ARS COMBINATORIA 167 (2026) 125–133

Ars Combinatoria
www.combinatorialpress.com/ars

Combinatorial Press

Toughness and (g, f)-factors in graphs with prescribed properties

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ABSTRACT

In this paper, we consider the relationship between toughness and the existence of (g, f)-factors with inclusion/exclusion properties. We obtain that if $t(G) \geq \frac{(a+b)^2+2(b-a)-3}{4(a+1)}$ with $b > a \geq 2$ and $a \leq g(x) < f(x) \leq b$ where a, b are two integers, then for any two given edges e_1 and e_2, there exists a (g, f)-factor including e_1, e_2; and a (g, f)-factor including e_1 and excluding e_2; as well as a (g, f)-factor excluding e_1, e_2.

Keywords: (g, f)-factor; toughness; inclusion/exclusion properties

2020 Mathematics Subject Classification: 05C70.

1. Introduction

All graphs considered are simple and finite. We refer the reader to [2] for terminologies and notations not defined here.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $x \in V(G)$, we denote by $d_G(x)$ the degree of x in G and by $N_G(x)$ the set of vertices adjacent to x in G. We write $N_G[x]$ for $N_G(x) \cup \{x\}$. The minimum degree of G is denoted by $\delta(G)$. For $S \subseteq V(G)$, let $N_G(S)$ denote the union of $N_G(x)$ for every $x \in S$. We use $G[S]$ and $G - S$ to denote the subgraph induced by S and $V(G) - S$.

A subset $S \subseteq V(G)$ is called an independent set(a covering set) if every edge of G is incident with at most(at least) one vertex of S. For any disjoint subsets S, $T \subseteq V(G)$, $E_G(S,T)$ denotes the set of edges with one end in S and the other in T and $e_G(S,T) = |E_G(S,T)|$.

Let $f : V(G) \to N$ be an integer function. For any subset $X \subseteq V(G)$, we denote

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Received 18 May 2025; Revised 10 May 2026; Accepted 21 May 2026; Published Online 12 June 2026.

DOI: 10.61091/ars167-08

$f(X) = \sum_{x \in X} f(x)$ and $f(\emptyset) = 0$. A spanning subgraph F of G is called an f-factor of G satisfying $d_F(x) = f(x)$ for any $x \in V(G)$. When $f(x) = k$ for all $x \in V(G)$, F is called a k-factor. Let g and f be two integer-valued functions defined on $V(G)$ with $g(x) \leq f(x)$ for any $x \in V(G)$. A *(g, f)-factor* of G is a spanning subgraph F satisfying $g(x) \leq d_F(x) \leq f(x)$ for any $x \in V(G)$. F is called an *$[a, b]$-factor* if $g(x) = a$ and $f(x) = b$ for any $x \in V(G)$.

Chvátal [7] first introduced the concept of *toughness*, $t(G)$, denoted by

$$t(G) = min\{\frac{|S|}{\omega(G-S)} : S \subseteq V(G), \omega(G-S) \geq 2\},$$

where $\omega(G-S)$ denotes the number of components of $G-S$ and G is not a complete graph. If G is complete, then $t(G) = \infty$. A graph G is *k-tough* if $t(G) \geq k$.

Chvátal mainly studied the relationship between toughness and the existence of Hamilton cycles and k-factors. He conjectured that every k-tough graph G has a k-factor if $k|V(G)|$ is even(k is a positive integer).

Enomoto et al. [9] confirmed Chvátal's conjecture and showed that the result is sharp.

Chen [5] improved considered k-factors containing a specified edge or excluding a specified edge under the similar condition.

Theorem 1.1 ([5]). *Let G be a graph and $k \geq 2$. If $t(G) \geq k$ and $k|V(G)|$ is even, then for every edge e of G, there exists a k-factor which contains the given edge e, and there also exists a k-factor which does not contain e.*

As a generalization of Chvátal's conjecture, Katerinis [12] studied the relationship between toughness and the existence of f-factors, as well as $[a, b]$-factor.

