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A sufficient condition for the square of a line graph to be perfect

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ABSTRACT

In this paper, we prove that if a graph does not contain any cycle of length greater than 4, then the square of its line graph is perfect. As an application, we give a concise proof of a known result: the strong chromatic index of a bipartite graph that does not contain any cycle of length greater than 4 is at most Δ^2, where Δ represents the maximum degree of the graph. This latter result provides a partial affirmative answer to some known conjectures on upper bounds for the strong chromatic index of graphs.

Keywords: square of a graph, line graph, perfect graph, strong chromatic index

2020 Mathematics Subject Classification: Primary: 05C76, 05C17; Secondary: 05C15, 05C38.

1. Introduction

Throughout this paper, by a graph we mean a finite undirected graph that has no loops or multiple edges. Given the graph G, we denote the set of vertices and the set of edges of G by $V(G)$ and $E(G)$, respectively. For terminology and concepts not given here, the reader may refer to [11] and [21].

We begin the paper by introducing the notion of an induced subgraph, which plays a central role in our discussion. An *induced subgraph* of a graph G is a graph H formed by taking $V(H)$ as a nonempty subset of $V(G)$ and $E(H)$ as all of the edges in G connecting pairs of vertices in $V(H)$. In particular, an *induced cycle* in G is a cycle of G that is an induced subgraph of G.

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1.1. *The chromatic and clique numbers*

A *proper vertex-coloring* of a graph G is an assignment of colors to the vertices of G in such a way that no two adjacent vertices share a common color. The minimum number of colors that allow a proper vertex-coloring of G is called the *chromatic number* of G and is denoted by $\chi(G)$. A list of well-studied coloring parameters of graphs can be found in [13].

A *clique* in the graph G is a subset of vertices in G such that every two distinct vertices in the clique are adjacent. A *maximum clique* is a clique that has the largest possible number of vertices. The *clique number* of G is the number of vertices in a maximum clique of G and is denoted by $\omega(G)$.

It is known that the problem of finding the parameters $\chi(G)$ and $\omega(G)$ is NP-hard (see [10]). As every vertex of a clique should get a different color, it is obvious that $\chi(G) \geq \omega(G)$. It is also easy to see that $\chi(G)$ may be greater than $\omega(G)$. For example, $\chi(C_5) = 3 > 2 = \omega(C_5)$, where C_5 is the cycle of length 5. Based on the above notions and observations, it seems natural to ask when does the equality $\chi(G) = \omega(G)$ hold. For a graph G, it is known that determining whether the equality $\chi(G) = \omega(G)$ holds is an NP-complete problem (see [15]). A graph G is called *perfect* provided $\chi(H) = \omega(H)$ holds true for every induced subgraph H of G. The *strong perfect graph theorem*, one of the foundational results in graph theory, says that a graph is perfect if and only if neither the graph nor its complement contains an induced odd cycle of length at least 5 (see [5]).

A graph is said to be 4-*chordal* if it does not contain induced cycles of length greater than 4. It is worthwhile to note that 4-chordal graphs are also called *long-hole-free* graphs (see, for example, [4]). A graph is said to be *chordal* if every cycle of length 4 or more has a chord, which is an edge that is not part of the cycle but connects two vertices of the cycle.

1.2. *The line graphs and their squares*

The *line graph* $L(G)$ of the graph G is the graph with vertex set $E(G)$ such that two vertices are adjacent in $L(G)$ if and only if the corresponding edges of G have a common vertex. A graph G is a *line graph* if it is isomorphic to the line graph $L(H)$ of some graph H.

For a graph G, its *square*, denoted by G^2, has vertex set $V(G)$ and two vertices are adjacent in G^2 exactly when they are joined by an edge or a path of two edges in G. The graph $L(G)^2$ is the square of the line graph of G, that is, the vertex set of $L(G)^2$ is $E(G)$ and two vertices are adjacent in $L(G)^2$ if and only if the corresponding edges of G share a vertex or are connected by an edge in G.

