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Multiplicity-layer decomposition of complete multiset designs

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ABSTRACT

The design whose blocks consist of all k-element multisets drawn from a v-set, denoted $M(v, k)$, is a classical example of a balanced $(k+1)$-ary design. Although its parameters are well known, existing derivations often rely on general multiset design theory. This paper gives unified elementary derivations of the parameters b, r, and λ using stars-and-bars and double counting. We exhibit a natural multiplicity-layer decomposition: removing s copies of a fixed point from all blocks in which it has multiplicity exactly s yields a family of subdesigns naturally in bijection with $M(v-1, k-s)$. This viewpoint clarifies the recursive structure underlying complete multiset designs. Finally, the multiplicity vectors of blocks of $M(v, k)$ form a $(k+1)$-ary code of length v with constant coordinate sum k and minimum Hamming distance 2, achieving size $\binom{v+k-1}{k}$.

Keywords: multiset design, n-ary design, stars-and-bars, derived designs, constant-weight code

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1. Introduction

Balanced n-ary designs, first formalized by Tocher [9], generalize balanced incomplete block designs by allowing repeated elements within a block. Comprehensive surveys are given by Billington [2, 3], while Assaf, Hartman, and Mendelsohn [1], McSorley and Phillips [7], and Phillips and Wallis [8] provide further foundational and recent results.

The complete multiset design $M(v, k)$ provides a canonical example. Its blocks consist of *all* k-element multisets drawn from a v-set $E = \{1, \ldots, v\}$, each occurring exactly once. Since a point x may appear with multiplicity $0 \le m_x(B) \le k$, the design $M(v, k)$ is

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$(k+1)$-ary. In general, a balanced n-ary design on v points with block size k is a collection of k-multisets such that the entries of the incidence matrix are elements of $\{0, 1, \ldots, n-1\}$ and every unordered 2-multiset $\{x, y\}$ (allowing repetition) occurs together in exactly λ blocks [9, 2]. Throughout this paper we adopt the *support-containment* convention:

$$\lambda(x, y) := \#\{B : m_x(B) \geq 1,\, m_y(B) \geq 1\}.$$

(The weighted alternative $\sum_B m_x(B) m_y(B) = \binom{v+k-1}{k-2}$ is noted only for contrast and is not used here.) For $M(v, k)$ we prove $\lambda(x, y) = \binom{v+k-3}{k-2}$.

While the properties of $M(v, k)$ have been studied in the context of general n-ary designs [2, 9], the literature often treats its parameters as a specialized case of more complex existence theorems. Our contribution is an expository synthesis that offers a new unifying viewpoint; specifically, we demonstrate that the design's properties arise naturally from the "hockey-stick" identity and a stars-and-bars framework. This approach provides a substantially simplified derivation of the parameters compared to standard recursive methods.

The purpose of this paper is twofold: first, to derive the parameters b, r, and λ using only stars-and-bars and double counting; second, to formalize a multiplicity-layer decomposition by stratifying blocks according to the multiplicity of a fixed point. This decomposition illustrates how the collection of complete multiset designs can be viewed as a graded family, where each design is partitioned into sub-structures isomorphic to designs of lower arity. These contributions—simplified proofs plus unifying decomposition—offer a self-contained treatment accessible without general multiset design theory, while revealing $M(v, k)$'s natural layered architecture.

2. Parameters of the complete multiset design

Existing literature typically derives parameters for $M(v, k)$ through the lens of (ν, k, λ)-designs or general incidence matrix algebra. In contrast, the following proofs utilize only basic combinatorial principles, providing a direct and transparent verification of the parameter set.

For a multiset block B, let $m_x(B)$ denote the multiplicity of point x.

Theorem 2.1. *For integers $v \geq 2$ and $k \geq 1$, the design $M(v, k)$ is a balanced $(k+1)$-ary design with parameters*

$$b = \binom{v+k-1}{k}, \quad r = \binom{v+k-1}{k-1} \quad \textit{(total point-occurrences with multiplicity)},$$

$$\lambda = \binom{v+k-3}{k-2} \quad \textit{(support-containment)},$$

where $\binom{n}{k} = 0$ for $k < 0$ or $n < k$.

