

A lower bound for the exponential Randić index of line graphs of trees

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ABSTRACT

The exponential Randić index of a graph G , denoted by $ER(G)$, is defined as $\sum_{uv \in E(G)} e^{\frac{1}{d(u)d(v)}}$, where $d(u)$ denotes the degree of a vertex u in G . The line graph $L(G)$ of a graph G is a graph in which each vertex represents an edge of G , and two vertices are adjacent in $L(G)$ if and only if their corresponding edges in G are incident to a common vertex. In this paper, we proved that for any tree T of order $n \geq 3$, $ER(L(T)) > \frac{n}{4}e^{\frac{1}{2}}$.

Keywords: exponential Randić index, line graph, tree

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1. Introduction

Let $G = (V, E)$ be a simple graph, where $V = V(G)$ denotes the set of vertices and $E = E(G)$ denotes the set of edges. For a vertex $u \in V(G)$, the degree of u , denoted by $d_G(u)$ or $d(u)$, is the number of edges incident to it. The neighborhood of u , denoted by $N(u)$, is the set of all vertices adjacent to u .

Chemical topological indices are applications of graph theory in chemistry that mathematically quantify key features of molecular structures to predict the physicochemical properties and biological activities of compounds. The Randić index is one of the most classical topological indices in chemical graph theory, developed by Milan Randić in 1975 [18]. For a simple connected graph $G = (V, E)$, its Randić index is defined as

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d(u)d(v)}}.$$

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This parameter is very useful in mathematical chemistry and has been studied extensively in the monograph [13]. For some results, we refer to [14, 20, 21, 22, 9, 4].

In 1998, Bollobás and Erdős generalized the Randić index in [3], known as the General Randić Index, which is defined as

$$R^\alpha(G) = \sum_{uv \in E(G)} (d(u)d(v))^\alpha,$$

where α is real. For detailed results, see [15, 5, 11, 10].

In 2019, Juan Rada proposed an exponential version of the Randić index in [17] that substantially enhanced its discriminative power. The exponential Randić index is formally defined as

$$ER(G) = \sum_{uv \in E(G)} e^{\frac{1}{\sqrt{d(u)d(v)}}}.$$

The generalized Randić index of a graph G is defined as

$$ER^\alpha(G) = \sum_{uv \in E(G)} e^{(d(u)d(v))^\alpha}.$$

Cruz, Monsalve and Rada [8] proved that the exponential Randić index attains its maximum value in the path P_n . Cruz et al. [7] established that over the set \mathcal{T}_n of trees with n vertices, the minimal value of the exponential of the generalized Randić index is attained in the path P_n when $\alpha > 0$, and in the star S_n when $\alpha < 0$. Lin and Zhu [16] gave the sharp lower and upper bounds on the exponential Randić index of unicyclic graphs. Bera and Das [2] characterize the minimal molecular trees in relation to the exponential Randić index.

The line graph $L(G)$ of a graph G is a graph whose vertex set corresponds to $E(G)$, where two vertices are adjacent if and only if their corresponding edges in G share a common vertex. The line graph transformation is a pivotal tool in chemical graph theory, with the ability to encode edge adjacency relationships into vertices. This transformation facilitates the characterization of bond-specific topological indices, offering unique insights into conjugated systems and reactive sites that are obscured in conventional vertex-based representations.

Ranjini et al. [19] used the concept of subdivision to give Zagreb indices for line graphs of tadpole, wheel, and step graphs. In order to enhance QSAR/QSPR modeling, Aslam et al. [1] developed topological indices for subdivided graph line graphs, in particular for complete bipartite graphs. The relation between Wiener index of a graph and that of its line graph was investigated in [12, 24, 6]. Wang and Wu [23] gave a lower bounds for the harmonic index for line graph of tree and unicyclic graph.

Recently, Zhang and Wu [25] proved that $R(L(T)) > \frac{n}{4}$ for any tree T of order $n \geq 3$. In this paper, we extend this line of inquiry and prove that for any tree T of order $n \geq 3$, $ER(L(T)) > \frac{n}{4}e^{\frac{1}{2}}$.

