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Factorization of spread polynomials

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ABSTRACT

We present a proof of a conjecture of Goh and Wildberger on the factorization of the spread polynomials. We indicate how the factors can be effectively calculated and exhibit a connection to the factorization of Fibonacci numbers into primitive parts.

Keywords: spread polynomials, cyclotomic polynomials, Fibonacci numbers

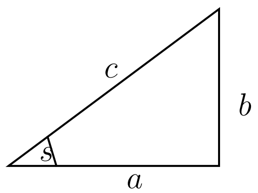
2020 Mathematics Subject Classification: Primary 11B83; Secondary 11B39, 97G40.

1. Introduction

The sequence of *spread polynomials* $(S_n(x))_{n\geq 1}$ of Norman J. Wildberger is uniquely defined by the requirement

$$S_n\left(\sin^2\theta\right) = \sin^2(n\theta). \tag{1}$$

The spread polynomials play a central role in Wildberger's *rational trigonometry* [8]. In order to make plane geometry free of transcendental expressions and square roots he suggests to replace lengths by their squares, which are referred to as *quadrances*, and angles by *spreads*. In the right angled triangle with sides a, b and c



the spread $s \in [0,1]$ is defined as the ratio $s = b^2/c^2$ of the quadrance of the opposite leg b by the quadrance of the hypothenuse c. Let s_n be the spread between the extremal

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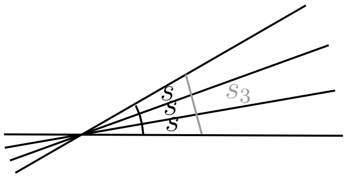
lines of an arrangement of $n+1$ lines which all meet in a single point and whose spreads of neighboring lines all coincide. Then

$$s_n = S_n(s), \tag{2}$$

states the geometric property of the spread polynomials that makes them play a central role in rational trigonometry.



The spread polynomials can be written in terms of the Lucas polynomials $L_n(x)$

$$S_n(x) = \frac{2 - L_n(2-4x)}{4}. \tag{3}$$

Following [3] we consider the closely related polynomials

$$Z_n(x) = 4S_n\left(\frac{x}{4}\right) = 2 - L_n(2-x). \tag{4}$$

A conjecture of Wildberger's honours student Shuxiang Goh (see [9]), stated in terms of $Z_n(x)$, claims the following.

Theorem 1.1. *There is a sequence of polynomials* $(\Psi_d(x))_{d\geq 1}$, $\Psi_d(x) \in \mathbb{Z}[x]$, *which are closely related to the cyclotomic polynomials* $\Phi_d(x)$, *such that* $Z_n(x) = \prod_{d|n} \Psi_d(x)$ *and* $\deg(\Psi_d) = \varphi(d)$, *where* $\varphi(d)$ *is Euler's totient function.*

The aim of this note is to prove the theorem as a consequence of (4) and known results about the Chebyshev polynomials obtained in [1], [2] and [7]. Moreover, we obtain some concrete facts about the polynomials $\Psi_d(x)$, some of which had already been conjectured in [3].

2. Some well-known facts about Lucas polynomials

The Lucas polynomials $L_n(x)$ are defined by the recursion

$$L_n(x) = xL_{n-1}(x) - L_{n-2}(x), \tag{5}$$

for $n \geq 2$ with initial values $L_0(x) = 2$ and $L_1(x) = x$.[2] The first terms are (cf. [6, entry A034807]):

[2] Up to the index $n=0$ the sequence of Lucas polynomials coincides with the sequence of the pyramidal polynomials from [3], the latter coming from a Riordan array.

n	0	1	2	3	4	5	6	$\cdots$
$L_n(x)$	2	x	$-2+x^2$	$-3x+x^3$	$2-4x^2+x^4$	$5x-5x^3+x^5$	$-2+9x^2-6x^4+x^6$	$\cdots$