We are now in the position to present the main result of this paper.

Theorem 1.1 (Theorem A)**.** *Let G be a graph that does not contain any cycle of length greater than 4. Then $L(G)^2$ is perfect.*

While chordal graphs are perfect by a theorem of Dirac [6], Theorem 1.1 (Theorem A) is not an immediate consequence: indeed $L(G)^2$ need not be chordal even when G does not contain any cycle of length greater than 4. The graph presented in Figure 1 provides a counterexample (see [19, Theorem 4.1]).

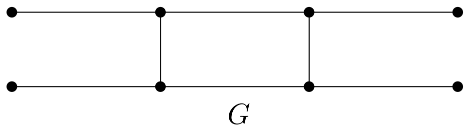

Fig. 1. A 4-chordal graph G where $L(G)^2$ is K_8 with two independent edges deleted, hence not chordal, showing Theorem 1.1 (Theorem A) does not follow directly from chordal graphs being perfect

It is worthwhile to note that previous work by Hatzel and Wiederrecht [12, Theorem 24] provided necessary and sufficient conditions for $L(G)^2$ to be perfect, including a necessary condition requiring the graph G to have no induced cycle of length at least 7. The absence of such long induced cycles is not sufficient by itself; additional forbidden configurations are required (see [12]). Theorem 1.1 (Theorem A), by contrast, shows that the stronger assumption that G contains no cycle of length greater than 4 is sufficient for $L(G)^2$ to be perfect.

The rest of the paper is organized as follows. In Section 2, we provide a proof of Theorem 1.1 (Theorem A). In Section 3, we apply Theorem 1.1 (Theorem A) to give a concise proof of a known upper bound for the strong chromatic index of bipartite graphs without cycles longer than 4.

2. Proof of Theorem 1.1 (Theorem A)

In a paper published in 1970, Beineke [1] obtained a characterization of line graphs in terms of nine forbidden induced subgraphs. Since the complete bipartite graph $K_{1,3}$ is one of those forbidden subgraphs, line graphs cannot contain it as an induced subgraph. Let us now formally state this result for future reference.

Lemma 2.1. *Let G be a graph. Then the line graph $L(G)$ cannot contain a $K_{1,3}$ as an induced subgraph.*

In what follows, when we deal with cycles of length n, we always take the indices modulo n. By this convention, we may give the following definition.

Definition 2.2. Let G be a graph with edges $e_1, \dots, e_n$ such that $C = e_1e_2 \dots e_ne_1$ is an induced cycle of length $n \geq 4$ in $L(G)^2$. Then, obviously, $e_1, \dots, e_n$ are all vertices of $L(G)$ and we may define the set $\mathfrak{A}(C)$ as follows:

$$\mathfrak{A}(C) = \{1 \leq i \leq n \mid e_i \text{ is not adjacent to } e_{i+1} \text{ in } L(G)\}.$$

We will need the following lemma to avoid repetition in the proofs later on.

Lemma 2.3. *Let G be a graph and $C = e_1e_2 \dots e_ne_1$ be an induced cycle of length $n \geq 4$ in $L(G)^2$. Then the following statements hold:*

(a) *For every $1 \leq i \leq n$, e_i is not adjacent to e_j in $L(G)$ for $j \neq i \pm 1$.*

(b) *The set* $\{1,\ldots,n\}\setminus\mathfrak{A}(C)$ *does not contain consecutive numbers and* $\lceil\frac{n}{2}\rceil\leq|\mathfrak{A}(C)|\leq n$.

(c) *For every* $i\in\mathfrak{A}(C)$, *there exists a vertex* x_i *of* $L(G)$ *which is adjacent to both* e_i *and* e_{i+1} *in* $L(G)$. *Moreover,*

(i) *the vertex* x_i *is not adjacent to* e_j *in* $L(G)$ *for* $j\neq i,i+1$, *and*

(ii) *if, for* $j\in\mathfrak{A}(C)$, x_i *is adjacent to* x_j *in* $L(G)$, *then* $j=i\pm 1$.