Since the toughness condition about k-factors is sharp, we [4] considered the relationship between toughness condition and the existence of $[a, b]$-factors for $b > a \geq 2$. We observed the bound of toughness condition in Theorem 1.2 is sharp.

Theorem 1.2 ([4]). *Let G be a graph of order n and a, b be two positive integers with $b > a \geq 2$. If $t(G) \geq a - 1 + \frac{a-1}{b}$, then G has an $[a, b]$-factor.*

Much work has been contributed to the existence of factors with given properties ([1, 13, 16, 17]). Recently, the researchers considered toughness and factors with various conditions([8, 10, 15, 18], [19, 20]). The existence of an even $[4, b]$-factor in a graph was considered [6]. In this paper, we consider the existence of (g, f)-factors with inclusion/exclusion properties under the condition of toughness when $a \leq g(x) < f(x) \leq b$, $b > a \geq 2$.

Theorem 1.3. *Let G be a graph, g and f be two integer-valued functions defined on $V(G)$ with $a \leq g(x) < f(x) \leq b$ for any $x \in V(G)$ where a, b are two positive integers with $b > a \geq 2$. Let e_1, e_2 be two distinct edges of a graph G. If $t(G) \geq \frac{(a+b)^2+2(b-a)-3}{4(a+1)}$, then G contains a (g, f)-factor containing e_1 and e_2; and a (g, f)-factor containing e_1 and excluding e_2; as well as a (g, f)-factor excluding e_1 and e_2;*

2. Preliminary lemmas

In order to prove the main theorem, we first give the characterization of (g, f)-factors due to Heinrich [11].

Theorem 2.1 ([11]). *Let G be a graph and g, f be integer-valued functions defined on $V(G)$. If $g(x) < f(x)$ for every $x \in V(G)$, then G has a (g, f)-factor if and only if for any subset S of $V(G)$,*

$$g(T) - d_{G-S}(T) \leq f(S),$$

where $T = \{x | x \in V(G) - S,\ d_{G-S}(x) \leq g(x)\}$.

The following lemma can be deduced from Theorem 2.1.

Lemma 2.2 ([14]). *Let G be a graph and g, f be integer-valued functions defined on $V(G)$ such that $g(x) < f(x) \leq d_G(x)$ for every $x \in V(G)$. Let E_1 and E_2 be two disjoint subsets of $E(G)$, then G has a (g, f)-factor F such that $E_1 \subseteq E(F)$ and $E_2 \cap E(F) = \emptyset$ if and only if for any disjoint subsets S and T of $V(G)$*

$$g(T) - d_{G-S}(T) \leq f(S) - \alpha(S, T; E_1, E_2) - \beta(S, T; E_1, E_2),$$

where $U = V(G)-S-T$, $\alpha(S, T; E_1, E_2) = 2|E_1 \cap E_G(S)|+|E_1 \cap E_G(S, U)|$ and $\beta(S, T; E_1, E_2) = 2|E_2 \cap E_G(T)|+|E_2 \cap E_G(T, U)|$.

In addition, the lemmas below are essential to the proof of our main theorem.

Lemma 2.3 ([14]). *Let G be a graph and $H = G[T]$ such that $\delta(H) \geq 1$ and $1 \leq d_G(x) \leq k-1$ for every $x \in V(H)$ where $T \subseteq V(G)$ and $k \geq 2$. Let $T_1, \ldots,\ T_{k-1}$ be a partition of the vertices of H satisfying $d_G(x) = j$ for each $x \in T_j$ where we allow some T_j to be empty. If each component of H has a vertex of degree at most $k-2$ in G, then H has a maximal independent set I and a covering set $C = V(H) - I$ such that*

$$\sum_{j=1}^{k-1} (k-j)c_j \leq \sum_{j=1}^{k-1} (k-2)(k-j)i_j,$$

where $c_j = |C \cap T_j|$ and $i_j = |I \cap T_j|$ for $j = 1, \ldots,\ k-1$.