Binet's formula gives

$$L_n(x) = (\alpha(x))^n + (\beta(x))^n, \tag{6}$$

with $\alpha(x) = \frac{x+\sqrt{x^2-4}}{2}$ and $\beta(x) = \frac{1}{\alpha(x)} = \frac{x-\sqrt{x^2-4}}{2}$. The Lucas polynomials are related to the Chebyshev polynomials of the first kind $T_n(x)$ by

$$L_n(x) = 2T_n\left(\frac{x}{2}\right), \tag{7}$$

and are characterized by

$$L_n\left(z + \frac{1}{z}\right) = z^n + \frac{1}{z^n}, \tag{8}$$

for $n \geq 1$ and $z \neq 0$ because $z^n + \frac{1}{z^n} = \left(z + \frac{1}{z}\right)\left(z^{n-1} + \frac{1}{z^{n-1}}\right) - \left(z^{n-2} + \frac{1}{z^{n-2}}\right)$. For $z = e^{\sqrt{-1}\theta}$ Eq. (8) implies

$$L_n(2\cos\theta) = 2\cos(n\theta). \tag{9}$$

Eq. (8) also implies

$$L_{mn}(x) = L_m(L_n(x)), \tag{10}$$

for all $m, n \geq 0$ since $z^{mn} + \frac{1}{z^{mn}} = (z^n)^m + \left(\frac{1}{z^n}\right)^m$.

Let us also mention that $z^{2n} + \frac{1}{z^{2n}} - 2 = \left(z^n + \frac{1}{z^n} - 2\right)\left(z^n + \frac{1}{z^n} + 2\right)$ implies

$$L_{2n}(x) - 2 = (L_n(x) - 2)(L_n(x) + 2), \tag{11}$$

and that

$$L_{2n}(x) + 2 = (L_n(x))^2, \tag{12}$$

because $z^{2n} + \frac{1}{z^{2n}} + 2 = \left(z^n + \frac{1}{z^n}\right)^2$.

3. Factorization of the polynomials $L_n(x) - 2$

Proposition 3.1. *For $n \geq 1$ the roots of $f_n(x) := L_n(x) - 2$ are $\nu_k = 2\cos\left(\frac{2\pi k}{n}\right)$ with $0 \leq k \leq \frac{n}{2}$. For odd $n = 2m+1$ the roots ν_k with $1 \leq k \leq m$ are double roots and $\nu_0 = 2$ is a simple root. For even $n = 2m$ the roots ν_k with $1 \leq k < m$ are double roots and ν_0, ν_m are simple roots.*

Proof. Since $f_n\left(2\cos\left(\frac{2\pi k}{n}\right)\right) = L_n\left(2\cos\left(\frac{2\pi k}{n}\right)\right) - 2 = 2\cos(2\pi k) - 2 = 0$ each $\nu_k = 2\cos\left(\frac{2\pi k}{n}\right)$ for $k \in \mathbb{Z}$ is a root of $f_n(x)$. All different roots of this form are given by $0 \leq k \leq \frac{n}{2}$.

We claim that in this way we obtain all the roots of $f_n(x)$ because the number of roots above, counted with multiplicity, in fact already amounts to n.

To show the claim for $n = 2m + 1$ it suffices to show that

$$(L_{2m+1}(x) - 2)(x - 2) = (L_{m+1}(x) - L_m(x))^2, \tag{13}$$

which is a square of a polynomial. For $n = 2m$ in turn it suffices to show the identity

$$(L_{2m}(x) - 2)(x - 2)(x + 2) = (L_{m+1}(x) - L_{m-1}(x))^2. \tag{14}$$

Since both sides of these identities are linear combinations of α^{2n}, β^{2n} and 1^n they satisfy the same recurrence of order 3. More precisely since

$$(z - \alpha^2)(z - \beta^2)(z - 1) = z^3 + (1 - x^2)z^2 + (x^2 - 1)z - 1, \tag{15}$$

they satisfy

$$f(n + 3) + (1 - x^2)f(n + 2) + (x^2 - 1)f(n + 1) - f(n) = 0. \tag{16}$$

Thus to prove these identities it suffices to verify them for $0 \leq n \leq 2$. □

Next we use the fact that a polynomial $p(x)$ with an algebraic number a as a root has the minimal polynomial of a as a factor.