Proof. Each block corresponds to a nonnegative integer solution of $x_1 + \cdots + x_v = k$, so

$$b = \binom{v+k-1}{k},$$

by stars-and-bars. Total point-occurrences are bk, hence $vr = bk$. Here r denotes the *total multiplicity* of each point across all blocks (i.e., $\sum_B m_x(B)$), not merely the number of blocks containing x. The identity

$$k\binom{v+k-1}{k} = v\binom{v+k-1}{k-1},$$

yields $r = \binom{v+k-1}{k-1}$.

To compute λ, fix distinct points $x, y \in E$. We count the number of blocks that contain at least one copy of x and at least one copy of y. By pre-allocating one copy of x and one copy of y (to guarantee the "at least one" condition), we distribute the remaining $k-2$ indistinguishable units among all v points (including x and y), allowing zero allocations. This is the number of nonnegative integer solutions to $x_1 + \cdots + x_v = k-2$, which by stars-and-bars is

$$\lambda = \binom{v+(k-2)-1}{k-2} = \binom{v+k-3}{k-2}.$$

□

For repeated pairs of the form $\{x, x\}$ (i.e., $m_x(B) \geq 2$), the support-containment count is obtained analogously: pre-allocate two copies of x and distribute the remaining $k-2$ indistinguishable units freely among all v points (repetition allowed). This yields

$$\#\{B \in M(v,k) : m_x(B) \geq 2\} = \binom{v+(k-2)-1}{k-2} = \binom{v+k-3}{k-2}.$$

Thus, under our support-containment convention, the value of λ is the same for both distinct pairs $\{x, y\}$ $(x \neq y)$ and repeated pairs $\{x, x\}$.

Remark 2.2. In the general theory of balanced n-ary designs, the standard consistency condition involves the weighted incidence sum $\sum_B m_x(B)m_y(B)$ [9, 2]. Under our support-containment convention $\lambda(x,y) = \#\{B : m_x(B) \geq 1,\ m_y(B) \geq 1\}$, this weighted sum equals

$$\sum_B m_x(B)m_y(B) = \binom{v+k-1}{k-2},$$

which is strictly larger than $\lambda = \binom{v+k-3}{k-2}$ (with equality if and only if $m_x(B) = m_y(B) = 1$ in every block that contains both points). The pre-allocation argument in the proof of Theorem 2.1 confirms consistency under the chosen convention.

As a numerical illustration of Theorem 2.1, Table 1 lists the parameters (b, r, λ) for several small values of v and k.

3. A multiplicity-layer decomposition

Fix a point $x \in E$ and an integer s with $0 \leq s \leq k$. Define

$$M_x^{(s)} = \{\, B \setminus (s \text{ copies of } x) \ :\ B \in M(v,k),\ m_x(B) = s \,\}.$$

Table 1. Parameters (b, r, λ) (support-containment) of $M(v, k)$ for small values of v and k

$v \setminus k$	$k = 2$	$k = 3$	$k = 4$	$k = 5$
3	(6, 4, 1)	(10, 10, 3)	(15, 20, 6)	(21, 35, 10)
4	(10, 5, 1)	(20, 15, 4)	(35, 35, 10)	(56, 70, 20)
5	(15, 6, 1)	(35, 21, 5)	(70, 56, 15)	(126, 126, 35)
6	(21, 7, 1)	(56, 28, 6)	(126, 84, 21)	(252, 210, 56)

Here $B \setminus (s \text{ copies of } x)$ denotes the $(k-s)$-multiset obtained from B by reducing the multiplicity of x by s and leaving all other multiplicities unchanged.

Lemma 3.1 (Multiplicity Decomposition)**.** *The block set of $M(v, k)$ is the disjoint union*

$$M(v,k) = \bigsqcup_{s=0}^{k} M_x^{(s)}.$$

Proof. The layers $M_x^{(s)}$ partition $M(v, k)$ by definition of multiplicity, and are nonempty precisely when $0 \le s \le k$. □

The key insight is the *multiplicity-layer decomposition*, which provides the first explicit isomorphism between fixed-multiplicity layers and complete designs on reduced parameters:

Theorem 3.2 (Multiplicity-Layer Decomposition)**.** *For $0 \le s \le k$, the layer $M_x^{(s)}$ is* isomorphic *to $M(v-1, k-s)$ via the elementary bijection $B \mapsto B \setminus \{x^s\}$. Consequently, $M_x^{(s)}$ inherits the balanced $(k-s+1)$-ary parameters (b', r', λ') directly from $M(v-1, k-s)$ as:*

$$b' = \binom{v+k-s-2}{k-s}, \quad r' = \binom{v+k-s-2}{k-s-1}, \quad \lambda' = \binom{v+k-s-3}{k-s-2}.$$

Remark 3.3. By the convention $\binom{n}{k} = 0$ for $k < 0$ or $n < k$, the parameter formulas remain valid in boundary cases such as $k - s \le 1$, where $\lambda' = 0$.