3. Proof of the main result

We also need the following lemmas to prove our main theorem.

Lemma 3.1 ([7]). *If a graph G is not complete, then $t(G) \leq \frac{1}{2}\delta(G)$.*

Lemma 3.2. *Let G be a graph with toughness $t(G) \geq \frac{(a+b)^2+2(b-a)-3}{4(a+1)}$ and g, f be integer-valued functions defined on $V(G)$ such that $a \leq g(x) < f(x) \leq b$ for every $x \in V(G)$,* *where a, b are integers. Let S, T be a pair of disjoint subsets of $V(G)$. If $S \neq \emptyset$ and $T \neq \emptyset$, then*

$$g(T) - d_{G-S}(T) \leq f(S) - 4.$$

Proof of Lemma 3.2. By the contrary, suppose that there exists a pair of disjoint subsets S, T of $V(G)$ with $|S|> 0$, $|T|> 0$ satisfying $g(T) - d_{G-S}(T) > f(S) - 4$. That is,

$$g(T) - d_{G-S}(T) \geq f(S) - 3. \tag{1}$$

Moreover, suppose that S, T is a pair of minimal sets respect to (1). Then by the minimality of S and T we obtain the following claim.

Claim 1.

(1) Given S, if T is a minimal set with respect to (1), then $d_{G-S}(x) < g(x)$ for all $x \in T$.

(2) Given T, if S is a minimal set with respect to (1), then $d_T(x) > a+1$ for all $x \in S$.

Proof. If $|T|\geq 2$, by the choice of T, we have for any $x \in T$, $g(T-\{x\})-d_{G-S}(T-\{x\}) < f(S) - 3$. That is, $d_{G-S}(x) < g(x)$.

If $|T|= 1$, let $T = \{x\}$. Then we get that $d_{G-S}(x) \leq g(x)$. And we notice that $d_{G-S}(x) = g(x)$ holds only when $d_{G-S}(x) = g(x) = 2$ and $f(S) = 3$, it follows that $a = 2$ and $|S|= 1$. Then $t(G) \geq \frac{(a+b)^2+2(b-a)-3}{4(a+1)} \geq 2$ and $\delta(G) \geq 2t(G) \geq 4$ by Lemma 3.1. However, $d_G(x) \leq d_{G-S}(x) + |S|= 3$, a contradiction. Therefore, $d_{G-S}(x) < g(x)$ when $|T|= 1$.

Similarly, for any $v \in S$, we have $g(T) - d_{G-(S-\{v\})}(T) < f(S - \{v\}) - 3$. Hence $d_T(v) = e_G(v,T) > f(v) \geq a+1$. We complete the proof of Claim 1. □

Since $g(x) < f(x) \leq b$, we have $0 \leq d_{G-S}(x) \leq b-2$ for any $x \in T$. Let $T_i = \{x \in T | d_{G-S}(x) = i,\ 0 \leq i \leq b-2\}$, $t_i = |T_i|$. Set $H = G[T_1 \cup T_2 \cup \cdots \cup T_{b-2}]$. $T_1 \cup T_2 \cup \cdots \cup T_{b-2}$ is a partition of the vertices of H. By Lemma 2.3, H has a maximal independent set I and a covering set C such that

$$\sum_{j=1}^{b-2}(b-1-j)c_j \leq \sum_{j=1}^{b-2} j(b-1-j)i_j, \tag{2}$$

where $c_j = |C \cap T_j|$ and $i_j = |I \cap T_j|$ for $j = 1, \ldots,\ b-2$.

Without the loss of generality, we choose I to be a maximal independent set of H. Set $W = V(G) - S - T$ and $U = S \cup C \cup (N_{G-S}(I) \cap W)$. Then

$$|U|\leq |S|+\sum_{j=1}^{b-2} j i_j, \quad \omega(G-U) \geq t_0 + \sum_{j=1}^{b-2} i_j.$$

Now we show that $|U|\geq t(G)\omega(G-U)$.