Proof. (a) Suppose that e_i and e_j are adjacent in $L(G)$ for $j\neq i\pm 1$. Then e_i and e_j are adjacent in $L(G)^2$ and e_ie_j is a chord of C, which is a contradiction.

(b) Let $1\leq i\leq n$ be given. Note that e_i may or may not be adjacent to e_{i+1} in $L(G)$. If e_i is adjacent to e_{i+1}, then e_{i+1} cannot be adjacent to e_{i+2}; otherwise, e_i would be adjacent to e_{i+2} in $L(G)^2$, which is a contradiction. Therefore, the set $\{1,\ldots,n\}\setminus\mathfrak{A}(C)$ does not contain consecutive integers and $\lceil\frac{n}{2}\rceil\leq|\mathfrak{A}(C)|\leq n$.

(c) For every $i\in\mathfrak{A}(C)$, since e_i is not adjacent to e_{i+1} in $L(G)$ but is adjacent to e_{i+1} in $L(G)^2$, there exists a vertex x_i of $L(G)$ such that x_i is adjacent to both e_i and e_{i+1} in $L(G)$. Moreover, we have the following:

(i) The vertices x_i and e_j are not adjacent in $L(G)$ for $j\neq i,i+1$; otherwise, either e_i is adjacent to e_j in $L(G)^2$ for $j\neq i,i\pm 1$, or e_{i+1} is adjacent to e_{i-1} in $L(G)^2$, both of which are contradictions.

(ii) Since adjacency in $L(G)$ means that the corresponding edges of G share a common vertex, the adjacency of x_i and x_j in $L(G)$ implies that the edges of G corresponding to x_i and x_j are incident to a common vertex.

Now x_i is adjacent in $L(G)$ to e_i and e_{i+1}, while x_j is adjacent in $L(G)$ to e_j and e_{j+1}. Moreover, since $i,j\in\mathfrak{A}(C)$, the pairs e_i,e_{i+1} and e_j,e_{j+1} are not adjacent in $L(G)$; hence, each pair consists of two edges of G with distinct endpoints. Therefore, if the edges corresponding to x_i and x_j share a vertex in G, that vertex must also be an endpoint of one of e_j or e_{j+1}. In other words, x_i must be adjacent in $L(G)$ to e_j or e_{j+1}. By part (i), x_i is adjacent only to e_i and e_{i+1} in $L(G)$. Hence, $j\in\{i,i+1\}$ or $j+1\in\{i,i+1\}$. Since $i\neq j$, it follows that $j=i\pm 1$. □

Definition 2.4. With the set-up of Definition 2.2, for each $i\in\mathfrak{A}(C)$, we call the vertex x_i given by Lemma 2.3(c) a *bridging vertex*.

In order to prove Theorem 1.1 (Theorem A), we will also need the following lemma which enables us to construct induced cycles in line graphs.

Lemma 2.5. *Let* G *be a graph. If* $C=e_1e_2\ldots e_ne_1$ *is an induced cycle of length* $n\geq 4$ *in* $L(G)^2$, *then* $L(G)$ *contains an induced cycle of length* $\geq n$.

Proof. For each $i\in\mathfrak{A}(C)$, let x_i be the bridging vertex of $L(G)$ given by Lemma 2.3(c) (see Definition 2.4). By part (ii) of Lemma 2.3(c), adjacency among the vertices $\{x_i\mid i\in\mathfrak{A}(C)\}$ can occur only between consecutive indices, that is, if x_ix_j are adjacent in $L(G)$, then $j = i \pm 1$. We will construct an induced cycle C' in $L(G)$ with length $\geq n$. In order to do this, consider

$$W = \{e_{i+1} \mid i, i+1 \in \mathfrak{A}(C) \text{ and } x_i \text{ is adjacent to } x_{i+1} \text{ in } L(G)\},$$

and define the set of vertices of C' by

$$V = (\{e_i \mid 1 \leq i \leq n\} \setminus W) \cup \{x_i \mid i \in \mathfrak{A}(C)\}.$$

We now define a cyclic ordering of V in $L(G)$. Note that in each case, any vertex belonging to W is omitted.