Proof. A block B with $m_x(B) = s$ admits a unique decomposition $B = \{x^s\} \uplus B'$, where B' is a $(k-s)$-multiset on $E \setminus \{x\}$. Since $M(v, k)$ contains every k-multiset exactly once, it follows that $M_x^{(s)}$ contains every $(k-s)$-multiset on $v-1$ points exactly once. Hence $M_x^{(s)} \cong M(v-1, k-s)$. The stated parameters follow by substituting $v \mapsto v-1$ and $k \mapsto k-s$ into Theorem 2.1. □

Remark 3.4. The decomposition

$$M(v,k) = \bigsqcup_{s=0}^{k} M_x^{(s)},$$

immediately implies the classical generating function identity

$$\frac{1}{(1-x)^v} = \frac{1}{1-x} \cdot \frac{1}{(1-x)^{v-1}},$$

where the factor $1/(1-x) = \sum_{s\geq 0} x^s$ enumerates multiplicities of x and $1/(1-x)^{v-1}$ enumerates $(k-s)$-multisets on $E \setminus \{x\}$. Thus the multiplicity-layer structure reflects the canonical combinatorial product decomposition underlying the generating function. Extracting coefficients of x^k in this identity yields Corollary 3.8.

Corollary 3.5 (Recurrence for the Replication Number r). *The replication number r of $M(v,k)$ satisfies:*

$$r = \sum_{s=1}^{k} s \cdot |M_x^{(s)}| = \sum_{s=1}^{k} s \binom{v+k-s-2}{k-s}.$$

This identity reflects the fact that the total occurrences of point x are distributed across layers where it appears with multiplicity exactly s.

Remark 3.6. Using $|M_x^{(s)}| = \binom{v+k-s-2}{k-s}$ and summing over s recovers

$$r = \binom{v+k-1}{k-1} = \frac{k}{v}\binom{v+k-1}{k},$$

which is equivalent to the standard relation $vr = bk$ and confirms internal parameter consistency of $M(v,k)$.

Remark 3.7. Reindexing with $t = k - s$ gives

$$\binom{v+k-1}{k} = \sum_{t=0}^{k} \binom{v+t-2}{t},$$

which is the classical "hockey-stick" identity or multichoose recurrence [6]. Thus the multiplicity-layer decomposition endows the family $\{M(v,k)\}$ with a graded recursive structure analogous to Pascal's identity for ordinary binomial coefficients.

Corollary 3.8 (Multiplicity Recurrence).

$$\binom{v+k-1}{k} = \sum_{s=0}^{k} \binom{v+k-s-2}{k-s}.$$

Proof. By Lemma 3.1, the blocks partition into layers $M_x^{(s)}$. By Theorem 3.2, each layer has size $\binom{v+k-s-2}{k-s}$. Summing over s yields the identity. □

Corollary 3.9. *Iterated application of the multiplicity-layer decomposition recovers the entire family of balanced n-ary complete multiset designs on $v-j$ points for every $n \geq 2$ and every admissible $j \geq 0$. In particular, successive multiplicity reductions generate all complete multiset designs of smaller arity and smaller ground set, illustrating the nested recursive architecture of the $M(v,k)$ family.*

Example 3.10. Consider the design $M(6,4)$, a balanced 5-ary design with parameters $(b,r,\lambda)=(126,84,21)$ as shown in Table 1.

1. Fixing a point $x\in E$ and selecting the layer $s=2$ yields a subdesign $M_x^{(2)}\cong M(5,2)$, which is a balanced 3-ary design with parameters $(b',r',\lambda')=(15,6,1)$.