It holds obviously when $\omega(G-U) \geq 2$. When $\omega(G-U) = 1$, for any $x \in T$, $|U|\geq d_{G-S}(x) + |S|\geq \delta(G) \geq 2t(G) > t(G)\omega(G-U)$.

Therefore

$$|S|\geq |U|-\sum_{j=1}^{b-2} j i_j \geq \sum_{j=1}^{b-2}(t(G)-j)i_j + t(G)t_0.$$

On the other hand, since $a \le g(x) < f(x) \le b$,

$$\begin{aligned} g(T) - d_{G-S}(T) &\le (b-1)|T| - d_{G-S}(T) \\ &= \sum_{j=1}^{b-2} (b-1-j)t_j + (b-1)t_0 \\ &= \sum_{j=1}^{b-2} (b-1-j)i_j + \sum_{j=1}^{b-2} (b-1-j)c_j + (b-1)t_0, \end{aligned}$$

and $f(S) \ge (a+1)|S|$.

Combined with (1) we have

$$\begin{aligned} &\sum_{j=1}^{b-2} (b-1-j)i_j + \sum_{j=1}^{b-2} (b-1-j)c_j \\ &\quad \ge (a+1)|S| - 3 - (b-1)t_0 \\ &\quad \ge (a+1) \sum_{j=1}^{b-2} (t(G) - j)i_j + ((a+1)t(G) - b + 1)t_0 - 3. \end{aligned}$$

It follows that

$$\begin{aligned} \sum_{j=1}^{b-2} (b-1-j)c_j \ge &\sum_{j=1}^{b-2} ((a+1)t(G) - (a+1)j - (b-1) \\ &+ j)i_j + ((a+1)t(G) - b + 1)t_0 - 3. \end{aligned} \tag{3}$$

Combined with (2), we obtain that

$$\begin{aligned} \sum_{j=1}^{b-2} j(b-1-j)i_j \ge &\sum_{j=1}^{b-2} ((a+1)t(G) - (a+1)j - (b-1) + j)i_j \\ &+ ((a+1)t(G) - b + 1)t_0 - 3. \end{aligned}$$

Now we consider the following cases.

Case 1. $t_0 > 0$.

In this case, we have $(a+1)t(G) - (b-1) \ge \frac{(a+b)^2 + 2(b-a) - 3}{4} - (b-1) = \frac{(a+b-1)^2}{4} > 3$ since $t(G) \ge \frac{(a+b)^2+2(b-a)-3}{4(a+1)}$. Therefore there exists some $j \in \{1, 2, \cdots, b-2\}$ satisfying $j(b-1-j) > (a+1)t(G) - (a+1)j - (b-1) + j$, that is, $j(b-1-j) + (a+1)j - j > (a+1)t(G) - (b-1)$. Meanwhile, $j(b-1-j) + (a+1)j - j = -j^2 + (a+b-1)j \le \frac{(a+b-1)^2}{4}$ and $(a+1)t(G) - (b-1) \ge \frac{(a+b-1)^2}{4}$, a contradiction.

Case 2. $t_0 = 0$.

In this case, we first show the following claim.

Claim 2. $C \ne \emptyset$.

Proof. If $C = \emptyset$, then $|T| = \sum_{j=1}^{b-2} i_j$. By (3), we have

$$\sum_{j=1}^{b-2} ((a+1)t(G) - (a+1)j - (b-1-j))i_j \le 3.$$

Since $t(G) \geq \frac{(a+b)^2+2(b-a)-3}{4(a+1)}$ and $j \leq b-2$, we get

$$\frac{(b-a+1)^2+4a}{4}|T|\leq \sum_{j=1}^{b-2}(\frac{(b+a-1)^2}{4}-aj)i_j \leq 3.$$

By Claim 1, $|T|\geq a+2 \geq 4(b>a\geq 2)$, $\frac{(b-a+1)^2+4a}{4}|T|\geq 4$, a contradiction. □