∘ If $i \notin \mathfrak{A}(C)$, retain the edge $e_i e_{i+1}$.
∘ If $i \in \mathfrak{A}(C)$ and $i+1 \notin \mathfrak{A}(C)$, replace $e_i e_{i+1}$ by $e_i x_i e_{i+1}$.
∘ If $i, i+1 \in \mathfrak{A}(C)$, then replace the segment through e_{i+1} as follows:

- if x_i is adjacent to x_{i+1} in $L(G)$, use $e_i x_i x_{i+1} e_{i+2}$;
- otherwise, use $e_i x_i e_{i+1} x_{i+1} e_{i+2}$.

By Lemma 2.3(a), the vertices $e_1, \ldots, e_n$ are pairwise nonadjacent in $L(G)$ except for consecutive indices. By Lemma 2.3(c), each bridging vertex x_i is adjacent precisely to e_i and e_{i+1}, and possible adjacencies among bridging vertices occur only between consecutive indices. Therefore, no additional adjacencies occur among the vertices of C' beyond those explicitly listed, and each vertex in the induced subgraph has degree exactly 2. Hence, the resulting graph C' is an induced cycle in $L(G)$.

Finally, consider the length of C'. Observe that whenever a vertex e_{i+1} is omitted (that is, when $e_{i+1} \in W$), it is replaced by at least two vertices x_i and x_{i+1}. In all other cases, either one vertex in inserted or the original adjacency is retained. Thus, every replacement described above either preserves the number of vertices in the segment or increases it. Hence, $|V(C')| \geq n$. This completes the proof. □

The following lemma will also be used repeatedly in the remainder of this paper.

Lemma 2.6 ([19], Lemma 3.1)**.** *Let G be a graph. Then G contains a cycle C of length at least 4 if and only if the line graph $L(G)$ contains an induced cycle C_L of the same length.*

It is worthwhile to note that the cycle C of G in Lemma 2.6 is not necessarily induced. Figure 2 presents such an example: here G is 4-chordal, yet its line graph $L(G)$ contains an induced cycle of length greater than 4.

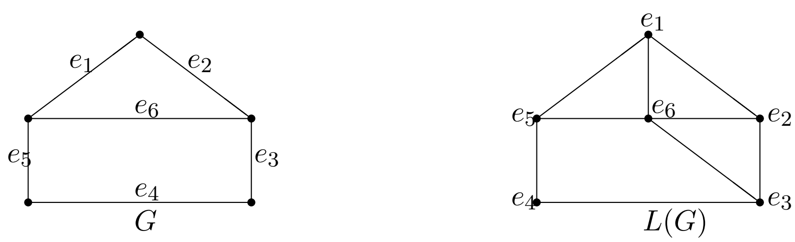

Fig. 2. The graph G is 4-chordal but $L(G)$ is not

Lemma 2.7. *Let G be a graph that does not contain any cycle of length greater than 4. If $C = e_1e_2e_3e_4e_1$ is an induced cycle in $L(G)^2$, then $|\mathfrak{A}(C)| = 4$.*

Proof. By Lemma 2.3(b), the set $\{1,2,3,4\} \setminus \mathfrak{A}(C)$ does not contain consecutive integers and also $|\mathfrak{A}(C)|$ is equal to 2, 3 or 4.

Case I. $|\mathfrak{A}(C)| = 2$. In this case, by the above observation, without loss of generality, we may assume that $\{1,2,3,4\} \setminus \mathfrak{A}(C) = \{1,3\}$, and so, $\mathfrak{A}(C) = \{2,4\}$. Therefore, by Lemma 2.3(c), there exist bridging vertices x_2 and x_4 of $L(G)$. By part (ii) of Lemma 2.3(c), x_2 is not adjacent to x_4 in $L(G)$, and so, by Definition 2.2 and parts (a) and (c) of Lemma 2.3, $e_1e_2x_2e_3e_4x_4e_1$ is an induced cycle in $L(G)$ with length 6. This, together with Lemma 2.6, implies that G has a cycle of length 6, a contradiction.