The minimal polynomial $\psi_n(x)$ of 2cos$\left(\frac{2\pi}{n}\right)$ is

$$\psi_n(x) = \prod_{\gcd(j,n)=1,\ 0<j<n/2} \left(x - 2\cos\frac{2\pi j}{n}\right), \tag{17}$$

for $n > 2$ with $\psi_1(x) = x - 2$ and $\psi_2(x) = x + 2$. The first examples are:

n	1	2	3	4	5	6	7	8	9	$\cdots$
$\psi_n(x)$	$-2+x$	$2+x$	$1+x$	x	$-1+x+x^2$	$-1+x$	$-1-2x+x^2+x^3$	$-2+x^2$	$1-3x+x^3$	$\cdots$

Theorem 3.2. *Let* $\alpha(x) = \frac{x+\sqrt{x^2-4}}{2}$ *and* $\Phi_n(x)$ *be the nth cyclotomic polynomial. Then for* $n \geq 3$ *the minimal polynomial of* $2\cos\left(\frac{2\pi}{n}\right)$ *is*

$$\psi_n(x) = \frac{\Phi_n(\alpha(x))}{(\alpha(x))^{\varphi(n)/2}}. \tag{18}$$

Proof. For $n \geq 3$ the cyclotomic polynomial $\Phi_n(x)$ is a symmetric polynomial with integer coefficients of even degree $\varphi(n)$. For example:

n	3	4	5	6	7	8
$\Phi_n(x)$	$1+x+x^2$	$1+x^2$	$1+x+x^2+x^3+x^4$	$1-x+x^2$	$1+x+\cdots+x^6$	$1+x^4$

9	$\cdots$
$1+x^3+x^6$	$\cdots$

Therefore $\frac{\Phi_n(x)}{x^{\varphi(n)/2}}$ is a sum of terms of the form $x^k+\frac{1}{x^k}$ with integer coefficients. Since $L_k=\alpha^k+\frac{1}{\alpha^k}$ we can write

$$\frac{\Phi_n(\alpha(x))}{(\alpha(x))^{\varphi(n)/2}}=\sum_{k=0}^{\varphi(n)/2}c_kL_k(x)\in\mathbb{Z}[x],$$

with integer coefficients c_k and note that this polynomial is of degree $\varphi(n)/2$. Since $\alpha(2\cos\theta)=\cos\theta+\sqrt{\cos^2\theta-1}=\cos\theta+\sqrt{-\sin^2\theta}=e^{\sqrt{-1}\theta}$ we get

$$\Phi_n\left(\alpha\left(2\cos\frac{2\pi}{n}\right)\right)=\Phi_n\left(e^{2\sqrt{-1}\pi/n}\right)=0.$$

Therefore $\frac{\Phi_n(\alpha(x))}{(\alpha(x))^{\varphi(n)/2}}$ is a monic polynomial of degree $\varphi(n)/2$ with integer coefficients and root $2\cos\frac{2\pi}{n}$. Consequently it must coincide with the minimal polynomial $\psi_n(x)$. □

For example, $\frac{\Phi_9(x)}{x^3}=1+x^3+\frac{1}{x^3}$ gives $\frac{\Phi_9(\alpha(x))}{(\alpha(x))^3}=1+(\alpha(x))^3+\frac{1}{(\alpha(x))^3}=1+L_3(x)=1-3x+x^3=\psi_9(x)$.

Corollary 3.3. *For $n\geq 1$ we have $\psi_{2^{n+2}}(x)=L_{2^n}(x)$.*

Proof. Since $\Phi_{2^{n+2}}(x)=x^{2^{n+1}}+1$ we get $\frac{\Phi_{2^{n+2}}(x)}{x^{2^n}}=x^{2^n}+\frac{1}{x^{2^n}}$ and $(\alpha(x))^{2^n}+\frac{1}{(\alpha(x))^{2^n}}=L_{2^n}(x)$. □

Proposition 3.4. [1, Proposition 2.2] *For $n\geq 1$ we have*

$$L_n(x)-2=\psi_1(x)\left(\psi_2(x)\right)^{e_n}\prod_{k|n,k\neq 1,2}\psi_k^2(x),$$

with $e_n=\left(1+(-1)^n\right)/2$.