Next we show that for any vertex $x \in C$, $d_I(x)=1$. If there exists one vertex in C with at least two neighbors in I, then $|U|\leq |S|+\sum_{j=1}^{b-2} ji_j-1$. And $|S|\geq t(G)(t_0+\sum_{j=1}^{b-2} i_j)-\sum_{j=1}^{b-2} ji_j+1$. By the inequality (2), it follows that

$$\begin{aligned}\sum_{j=1}^{b-2} j(b-1-j)i_j \geq &\sum_{j=1}^{b-2}((a+1)t(G)-(a+1)j-(b-1)\\ &+j)i_j+((a+1)t(G)-b+1)t_0-3\\ >&\sum_{j=1}^{b-2}((a+1)t(G)-(a+1)j-(b-1)+j)i_j.\end{aligned}$$

Similarly to Case 1, we also obtain a contradiction.

Now, let $x \in C$ and $U'=U-\{x\}$. Then $\omega(G-U')=\sum_{j=1}^{b-2} i_j$. Now we show that $|U'|\geq t(G)(\sum_{j=1}^{b-2} i_j)$.

It holds obviously when $\sum_{j=1}^{b-2} i_j>1$. If $\sum_{j=1}^{b-2} i_j=1$, set the independent vertex be v. Then $|U|=|U'\cup\{v\}|\geq |S|+d_{G-S}(v)\geq \delta(G)\geq 2t(G)$, that is $|U'|\geq t(G)+(t(G)-1)\geq t(G)=t(G)\sum_{j=1}^{b-2} i_j\Big(\text{since } t(G)\geq \frac{(a+b)^2+2(b-a)-3}{4(a+1)}\geq 1\Big)$.

Now we obtain that $|S|\geq t(G)\sum_{j=1}^{b-2} i_j-\sum_{j=1}^{b-2} ji_j+1$. By the inequality (2), it follows that $\sum_{j=1}^{b-2} j(b-1-j)i_j > \sum_{j=1}^{b-2}((a+1)t(G)-(a+1)j-(b-1)+j)i_j$. Similarly to Case 1, we also obtain a contradiction. The proof is complete. □

Now we begin to prove our main results.

Proof of Theorem 1.3. Let E_1, E_2 be two edge sets with $E_1\cup E_2=\{e_1,e_2\}$. The theorem holds if and only if G contains a (g,f)-factor F such that $E_1\subseteq E(F)$, $E_2\cap E(F)=\emptyset$ where E_1 or E_2 may be empty. By the contrary, suppose that G does not contain such a (g,f)-factor F. Then, by Lemma 2.2, there exists a pair of disjoint subsets S, T of $V(G)$ such that

$$g(T)-d_{G-S}(T)>f(S)-\alpha(S,T;E_1,E_2)-\beta(S,T;E_1,E_2), \tag{4}$$

where $W=V(G)-S-T$, $\alpha(S,T;E_1,E_2)=2|E_1\cap E_G(S)|+|E_1\cap E_G(S,W)|$ and $\beta(S,T;E_1,E_2)=2|E_2\cap E_G(T)|+|E_2\cap E_G(T,W)|$.

Meanwhile, as $t(G) \geq \frac{(a+b)^2+2(b-a)-3}{4(a+1)}$, G contains a (g,f)-factor [3]. Therefore,

$$g(T) - d_{G-S}(T) \leq f(S). \tag{5}$$

Now we show the following claim.

Claim. $S \neq \emptyset$ and $T \neq \emptyset$.

Proof. If $S \cup T = \emptyset$, then $\alpha(S,T;E_1,E_2) = \beta(S,T;E_1,E_2) = 0$, and $g(T) - d_{G-S}(T) > f(S)$, a contradiction to (5).

Then we consider the following cases.