Case II. $|\mathfrak{A}(C)| = 3$. In this case, without loss of generality, we may assume that $\mathfrak{A}(C) = \{2,3,4\}$. Therefore, by Lemma 2.3(c), there exist bridging vertices x_2, x_3 and x_4 of $L(G)$. If none of the x_i's are adjacent in $L(G)$, then by Definition 2.2 and parts (a) and (c) of Lemma 2.3, $e_1e_2x_2e_3x_3e_4x_4e_1$ is an induced cycle of length 7 in $L(G)$. This, together with Lemma 2.6, implies that G has a cycle of length 7, a contradiction. Note that, by part (ii) of Lemma 2.3(c), x_2 is not adjacent to x_4 in $L(G)$. If x_2 is adjacent to x_3 and x_3 is not adjacent to x_4 in $L(G)$, then by Definition 2.2 and parts (a) and (c) of Lemma 2.3, $e_1e_2x_2x_3e_4x_4e_1$ is an induced cycle of length 6 in $L(G)$. Again, by Lemma 2.6, this implies that G has a cycle of length 6, a contradiction. If x_2 is not adjacent to x_3 but x_3 is adjacent to x_4 in $L(G)$, then by the same argument as above we are led to a contradiction. Finally, if x_2 is adjacent to x_3 and x_3 is adjacent to x_4 in $L(G)$, then by Lemma 2.3(c), $e_1e_2x_2x_3x_4e_1$ is an induced cycle of length 5 in $L(G)$. By Lemma 2.6, this implies that G has a cycle of length 5, again a contradiction.

Since the above two cases lead to a contradiction, we conclude that $|\mathfrak{A}(C)| = 4$. □

Lemma 2.8. *Let G be a graph that does not contain any cycle of length greater than 4, and let $C = e_1e_2e_3e_4e_1$ be an induced cycle in $L(G)^2$. Then the following statements hold:*

(a) *The edges e_1, e_2, e_3, e_4 are pairwise nonadjacent in $L(G)$.*

(b) *For each $i \in \{1,2,3,4\}$, there exists a vertex x_i of $L(G)$ such that*

  (i) *x_i is adjacent to both e_i and e_{i+1} in $L(G)$, and*

  (ii) *x_i is adjacent to neither e_{i+2} nor e_{i+3} in $L(G)$.*

(c) *Among the vertices x_1, x_2, x_3, x_4, we have*

  (i) *x_1 is not adjacent to x_3 in $L(G)$,*

  (ii) *x_2 is not adjacent to x_4 in $L(G)$, and*

  (iii) *for each $i \in \{1,2,3,4\}$, the vertices x_i and x_{i+1} are adjacent in $L(G)$.*

Proof. First, note that, by Lemma 2.7, we have $|\mathfrak{A}(C)| = 4$, and hence, $\mathfrak{A}(C) = \{1,2,3,4\}$.

(a) By Definition 2.2 and Lemma 2.3(a), none of the e_i's are adjacent to one another in $L(G)$.

(b) By Lemma 2.3(c) and its first part, there exist bridging vertices x_1, x_2, x_3 and x_4 of $L(G)$ such that for every $i \in \{1,2,3,4\}$, x_i is adjacent to both e_i and e_{i+1} in $L(G)$ but is not adjacent to e_{i+2} or e_{i+3}.

(c) By part (ii) of Lemma 2.3(c), x_1 is not adjacent to x_3 and x_2 is not adjacent to x_4 in $L(G)$. What remains is to prove that for every $i \in \{1,2,3,4\}$, x_i and x_{i+1} are adjacent in $L(G)$. We now distinguish four cases; in each case we shall obtain a contradiction.