Proof. This follows from the multiplicities of the roots of $L_n(x)-2$ and the fact that for $\gcd(j,n)=d$ the minimal polynomial of $2\cos(2\pi j/n)$ is $\psi_{n/d}(x)$. □

4. Some properties of $Z_n(x)$

The formula $Z_n(x)=2-L_n(2-x)$ shows that $Z_n(x)$ is closely related to the Lucas polynomials $L_n(x)$. The first terms are:

n	1	2	3	4
$Z_n(x)$	x	$4x-x^2$	$9x-6x^2+x^3$	$16x-20x^2+8x^3-x^4$

5	$\cdots$
$25x-50x^2+35x^3-10x^4+x^5$	$\cdots$

Note that $-Z_n(-x)$ is a monic polynomial of degree n with positive coefficients. Since $L_n(2) = 2$ we have $Z_n(0) = 0$.

Let $c(n,k) = [x^k]\, L_n(2+x)$. From $L_n(2+x) = (2+x)L_{n-1}(2+x) - L_{n-2}(2+x)$ we get $c(n,k) = 2c(n-1,k) + c(n-1,k-1) - c(n-2,k)$. Since $c(0,1) = 0$ and $c(1,1) = 1$ we get with induction that $c(n,1) = n^2$. In the same way $c(n,k) = \binom{n+k-1}{n-k}\frac{n}{k}$. Therefore, we get

$$Z_n(x) = \sum_{k=1}^{n} (-1)^{k-1} \frac{n}{k} \binom{n+k-1}{n-k} x^k. \tag{19}$$

With $\lambda(x) = \alpha(2-x) = \frac{2-x+\sqrt{x^2-4x}}{2}$ we get

$$Z_n(x) = -\frac{(\lambda(x)^n - 1)^2}{\lambda(x)^n}, \tag{20}$$

because $Z_n(x) = 2 - L_n(2-x) = 2 - \left((\lambda(x))^n + \frac{1}{(\lambda(x))^n}\right) = -\frac{((\lambda(x))^n-1)^2}{(\lambda(x))^n}$.

The polynomials $Z_n(x)$ satisfy the recursion

$$Z_{n+3}(x) + (x-3)Z_{n+2}(x) + (3-x)Z_{n+1}(x) - Z_n(x) = 0, \tag{21}$$

because they are a linear combination of $\lambda(x))^n$, $\frac{1}{(\lambda(x))^n}$, 1^n and

$$(z - \lambda(x))(z - \frac{1}{\lambda(x)})(z-1) = z^3 + (x-3)z^2 + (3-x)z - 1. \tag{22}$$

Another connection with the Lucas polynomials is the following.

Proposition 4.1.

$$Z_{2n+1}\left(x^2\right) = (L_{2n+1}(x)^2,$$
$$Z_{2n}\left(x^2\right) = 4 - (L_{2n}(x))^2.$$

Proof. For odd m we have $Z_m\left(x^2\right) = 2 - L_m\left(2 - x^2\right) = 2 - L_m(-L_2(x)) = 2 + L_{2m}(x) = (L_m(x))^2$ because of Eq. (12). The second identity follows from

$$Z_{2n}\left(x^2\right) = 2 - L_{2n}\left(2 - x^2\right) = 2 - L_{2n}\left(2 - \left(z + \frac{1}{z}\right)^2\right) = 2 - L_{2n}\left(z^2 + \frac{1}{z^2}\right)$$
$$= 2 - \left(z^{4n} + \frac{1}{z^{4n}}\right) = 4 - \left(z^{2n} + \frac{1}{z^{2n}}\right)^2 = 4 - (L_{2n}(x))^2.$$

