since $\mathcal{S}'$ is nontrivial, $\mathcal{S}'$ must intersect nontrivially with either $\mathcal{H}$, $\mathcal{J}$, $\mathcal{F}_1$, or $\mathcal{F}_2$. If $\mathcal{S}' \cap \mathcal{H}$ is nonempty, then $\mathcal{H} \subseteq \mathcal{S}'$ by Lemma 2.6, meaning $S \in \mathcal{S}$, which is a contradiction. Similarly, if $\mathcal{S} \cap \mathcal{J}$ is nonempty, it follows that $T \in \mathcal{S}'$, which also contradicts. So $\mathcal{S}' \subseteq (\mathcal{F}_1 \cup \mathcal{F}_2 \cup \{E\})$, when then by an identical argument used earlier, $W_1 \subseteq V(\mathcal{S}')$, $W_2 \subseteq V(\mathcal{S}')$, or $V \subseteq V(\mathcal{S}')$, and hence either $\mathcal{S}' \cap \mathcal{J}$ or $\mathcal{S}' \cap \mathcal{H}$ is nonempty, which is a contradiction. Therefore $\mathcal{S}' \cap \mathcal{K}'$ is nonempty.

Using the identical argument given in the proof of Lemma 3.4, it follows that $\mathcal{K}' \subseteq \mathcal{S}'$, which satisfies condition (e) in Lemma 2.8. Hence $\mathcal{G}'$ is primitive, and therefore K_{a+2m} has a primitive m-star decomposition. $\square$

To complete the proof of Theorem 1.4, we must first state a special case of the result proved by Dinavahi and Rodger. [4]

Theorem 4.6. *Let n be a positive integer. Let P_2 denote a path with two edges. Then K_n has a primitive P_2-decomposition if and only if $n = 1$ or $(2, n)$ is valid.*

Since a 2-star is isomorphic to P_2, by combining the results of Theorems 1.3 and 4.6 with Lemmas 3.2, 4.3, and 4.5, we admit the following classification.

Theorem 4.7. *Let $m \geq 2$ be an even integer. Then K_n has a primitive m-star decomposition if and only if (m, n) is valid.*

Finally, observe that a 1-star is isomorphic to K_2, which implies that each subgraph in a 1-star decomposition gives rise to a subdecomposition. This fact, along with Theorems 3.5 and 4.7, imply Theorem 1.4.

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