Case I. The vertices x_i and x_{i+1} are not adjacent in $L(G)$ for every $i \in \{1,2,3,4\}$. Then $e_1x_1e_2x_2e_3x_3e_4x_4e_1$ is an induced cycle of length 8 in $L(G)$. This, together with Lemma 2.6, implies that G has a cycle of length 8, a contradiction.

Case II. The vertices x_i and x_{i+1} are adjacent in $L(G)$ for exactly one $i \in \{1,2,3,4\}$. Without loss of generality, we may assume that x_1 and x_2 are adjacent in $L(G)$ and this is the only edge between the x_i's. Then $e_1x_1x_2e_3x_3e_4x_4e_1$ is an induced cycle of length 7 in $L(G)$. This, together with Lemma 2.6, implies that G has a cycle of length 7, a contradiction.

Case III. The vertices x_i and x_{i+1} are adjacent in $L(G)$ for exactly two $i \in \{1,2,3,4\}$. Without loss of generality, we may assume that the two edges are either x_1x_2 and x_2x_3 or x_1x_2 and x_3x_4. If the former occurs, then $e_1x_1x_2x_3e_4x_4e_1$ is an induced cycle of length 6 in $L(G)$. On the other hand, if the two edges are x_1x_2 and x_3x_4, then $e_1x_1x_2e_3x_3x_4e_1$ is an induced cycle of length 6 in $L(G)$. In either of the two possibilities, we have by Lemma 2.6 that G has a cycle of length 6, a contradiction.

Case IV. The vertices x_i and x_{i+1} are adjacent in $L(G)$ for exactly three $i \in \{1,2,3,4\}$. Without loss of generality, we may assume that x_1 is adjacent to x_2, x_2 is adjacent to x_3 and x_3 is adjacent to x_4 in $L(G)$, and these three edges are the only edges between the x_i's. Then $e_1x_1x_2x_3x_4e_1$ is an induced cycle of length 5 in $L(G)$. This, together with Lemma 2.6, implies that G has a cycle of length 5, a contradiction.

From the above cases, we deduce that the only possibility is that x_i and x_{i+1} are adjacent in $L(G)$ for every $i \in \{1,2,3,4\}$. □

We prove two more lemmas before giving the proof of Theorem 1.1 (Theorem A).

Lemma 2.9. *Let G be a graph that does not contain any cycle of length greater than 4. Then $L(G)^2$ cannot contain the graph H, presented in Figure 3, as an induced subgraph.*

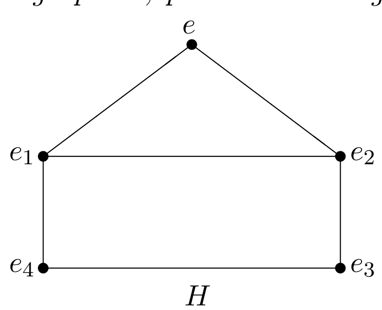

Fig. 3. The graph $L(G)^2$ cannot contain H as an induced subgraph

Proof. Suppose that $L(G)^2$ contains the graph H as an induced subgraph. Therefore,

$L(G)^2$ contains the induced cycle $e_1e_2e_3e_4e_1$. Now, by Lemma 2.8, there exist the bridging vertices x_1, x_2, x_3 and x_4 of $L(G)$ which satisfy the conditions of Lemma 2.8. Note that e is a vertex of $L(G)$ and we claim the following:

Claim 1. The vertex e is not adjacent to x_i in $L(G)$ for $1 \leq i \leq 4$.

Proof of Claim 1. First, note that e is adjacent to none of x_2, x_3 and x_4 in $L(G)$; otherwise, since x_2 and x_3 are adjacent to e_3 and x_4 is adjacent to e_4 in $L(G)$, e is adjacent to either e_3 or e_4 in $L(G)^2$, which is in contradiction to H being an induced subgraph.