□

Proposition 4.2. *For arbitrary $u \neq 0$ we have*

$$Z_n\left(-\left(u - \frac{1}{u}\right)^2\right) = -\left(u^n - \frac{1}{u^n}\right)^2, \tag{23}$$

$$Z_{mn}(x) = Z_m(Z_n(x)). \tag{24}$$

Proof. Note that $Z_n\left(-\left(u-\frac{1}{u}\right)^2\right) = 2 - L_n\left(2+\left(u-\frac{1}{u}\right)^2\right) = 2 - L_n\left(u^2+\frac{1}{u^2}\right) = 2 - \left(u^{2n}+\frac{1}{u^{2n}}\right) = -\left(u^n-\frac{1}{u^n}\right)^2$. On the other hand, $Z_{mn}(x) = 2-L_{mn}(2-x) = 2-L_m(L_n(2-x)) = 2-L_m(2-Z_n(x)) = Z_m(Z_n(x))$. □

5. Proof of Theorem 1.1

Let $z_n(x) = (-1)^{n-1}Z_n(x) = (-1)^{n-1}(2-L_n(2-x))$ be the monic versions of the $Z_n(x)$. The first terms are:

n	1	2	3	4	5	$\cdots$
$z_n(x)$	x	$-4x+x^2$	$9x-6x^2+x^3$	$-16x+20x^2-8x^3+x^4$	$25x-50x^2+35x^3-10x^4+x^5$	$\cdots$

Proposition 3.4 gives

$$z_n(x) = \psi_1(2-x)(\psi_2(2-x))^{e_n}\prod_{k|n,\ k\neq 1,2}(\psi_k(2-x))^2. \tag{25}$$

Since all the $z_n(x)$ are monic we can replace all the polynomials of the right hand side by their monic versions. This gives

$$z_n(x) = \phi_1(x)(\phi_2(x))^{e_n}\prod_{k|n,\ k\neq 1,2}(\phi_k(x))^2, \tag{26}$$

with $\phi_1(x) = x,\ \phi_2(x) = x-4,\ \phi_n(x) = (-1)^{\varphi(n)/2}\psi_n(2-x)$.

Observing that $2-2\cos\left(\frac{2\pi k}{n}\right) = 4\sin^2\left(\frac{k\pi}{n}\right)$ we get from Eq. (17) by changing $x\mapsto 2-x$

$$\phi_n(x) = \prod_{\gcd(k,n)=1,\ 0<k<n/2}\left(x-4\sin^2\left(\frac{k\pi}{n}\right)\right) = (-1)^{\varphi(n)/2}\frac{\Phi_n(\lambda(x))}{(\lambda(x))^{\varphi(n)/2}}, \tag{27}$$

for $n\geq 3$ with $\lambda(x) = \alpha(2-x) = \frac{2-x+\sqrt{x^2-4x}}{2}$. Thus $\phi_n(x)$ is the minimal polynomial of $4\sin^2\left(\frac{\pi}{n}\right)$. The first terms are:

n	1	2	3	4	5	6	7	8	9	$\cdots$
$\phi_n(x)$	x	$-4+x$	$-3+x$	$-2+x$	$5-5x+x^2$	$-1+x$	$-7+14x-7x^2+x^3$	$2-4x+x^2$	$-3+9x-6x^2+x^3$	$\cdots$

(see [6, entry A232633]).

Let

$$\Psi_n(x) := (\phi_n(x))^2 \text{ for } n\geq 3, \tag{28}$$
$$\Psi_1(x) := x,\ \Psi_2(x) := 4-x = -\phi_2(x),$$

such that $\phi_1(x)(\phi_2(x))^{e_n} = (-1)^{n-1}\Psi_1(x)(\Psi_2(x))^{e_n}$. Then $\Psi_n(x)\in\mathbb{Z}[x]$ with $\deg\Psi_n = \varphi(n)$.

With this the proof of Theorem 1.1 is a consequence of Eq. (26).