2. Preliminary results

Lemma 2.1. *For any positive integers $x, y \in \mathbb{Z}^+$ define*

$$f(x, y) = xe^{\frac{1}{x}} - xe^{\frac{1}{\sqrt{(x+y)x}}} + ye^{\frac{1}{y}} - ye^{\frac{1}{\sqrt{(x+y)y}}}.$$

Then the following inequality holds:

$$f(x, y) < e.$$

Proof. For any positive integers $t, a \in \mathbb{Z}^+$, we define the function $g(t, a) = t(e^{\frac{1}{t}} - e^{\frac{1}{\sqrt{t(t+a)}}})$. Then $f(x, y) = g(x, y) + g(y, x)$.

By the Fundamental Theorem of Calculus, $g(t, a)$ can be expressed as the following integral

$$g(t, a) = t \int_{\frac{1}{\sqrt{t(t+a)}}}^{\frac{1}{t}} e^s ds.$$

Since e^s is increasing on the interval, we have

$$g(t, a) \leq te^{\frac{1}{t}} \left(\frac{1}{t} - \frac{1}{\sqrt{t(t+a)}} \right) = e^{\frac{1}{t}} \left(1 - \sqrt{\frac{t}{t+a}} \right).$$

Applying this bound to $f(x, y)$, and noting that $e^{\frac{1}{x}} \leq e$ and $e^{\frac{1}{y}} \leq e$ for all $x, y \geq 1$, we obtain

$$\begin{aligned} f(x, y) &< e^{\frac{1}{x}} \left(1 - \sqrt{\frac{x}{x+y}} \right) + e^{\frac{1}{y}} \left(1 - \sqrt{\frac{y}{x+y}} \right) \\ &< e \left(1 - \sqrt{\frac{x}{x+y}} \right) + e \left(1 - \sqrt{\frac{y}{x+y}} \right) \\ &= e \left(2 - \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x+y}} \right). \end{aligned}$$

To complete the proof, observe that for any $x, y > 0$, the triangle inequality or simple squaring yields $\sqrt{x} + \sqrt{y} > \sqrt{x+y}$, which implies $\frac{\sqrt{x} + \sqrt{y}}{\sqrt{x+y}} > 1$. Consequently,

$$2 - \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x+y}} < 1,$$

which leads to the desired result

$$f(x, y) < e \cdot 1 = e.$$

The proof is completed. □

Lemma 2.2. *The function $f(x) = e^{\frac{1}{\sqrt{px}}} - e^{\frac{1}{\sqrt{(p+a)x}}}$ is monotonically decreasing for $x > 0$, where $p > 0, q > 0$.*

Proof. We define the function:

$$g(t) = \frac{1}{\sqrt{t}} e^{\frac{1}{\sqrt{tx}}}.$$

Taking the derivative of $g(t)$ with respect to t , we obtain:

$$\frac{dg(t)}{dt} = -\frac{1}{2t\sqrt{t}} e^{\frac{1}{\sqrt{xt}}} \left(1 + \frac{1}{\sqrt{xt}}\right) < 0,$$

which implies that $g(t)$ is strictly decreasing for $t > 0$. Since $p+q > p > 0$, it follows that

$$\frac{1}{\sqrt{p+q}} e^{\frac{1}{\sqrt{(p+q)x}}} < \frac{1}{\sqrt{p}} e^{\frac{1}{\sqrt{px}}},$$

which further implies that for any $x > 0$,

$$\frac{df(x)}{dx} = \frac{1}{2x\sqrt{x}} \left(\frac{1}{\sqrt{p+q}} e^{\frac{1}{\sqrt{(p+q)x}}} - \frac{1}{\sqrt{p}} e^{\frac{1}{\sqrt{px}}} \right) < 0.$$

Therefore, the function $f(x) = e^{\frac{1}{\sqrt{px}}} - e^{\frac{1}{\sqrt{(p+q)x}}}$ is monotonically decreasing for $x > 0$, where $p > 0$ and $q > 0$. The proof is completed. \square

Lemma 2.3. *Let G_1 and G_2 be two non-trivial connected disjoint graphs. If G is a graph formed by identifying $u \in V(G_1)$ and $v \in V(G_2)$, then*

$$ER(G) = ER(G_1) + ER(G_2) - c, \quad (1)$$

where

$$c = \sum_{u_i \in N_{G_1}(u)} e^{\frac{1}{\sqrt{d_{G_1}(u) \cdot d_{G_1}(u_i)}}} + \sum_{v_j \in N_{G_2}(v)} e^{\frac{1}{\sqrt{d_{G_2}(v) \cdot d_{G_2}(v_j)}}} - \left(\sum_{u_i \in N_{G_1}(u)} e^{\frac{1}{\sqrt{(d_{G_1}(u) + d_{G_2}(v)) \cdot d_{G_1}(u_i)}}} + \sum_{v_j \in N_{G_2}(v)} e^{\frac{1}{\sqrt{(d_{G_1}(u) + d_{G_2}(v)) \cdot d_{G_2}(v_j)}}} \right),$$

$u_i \in N_{G_1}(u)$ and $v_j \in N_{G_2}(v)$.