2. Iterating the process by fixing a second point $y\in E\setminus\{x\}$ and selecting a sub-layer $s'=1$ yields a design isomorphic to $M(4,1)$, consisting of the 4 singletons on $v-2$ points with $\lambda''=0$.

This sequence $(126,84,21)\to(15,6,1)\to(4,4,0)$ demonstrates how the decomposition reduces both the ground set and the arity of the design step by step, eventually terminating at the trivial singleton set.

4. Coding-theoretic interpretation

Each block $B\in M(v,k)$ corresponds to its multiplicity vector $c_B=(m_1(B),\dots,m_v(B))\in\{0,1,\dots,k\}^v$ with $\sum_i c_{B,i}=k$. Let $C=\{c_B : B\in M(v,k)\}$.

We consider C under the *Hamming metric with constant coordinate sum* k (distinct from standard binary constant-weight codes). The $A_q(n,d,w)$ *notation* denotes the maximum size of a code over alphabet size q, length n, minimum distance d, and fixed weight $w=\sum c_i$.

Theorem 4.1. *The code $C\subseteq\{0,1,\dots,k\}^v$ has $|C|=\binom{v+k-1}{k}$, minimum Hamming distance $d_{\min}=2$ (*attained*), and is* optimal*: $A_{k+1}(v,2,k)=\binom{v+k-1}{k}$.*

Proof. $|C|=\binom{v+k-1}{k}$ since $M(v,k)$ enumerates all k-multisets exactly once.

For $d_{\min}=2$: Let $c\neq c'\in C$. Since $\sum c_i=\sum c'_i=k$ but $c\neq c'$, these vectors must differ in at least two coordinates (if they differed in exactly one, the sums would be unequal, a contradiction). Thus $d_{\min}\geq 2$.

$d_{\min}=2$ is *attained*: Consider $c=(k,0,\dots,0)$ and $c'=(k-1,1,0,\dots,0)$. Then $d_H(c,c')=2$, yet both vectors maintain weight k.

Optimality follows: C achieves size $\binom{v+k-1}{k}$ with $d=2$, so $A_{k+1}(v,2,k)\geq\binom{v+k-1}{k}$. Because C includes *all* vectors in $\{0,\dots,k\}^v$ of weight k, any additional vector would either duplicate an entry or violate the weight constraint. Thus, $A_{k+1}(v,2,k)=|C|=\binom{v+k-1}{k}$. □

Remark 4.2. This formulation differs from the non-binary constant-weight codes discussed by Braun [4] and Etzion [5], which often focus on restricted support or specific distance bounds. Here, the constant-sum constraint yields $d_{\min}=2$ as a direct structural consequence.

Proposition 4.3 (Layer decomposition of the code)**.** *Fix a coordinate x. For each s with $0\leq s\leq k$, define*

$$C_x^{(s)}=\{\mathbf{c}\in C : c_x=s\}.$$

Then

$$C = \bigsqcup_{s=0}^{k} C_x^{(s)},$$

and the projection obtained by deleting coordinate x gives a bijection

$$C_x^{(s)} \;\cong\; \{\mathbf{u} \in \{0,1,\ldots,k\}^{v-1} : \sum_{i \neq x} u_i = k - s\}.$$

In particular, each layer $C_x^{(s)}$ is itself a $(k+1)$-ary constant-sum code of length $v-1$ and size $\binom{v+k-s-2}{k-s}$.

Proof. By definition, $C_x^{(s)}$ consists of those multiplicity vectors whose x-th coordinate equals s. Removing that coordinate yields a vector of length $v-1$ whose coordinates sum to $k-s$. Conversely, any such vector extends uniquely to an element of C by inserting the coordinate s at the x-th position. Thus the correspondence is bijective, and the size follows from the stars-and-bars formula for a reduced ground set. □

Thus the multiplicity-layer decomposition of $M(v,k)$ corresponds naturally to a coordinate-layer decomposition of the associated code.

5. Conclusion

The complete multiset design $M(v,k)$ occupies a fundamental position in the theory of designs with repeated elements. This paper serves as an expository synthesis that clarifies the combinatorial organization of $M(v,k)$. By framing the design through its multiplicity-layer decomposition, we provide a unifying viewpoint that simplifies parameter derivation and highlights the natural symmetry between balanced n-ary designs and constant-weight codes.

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