6. How to calculate the $\phi_k(x)$

6.1. *Odd index*

For odd m we know from Proposition 4.1 that $Z_m(x^2) = (L_m(x))^2$. This implies

$$L_m(x) = x \prod_{d|m,\ d>1} \phi_d\left(x^2\right). \tag{29}$$

The Möbius inversion formula then gives for odd $m > 1$

$$\phi_m\left(x^2\right) = \prod_{d|m} \left(L_{\frac{m}{d}}(x)\right)^{\mu(d)}. \tag{30}$$

6.2. *The case indexed by powers of 2*

Observing that $Z_{2^k}(x) = \prod_{j=0}^{k} \Psi_{2^j}(x)$ and $\Psi_{2^k}(x) = \phi_{2^k}^2(x)$ for $k \geq 2$ we get

$$\Psi_{2^k}(x) = \frac{Z_{2^k}(x)}{Z_{2^{k-1}}(x)} = \frac{1}{\lambda^{2^{k-1}}(x)} \left(\frac{\lambda^{2^k}(x) - 1}{\lambda^{2^{k-1}}(x) - 1}\right)^2 = \frac{\left(\lambda^{2^{k-1}}(x) + 1\right)^2}{\lambda^{2^{k-1}}(x)}$$
$$= \left(\lambda^{2^{k-2}}(x) + \frac{1}{\lambda^{2^{k-2}}(x)}\right)^2,$$

which gives

$$\phi_{2^k}(x) = L_{2^{k-2}}\left(\lambda(x) + \frac{1}{\lambda(x)}\right) = L_{2^{k-2}}(2 - x), \tag{31}$$

for $k \geq 3$. From $L_{2^{n+1}}(x) = L_2\left(L_{2^n}(x)\right) = L_{2^n}^2(x) - 2$, we get the recursion

$$\phi_{2^{k+1}}(x) = \phi_{2^k}^2(x) - 2. \tag{32}$$

It should also be noted that

$$\phi_{2^{n+1}}\left(x^2\right) = L_{2^n}(x), \tag{33}$$

for $n \geq 1$. This follows from

$$\phi_{2^{n+1}}\left(x^2\right) = L_{2^{n-1}}\left(2 - x^2\right) = L_{2^{n-1}}\left(x^2 - 2\right) = L_{2^{n-1}}\left(L_2(x)\right) = L_{2^n}(x).$$

6.3. *General case*

Theorem 6.1. *For odd $m \geq 3$ and $k \geq 2$*

$$\phi_{2m}(x) = (-1)^{\varphi(m)/2}\phi_m(4 - x) = (-1)^{\varphi(m)/2}(\phi_m \circ \Psi_2)(x), \tag{34}$$
$$\phi_{2^k m}(x) = \phi_m\left(\phi_{2^k}^2(x)\right) = (\phi_m \circ \Psi_{2^k})(x). \tag{35}$$

The main idea of the following proof is due to Tri Nguyen [4].

Proof. To prove (34), observe that $\phi_{2m}(x)$ and $(\phi_m \circ \Psi_2)(x)$ have the same degree $\frac{\varphi(m)}{2}$ and that $\phi_{2m}(x)$ is irreducible with root $\alpha = 4\sin^2\left(\frac{\pi}{2m}\right)$. If we show that α is also a root of $(\phi_m \circ \Psi_2)(x) = \phi_m(4-x)$, then $\phi_m(4-x) = (-1)^{\frac{\varphi(m)}{2}}\phi_{2m}(x)$. This follows from

$$\begin{aligned}\phi_m(4-\alpha) &= \phi_m\left(4-4\sin^2\left(\frac{\pi}{2m}\right)\right) = \phi_m\left(4\cos^2\left(\frac{\pi}{2m}\right)\right)\\ &= \phi_m\left(4\sin^2\left(\frac{\pi}{2}-\frac{\pi}{2m}\right)\right) = \phi_m\left(4\sin^2\left(\frac{(m-1)\pi}{2m}\right)\right) = 0.\end{aligned}$$

For the last step, observe that $m-1 = 2k$ is even and $\gcd(k,n) = 1$. Therefore,

$$\phi_m\left(4\sin^2\left(\frac{(m-1)\pi}{2m}\right)\right) = \phi_m\left(4\sin^2\left(\frac{k\pi}{m}\right)\right) = 0.$$