Furthermore, if $G_1 = L(T_1)$ and $G_2 = L(T_2)$ are line graphs of trees, and the identified vertices u and v correspond to pendant edges in T_1 and T_2 , respectively, then $c < e$.

Proof. For convenience, let $d_{G_1}(u) = k$, $d_{G_2}(v) = l$ and $N_{G_1}(u) = \{u_1, u_2, \dots, u_k\}$, $N_{G_2}(v) = \{v_1, v_2, \dots, v_l\}$ with $d_{G_1}(u_i) = k_i$, $d_{G_2}(v_j) = l_j$ for each $i \in \{1, 2, \dots, k\}$, $j \in \{1, 2, \dots, l\}$.

In addition, given that both u and v are non-isolated vertices, it can be deduced that $k \geq 1$ and $l \geq 1$. Furthermore, since $G_1 = L(T_1)$ and $G_2 = L(T_2)$ are line graphs of trees, we choose u and v corresponding to pendant edges in T_1 and T_2 , respectively. Let $e = xy$ be a pendant edge in a tree T_1 , where x is a leaf. Then all edges adjacent to e are incident with y . Hence, in $G_1 = L(T_1)$, the neighborhood $N_{G_1}(u)$ induces a clique.

Similarly, $N_{G_2}(v)$ is a clique. Consequently, for any i , there is $k_i \geq k$ and for any j , there is $l_j \geq l$.

$$\begin{aligned}
 c &= \sum_{i=1}^k e^{\frac{1}{\sqrt{k \cdot k_i}}} + \sum_{j=1}^l e^{\frac{1}{\sqrt{l \cdot l_j}}} - \left(\sum_{i=1}^k e^{\frac{1}{\sqrt{(k+l) \cdot k_i}}} + \sum_{j=1}^l e^{\frac{1}{\sqrt{(k+l) \cdot l_j}}} \right) \\
 &= \sum_{i=1}^k \left(e^{\frac{1}{\sqrt{k \cdot k_i}}} - e^{\frac{1}{\sqrt{(k+l) \cdot k_i}}} \right) + \sum_{j=1}^l \left(e^{\frac{1}{\sqrt{l \cdot l_j}}} - e^{\frac{1}{\sqrt{(k+l) \cdot l_j}}} \right) \\
 &\leq \sum_{i=1}^k \left(e^{\frac{1}{k}} - e^{\frac{1}{\sqrt{(k+l) \cdot k}}} \right) + \sum_{j=1}^l \left(e^{\frac{1}{l}} - e^{\frac{1}{\sqrt{(k+l) \cdot l}}} \right) \\
 &= k \left(e^{\frac{1}{k}} - e^{\frac{1}{\sqrt{(k+l) \cdot k}}} \right) + l \left(e^{\frac{1}{l}} - e^{\frac{1}{\sqrt{(k+l) \cdot l}}} \right) \\
 &= f(k, l).
 \end{aligned}$$

By Lemma 2.1 and Lemma 2.2, it follows that $c \leq f(k, l) < 2$. The proof is completed. \square

Lemma 2.4. *Let G be a simple graph of order n . If $d(v) = 1$, then*

$$ER(G) - ER(G - v) \geq 0.$$

Proof. First, we assume that v is a leaf vertex of graph G , and u is the neighbor of v . For the convenience of subsequent proof, let $d_G(u) = x$, $N_G(u) = \{v, u_1, u_2, \dots, u_{x-1}\}$, and $d_G(u_i) = x_i$ for $i = 1, 2, \dots, x - 1$.