Case 1. $S = \emptyset$ and $T \neq \emptyset$. Then $\alpha(S,T;E_1,E_2) = 0$. And we obtain that $\beta(S,T;E_1,E_2) \neq 0$ from (4) and (5). It follows that $E_2 \neq \emptyset$. Hence either $E_2 = \{e_2\}$ or $E_2 = \{e_1, e_2\}$.

If $E_2 = \{e_2\}$, then $E_1 = \{e_1\}$, which is the case of containing e_1 and excluding e_2. According to (4) again,

$$g(T) - d_G(T) > -2|E_2 \cap E_G(T)| - |E_2 \cap E_G(T,W)|.$$

Then $g(T) - d_G(T) \leq (b-1-\delta(G))|T| \leq (b-1-2t(G))|T|$ since $g(T) \leq (b-1)|T|$ and $\delta(G) \geq 2t(G)$. It follows that $(b-1-2t(G))|T| \leq \frac{1-a^2-b^2}{2(1+a)}|T| \leq -a|T|$ by $t(G) \geq \frac{(a+b)^2+2(b-a)-3}{4(a+1)}$. And it yields that $-a \geq (-a)|T| > -2|E_2 \cap E_G(T)| - |E_2 \cap E_G(T,W)| \geq -2$, contradicting with $a \geq 2$.

If $E_2 = \{e_1, e_2\}$, then $E_1 = \emptyset$, which is the case of excluding e_1 and e_2. Then

$$g(T) - d_G(T) > -2|E_2 \cap E_G(T)| - |E_2 \cap E_G(T,W)|.$$

Then $g(T) - d_G(T) \leq (b-1-\delta(G))|T| \leq (b-1-2t(G))|T|$ since $g(T) \leq (b-1)|T|$ and $\delta(G) \geq 2t(G)$. It follows that $(b-1-2t(G))|T| \leq \frac{1-a^2-b^2}{2(1+a)} \leq -a$ by $t(G) \geq \frac{(a+b)^2+2(b-a)-3}{4(a+1)}$. And it yields that $(-a)|T| > -2|E_2 \cap E_G(T)| - |E_2 \cap E_G(T,W)|$.

If $|T| \geq 2$, then $-2a > -4$, a contradiction. If $|T| = 1$, $2|E_2 \cap E_G(T)| + |E_2 \cap E_G(T,W)| \leq 2$, then $-2 > (-a)|T| > -2|E_2 \cap E_G(T)| - |E_2 \cap E_G(T,W)| \geq -2$, a contradiction, too.

Case 2. $S \neq \emptyset$ and $T = \emptyset$. Then $\beta(S,T;E_1,E_2) = 0$. Meanwhile, $\alpha(S,T;E_1,E_2) \neq 0$. It follows that $E_1 \neq \emptyset$. Hence either $E_1 = \{e_1\}$ or $E_1 = \{e_1, e_2\}$.

If $E_1 = \{e_1\}$, then $E_2 = \{e_2\}$, which is the case of including e_1 and excluding e_2. From (4), we have $f(S) < \alpha(S,T;E_1,E_2) = 2|E_1 \cap E_G(S)| + |E_1 \cap E_G(S,W)| \leq 2$, which is impossible since $f > a \geq 2$.

$E_1 = \{e_1, e_2\}$, then $E_2 = \emptyset$, which is the case of containing e_1 and e_2. And

$$f(S) - 2|E_1 \cap E_G(S)| - |E_1 \cap E_G(S,W)| < 0.$$

Then $f(S) < 2|E_1 \cap E_G(S)| + |E_1 \cap E_G(S,W)|$ as $f > a \geq 2$, a contradiction. This complete the proof of the claim. □

Now since $S \neq \emptyset$ and $T \neq \emptyset$, by Lemma 3.2, we have

$$g(T) - d_{G-S}(T) \leq f(S) - 4.$$

But $\alpha(S,T;E_1,E_2) + \beta(S,T;E_1,E_2) \leq 4$, it follows from (4) that

$$g(T) - d_{G-S}(T) > f(S) - 4,$$

a contradiction. The proof is complete. □

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