For the proof of (35), it is sufficient to show that the root $\gamma_k = 4\sin^2\left(\frac{\pi}{2^k m}\right)$ of $\phi_{2^k m}$ is also a root of $\phi_m\left(\phi_{2^k}^2(x)\right)$ as $\phi_{2^k m}(x)$ is irreducible and

$$\deg \phi_m\left(\phi_{2^k}^2(x)\right) = \frac{\varphi(m)}{2}2^{k-1} = \frac{\varphi\left(2^k m\right)}{2} = \deg\left(\phi_{2^k m}(x)\right).$$

Using the formula $L_n(2\cos\theta) = 2\cos(n\theta)$, we get

$$\begin{aligned}\phi_{2^k}(\gamma_k) = L_{2^{k-2}}(2-\gamma_k) &= L_{2^{k-2}}\left(2-4\sin^2\left(\frac{\pi}{2^k m}\right)\right)\\ &= L_{2^{k-2}}\left(2\cos\left(\frac{\pi}{2^{k-1}m}\right)\right) = 2\cos\left(\frac{\pi}{2m}\right).\end{aligned}$$

By (34)

$$\begin{aligned}\phi_m \circ \phi_{2^k}^2(\gamma_k) &= \phi_m\left(4\cos^2\left(\frac{\pi}{2m}\right)\right)\\ &= \phi_m\left(4-4\sin^2\left(\frac{\pi}{2m}\right)\right) = (-1)^{\frac{\varphi(m)}{2}}\phi_{2m}\left(4\sin^2\left(\frac{\pi}{2m}\right)\right) = 0.\end{aligned}$$

□

7. Final remarks

Let us finally show some connection with the Fibonacci numbers $(F_n)_{n\geq 0} = (0,1,1,2,3,5,8,13,21,\ldots)$. They satisfy $F_n = F_{n-1}+F_{n-2}$ for $n \geq 2$ with $F_0 = 0$, $F_1 = 1$ and Binet's Formula gives

$$F_n = \frac{\rho^n - \sigma^n}{\rho - \sigma}, \tag{36}$$

with $\rho = \frac{1+\sqrt{5}}{2}$ and $\sigma = \frac{-1}{\rho} = \frac{1-\sqrt{5}}{2}$. It is known (see [5]) that the Fibonacci numbers form what is called a divisibility sequence.

Since $\lambda(5) = \frac{2-5+\sqrt{25-20}}{2} = \frac{-3+\sqrt{5}}{2} = -\left(\frac{1-\sqrt{5}}{2}\right)^2 = -\sigma^2$ we get $(\lambda(5))^n + \frac{1}{(\lambda(5))^n} = (-1)^n\left(\sigma^{2n}+\rho^{2n}\right) = (-1)^n\left(5\left(\frac{\rho^n-\sigma^n}{\sqrt{5}}\right)+2(-1)^n\right) = 5(-1)^n F_n^2 + 2$, which implies

$$Z_n(5) = (-1)^{n-1}5F_n^2. \tag{37}$$

Theorem 1.1 gives then the factorization of the Fibonacci numbers into primitive parts (cf. [6, entry A061446] and [5])

$$F_n = \prod_{d|n} p_d,$$

with $p_1 = 1$ and $p_n = |\phi_n(5)|$ for $n \geq 2$. We get:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	
$\phi_n(5)$	5	1	-2	-3	5	-4	-13	7	-17	11	-89	6	233	-29	61	47	$\cdots$

For example

n	1	2	3	4	5	6	7	8	9
$F_n = \prod_{d\mid n} p_d$	1	$1 \cdot 1$	$1 \cdot 2$	$1 \cdot 1 \cdot 3$	$1 \cdot 5$	$1 \cdot 1 \cdot 2 \cdot 4$	$1 \cdot 13$	$1 \cdot 1 \cdot 3 \cdot 7$	$1 \cdot 2 \cdot 17$

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