$$\begin{aligned}
 ER(G) - ER(G - v) &= \sum_{i=1}^{x-1} e^{\frac{1}{\sqrt{x x_i}}} + e^{\frac{1}{\sqrt{x}}} - \sum_{i=1}^{x-1} e^{\frac{1}{\sqrt{(x-1) x_i}}} \\
 &= \sum_{i=1}^{x-1} \left(e^{\frac{1}{\sqrt{x x_i}}} - e^{\frac{1}{\sqrt{(x-1) x_i}}} \right) + e^{\frac{1}{\sqrt{x}}} \\
 &= \sum_{i=1}^{x-1} m_i + e^{\frac{1}{\sqrt{x}}}.
 \end{aligned}$$

By the Lagrange Mean Value Theorem, we have:

$$m_i = e^{\xi_i} \left(\frac{1}{\sqrt{x x_i}} - \frac{1}{\sqrt{(x-1) x_i}} \right) = e^{\xi_i} \frac{\sqrt{x-1} - \sqrt{x}}{\sqrt{x_i x (x-1)}},$$

where $\xi_i \in \left(\frac{1}{\sqrt{x x_i}}, \frac{1}{\sqrt{(x-1) x_i}} \right)$. Based on the condition $x_i \geq 1$, we derive $\frac{1}{\sqrt{x_i}} \leq 1$, which implies $e^{\xi_i} < e^{\frac{1}{\sqrt{(x-1) x_i}}} \leq e^{\frac{1}{\sqrt{x_i}}} \leq e$. Consequently, we obtain:

$$ER(G) - ER(G - v) = e^{\frac{1}{\sqrt{x}}} - \frac{\sqrt{x} - \sqrt{x-1}}{\sqrt{x(x-1)}} \sum_{i=1}^{x-1} \frac{1}{\sqrt{x_i}} e^{\xi_i}$$

$$\begin{aligned}
&> e^{\frac{1}{\sqrt{x}}} - \frac{\sqrt{x} - \sqrt{x-1}}{\sqrt{x(x-1)}} \sum_{i=1}^{x-1} \frac{1}{\sqrt{x_i}} e^{\frac{1}{\sqrt{x_i}}} \\
&> e^{\frac{1}{\sqrt{x}}} - \frac{\sqrt{x} - \sqrt{x-1}}{\sqrt{x(x-1)}} (x-1)e \\
&= e^{\frac{1}{\sqrt{x}}} - \left(\sqrt{x-1} - \sqrt{x} + \frac{1}{\sqrt{x}} \right) e.
\end{aligned}$$

Let $h(x) = e^{\frac{1}{\sqrt{x}}} - e \left(\sqrt{x-1} - \sqrt{x} + \frac{1}{\sqrt{x}} \right)$. The proof now reduces to showing that $h(x) > 0$ for all $x \in \mathbb{Z}^+$ with $x > 1$.

The function $h(x)$ can then be reformulated as a function of t , denoted by $H(t)$:

$$H(t) = e^t - e \left(\frac{\sqrt{1-t^2}}{t} - \frac{1}{t} + t \right) = e^t - e \frac{\sqrt{1-t^2} - 1 + t^2}{t}. \quad (2)$$

To prove $H(t) > 0$ for $t \in (0, 1)$, it is equivalent to show that

$$e^{t-1} > \frac{\sqrt{1-t^2} - (1-t^2)}{t}. \quad (3)$$

We apply a scaling argument to the right-hand side of (3). We can obtain

$$\sqrt{1-t^2} < 1 - \frac{t^2}{2}. \quad (4)$$

Consequently, the following upper bound holds for the expression in (3):

$$\frac{\sqrt{1-t^2} - 1 + t^2}{t} < \frac{(1 - \frac{t^2}{2}) - 1 + t^2}{t} = \frac{t}{2}. \quad (5)$$

Therefore, it suffices to demonstrate that $e^{t-1} > \frac{t}{2}$ for $t \in (0, 1)$. Let $\phi(t) = e^{t-1} - \frac{t}{2}$. A straightforward calculation shows that $\phi'(t) = e^{t-1} - \frac{1}{2}$, and the minimum value of $\phi(t)$ on $(0, 1)$ is attained at $t_0 = 1 - \ln 2$. At this point, $\phi(t_0) = \frac{\ln 2}{2} > 0$.

Thus, $H(t) > 0$ is established, which completes the proof. \square

Lemma 2.5. *Let $p \geq 1, q \geq 2$ be integers,*

$$g(p, q) = \frac{2(p+q+1)(p+q) - p - 2q}{4} e^{\frac{1}{2}} - e.$$

Then $g(p, q) > 0$ always holds.

Proof. Let $s = p + q$. Given $p \geq 1$ and $q \geq 2$, it follows that $s \geq 3$. We rewrite the numerator of $g(p, q)$.

$$g(p, q) = \frac{2(p+q+1)(p+q) - p - 2q}{4} e^{\frac{1}{2}} - e = \frac{2s^2 + s - q}{4} e^{\frac{1}{2}} - e,$$

For a fixed s , $g(p, q)$ is strictly decreasing with respect to q . Since $p \geq 1$, we have $q = s - p \leq s - 1$. The lower bound for any fixed s is attained at $q = s - 1$:

$$g(p, q) \geq g(1, s-1) = \frac{2s^2 + s - (s-1)}{4} e^{\frac{1}{2}} - e = \frac{2s^2 + 1}{4} e^{\frac{1}{2}} - e.$$

Define $h(s) = \frac{2s^2+1}{4}e^{\frac{1}{2}} - e$. The derivative $h'(s) = se^{\frac{1}{2}}$ is clearly positive for $s \geq 3$, implying that $h(s)$ is monotonically increasing on $[3, +\infty)$. Consequently, $g(p, q) \geq h(3) = \frac{19}{4}e^{\frac{1}{2}} - e > 0$. This completes the proof. \square

3. Main results

Theorem 3.1. *For any tree T of order $n \geq 3$, $ER(L(T)) > \frac{n}{4} \cdot e^{\frac{1}{2}}$.*

Proof. First, we start with trees of special structures and consider the exponential Randić index of the line graphs of star S_n and path P_n . The line graph of the star graph S_n is the complete graph K_{n-1} of order $n - 1$. According to the definition of the exponential Randić index, we obtain:

$$ER(L(S_n)) = ER(K_{n-1}) = \frac{(n-1)(n-2)}{2}e^{\frac{1}{n-2}}.$$

The line graph of the path graph P_n is the path graph P_{n-1} of order $n - 1$, and its exponential Randić index is easily derived as:

$$ER(L(P_n)) = ER(P_{n-1}) = 2e^{\frac{1}{\sqrt{2}}} + (n-4)e^{\frac{1}{2}}.$$

From this, it can be seen that the exponential Randić indices of the line graphs of the n -vertex star graph S_n and the n -vertex path graph P_n are both greater than $\frac{n}{4}e^{\frac{1}{2}}$, satisfying the theorem.

Next, we proceed by induction on n . Assume that T_n is an n -vertex tree graph that is neither isomorphic to S_n nor to the path P_n , where $n \geq 5$.

Let $P_t = v_0v_1 \dots v_{t-1}$ be a longest path in T_n . Let T_{v_1} be the tree containing v_1 after removing the edge v_1v_2 , and T_{v_2} be the tree containing v_2 after removing the edge v_1v_2 . Let $T_1 = T_{v_1} + v_1v_2$ and $T_2 = T_{v_2} + v_1v_2$. The degree of v_1 in the tree T_n is denoted as $d_T(v_1)$. For convenience in subsequent proofs, we set $d_1 = d_T(v_1) - 1$. Since P_t is the longest path, all neighbors of v_1 other than v_2 must be leaf vertices; otherwise, it would contradict the assumption that P_k is the longest path. Therefore, T_1 is a star graph, and:

$$ER(L(T_1)) = \frac{d_1(d_1 + 1)}{2}e^{\frac{1}{d_1}}.$$

Meanwhile, by the induction hypothesis, we have:

$$ER(L(T_2)) > \frac{n - d_1}{4}e^{\frac{1}{2}}.$$

Case 1. $d_1 \geq 2$.

According to Lemma 2.3, we know that $c < e$. Therefore, from Eq. (1) we can directly obtain:

$$\begin{aligned} ER(L(T)) &= ER(L(T_1)) + ER(L(T_2)) - c \\ &> \frac{d_1(d_1 + 1)}{2}e^{\frac{1}{d_1}} + \frac{n - d_1}{4}e^{\frac{1}{2}} - e \end{aligned}$$

$$= \frac{n}{4}e^{\frac{1}{2}} + \frac{d_1(d_1 + 1)}{2}e^{\frac{1}{d_1}} - \frac{d_1}{4}e^{\frac{1}{2}} - e.$$

We define the function $g(x) = \frac{x(x+1)}{2}e^{\frac{1}{x}} - \frac{x}{4}e^{\frac{1}{2}} - e$, To determine the monotonicity of $g(x)$, we take its derivative $g'(x) = (x - \frac{1}{2x})e^{\frac{1}{x}} - \frac{1}{4}e^{\frac{1}{2}}$. Since $d_1 \geq 2$, it is clear that when $x \geq 2$, $g'(x) > 0$, $g(x) = \frac{x(x+1)}{2}e^{\frac{1}{x}} - \frac{x}{4}e^{\frac{1}{2}} - e$ is monotonically increasing and $g(x) \geq g(2) > 0$. Thus, $g(d_1) = \frac{d_1(d_1+1)}{2}e^{\frac{1}{d_1}} - \frac{d_1}{4}e^{\frac{1}{2}} - e > 0$. Consequently, when $d_1 \geq 2$, we have $ER(L(T)) > \frac{n}{4}e^{\frac{1}{2}}$.

Case 2. $d_1 = 1$.

Since $d_1 = 1$, we have $d_T(v_1) = 2$, which means vertex v_1 has no neighbors other than v_0 and v_2 . We now examine the neighborhood distribution of v_2 . Let $d_T(v_2)$ denote the degree of v_2 in T_n , and set $d_2 = d_T(v_2) - 1$. As P_k is the longest path, all neighbors of v_2 except v_1 and v_3 must be vertices of degree 1 or 2.

Let T'_{v_2} be the subtree containing v_2 after removing edge v_2v_3 , and T'_{v_3} the subtree containing v_3 after removing v_2v_3 . Define $T'_1 = T'_{v_2} + v_2v_3$ and $T'_2 = T'_{v_3} + v_2v_3$. The frame diagrams of T_n and T'_1 are presented in Figures 1 and 2, respectively.

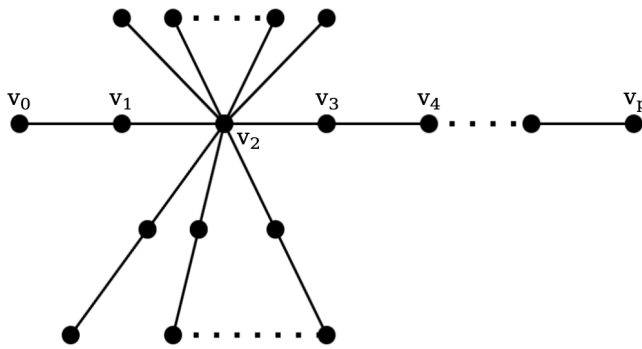


Fig. 1. T_n

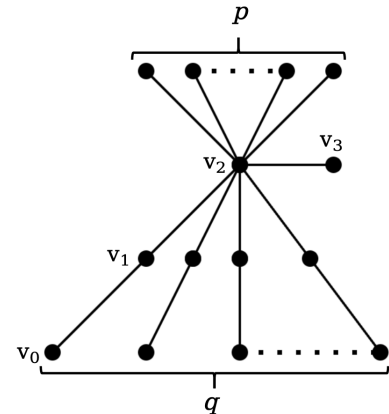


Fig. 2. T'_1

Denote T'_{v_2} by $S_{p,q}$, where p and q be the numbers of neighbors of v_2 having degrees 1 and 2, respectively. Then the exponential Randić index of $L(T'_1)$ can be expressed as:

$$ER(L(T'_1)) = qe^{\frac{1}{\sqrt{p+q+1}}} + \binom{q}{2}e^{\frac{1}{p+q+1}} + \binom{p+1}{2}e^{\frac{1}{p+q}} + (p+1)qe^{\frac{1}{\sqrt{(p+q)(p+q+1)}}}. \quad (6)$$

By Lemma 2.4, we obtain:

$$ER(L(T'_1)) > ER(K_{p+q+1}) = \frac{(p+q+1)(p+q)}{2}e^{\frac{1}{p+q}}.$$

Meanwhile, from the induction hypothesis we have:

$$ER(L(T'_2)) > \frac{n-p-2q}{4}e^{\frac{1}{2}}.$$

Subcase 1. $d_2 = 1$.

In this case, $d_T(v_2) = 2$, implying vertex v_2 has no neighbors other than v_1 and v_3 . Substituting $d_1 = 1$ and $d_2 = 1$ into the computation:

$$\begin{aligned} ER(L(T)) &= ER(L(T'_1)) + ER(L(T'_2)) - c \\ &> e + \frac{n-1}{4}e^{\frac{1}{2}} - \left(d_1 \left(e^{\frac{1}{d_1}} - e^{\frac{1}{\sqrt{(d_1+d_2) \cdot d_1}}} \right) + d_2 \left(e^{\frac{1}{d_2}} - e^{\frac{1}{\sqrt{(d_1+d_2) \cdot d_2}}} \right) \right) \\ &= \frac{n}{4}e^{\frac{1}{2}} + 2e^{\frac{1}{\sqrt{2}}} - e - \frac{1}{4}e^{\frac{1}{2}} \\ &> \frac{n}{4}e^{\frac{1}{2}}. \end{aligned}$$

Thus, the theorem holds when $d_2 = 1$.

Subcase 2. $d_2 \geq 2$.

Subcase 2.1. $p \geq 2$.

By Lemma 2.5, we obtain $g(p, q) = \frac{2(p+q+1)(p+q)-p-2q}{4}e^{\frac{1}{2}} - e > 0$, when p and q are positive integers with $p \geq 2$ and $q \geq 1$. Applying Lemma 2.4 we have

$$\begin{aligned} ER(L(T)) &= ER(L(T'_1)) + ER(L(T'_2)) - c \\ &> \frac{(p+q+1)(p+q)}{2}e^{\frac{1}{p+q}} + \frac{n-p-2q}{4}e^{\frac{1}{2}} - e \\ &> \frac{n}{4}e^{\frac{1}{2}} + \frac{2(p+q+1)(p+q)-p-2q}{4}e^{\frac{1}{2}} - e \\ &> \frac{n}{4}e^{\frac{1}{2}}. \end{aligned}$$

Therefore, the theorem holds in this case.

Subcase 2.2. $p = 1$.

In this case, with $p = 1$ and $q \geq 1$, substituting p and q into equation(6) we have that

$$ER(L(T'_1)) = qe^{\frac{1}{\sqrt{q+2}}} + \binom{q}{2}e^{\frac{1}{q+2}} + e^{\frac{1}{q+1}} + 2qe^{\frac{1}{\sqrt{(q+1)(q+2)}}} > \frac{1+2q}{4}e^{\frac{1}{2}} + e.$$

Consequently, we obtain that

$$\begin{aligned} ER(L(T)) &= ER(L(T'_1)) + ER(L(T'_2)) - c \\ &> ER(L(T'_1)) + \frac{n-1-2q}{4}e^{\frac{1}{2}} - e \\ &= \frac{n}{4}e^{\frac{1}{2}} + ER(L(T'_1)) - \frac{1+2q}{4}e^{\frac{1}{2}} - e \\ &> \frac{n}{4}e^{\frac{1}{2}}. \end{aligned}$$

Subcase 2.3. $p = 0$.

In this case, v_2 has only degree-2 neighbors. Substituting $p = 0$ into equation(6), we obtain that

$$ER(L(T'_1)) = qe^{\frac{1}{\sqrt{q+1}}} + \binom{q}{2}e^{\frac{1}{q+1}} + qe^{\frac{1}{\sqrt{q(q+1)}}} > \frac{1}{2}qe^{\frac{1}{2}} + e.$$

Since $d_2 = p + q$ and $p = 0$, we have $d_2 = q \geq 2$. Therefore,

$$ER(L(T)) = ER(L(T'_1)) + ER(L(T'_2)) - c$$

$$\begin{aligned}
&> ER(L(T'_1)) + \frac{n-2q}{4}e^{\frac{1}{2}} - e \\
&= \frac{n}{4}e^{\frac{1}{2}} + ER(L(T'_1)) - \frac{1}{2}qe^{\frac{1}{2}} - e \\
&> \frac{n}{4}e^{\frac{1}{2}}.
\end{aligned}$$

The proof is completed. □

4. Conclusions

In this paper, we have studied a lower bound on the exponential Randić index for the line graph of any tree T_n and proved that for any tree T of order $n \geq 3$, $ER(L(T)) > \frac{n}{4} \cdot e^{\frac{1}{2}}$. Our findings contribute to a deeper understanding of how structural graph invariants behave under line graph transformations.

In future work, it would be interesting to explore whether similar bounds hold for broader graph classes such as unicyclic graphs, potentially offering new insights and potential applications in the field of chemical graph theory.

Conflict of Interest

The authors declare that they have no conflict of interest.

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