We also note that e is not adjacent to x_1 in $L(G)$; otherwise, by the observation in the previous paragraph and Lemma 2.8(c), $L(G)$ contains a $K_{1,3}$ as an induced subgraph with $\{x_1\}$ and $\{e, x_2, x_4\}$ as parts of the bipartition, which is in contradiction to Lemma 2.1. □

We now consider the three cases: (I) the vertex e is adjacent to both e_1 and e_2 in $L(G)$; (II) the vertex e is adjacent to only one of e_1 and e_2 in $L(G)$; (III) the vertex e is adjacent to neither e_1 nor e_2 in $L(G)$. We show below that each of these cases gives rise to a contradiction.

Case I. The vertex e is adjacent to both e_1 and e_2 in $L(G)$. In this case, by Claim 1 and parts (a), (b) and (c) of Lemma 2.8, $e_1ee_2x_2x_3x_4e_1$ is an induced cycle of length 6 in $L(G)$. This, together with Lemma 2.6, implies that G has a cycle of length 6, a contradiction.

Case II. The vertex e is adjacent to only one of e_1 and e_2 in $L(G)$. In this case, we may suppose without loss of generality that e is not adjacent to e_1 in $L(G)$ but is adjacent to e_2 in $L(G)$. Since e is adjacent to e_1 in $L(G)^2$, there exists a vertex y_1 of $L(G)$ such that y_1 is adjacent to both e_1 and e in $L(G)$. The next two claims will be useful in concluding that Case II is not possible.

Claim 2. The vertex y_1 is adjacent to neither x_2 nor x_3 in $L(G)$.

Proof of Claim 2. Suppose that y_1 is adjacent to x_i in $L(G)$ ($i = 2$ or 3). Then by Claim 1 and Lemma 2.8(b), $L(G)$ contains a $K_{1,3}$ as an induced subgraph with $\{y_1\}$ and $\{x_i, e_1, e\}$ as parts of the bipartition, which is in contradiction to Lemma 2.1. □

Claim 3. The vertex y_1 is not adjacent to e_2 in $L(G)$.

Proof of Claim 3. Suppose that y_1 is adjacent to e_2 in $L(G)$. There are two possibilities: either y_1 and x_4 are adjacent in $L(G)$ or they are not.

First, suppose that y_1 and x_4 are adjacent in $L(G)$. Then by Claim 2 as well as parts (b) and (c) of Lemma 2.8, $y_1e_2x_2x_3x_4y_1$ is an induced cycle of length 5 in $L(G)$. This, together with Lemma 2.6, implies that G has a cycle of length 5, a contradiction.

Now, suppose that y_1 and x_4 are not adjacent in $L(G)$. Then again by Claim 2 as well as parts (a), (b) and (c) of Lemma 2.8, $y_1e_2x_2x_3x_4e_1y_1$ is an induced cycle of length 6 in $L(G)$. By Lemma 2.6, this implies that G has a cycle of length 6, again a contradiction.

Since both cases give us a contradiction, Claim 3 holds true. □

Now, there are two possibilities: either y_1 and x_4 are adjacent in $L(G)$ or they are not.

If y_1 and x_4 are adjacent in $L(G)$, then it follows by Claims 1, 2 and 3 as well as parts (b) and (c) of Lemma 2.8 that $y_1ee_2x_2x_3x_4y_1$ is an induced cycle of length 6 in $L(G)$. By Lemma 2.6, this gives us a contradiction as it implies that G has a cycle of length 6. On the other hand, if y_1 is not adjacent to x_4 in $L(G)$, then again by Claims 1, 2 and 3 as well as parts (a), (b) and (c) of Lemma 2.8, we have that $y_1ee_2x_2x_3x_4e_1y_1$ is an induced cycle of length 7 in $L(G)$. Again, by Lemma 2.6, we have a contradiction as this implies that G has a cycle of length 7.

These contradictions tell us that Case II does not occur. This leaves us with one remaining case to consider.

Case III. The vertex e is adjacent to neither e_1 nor e_2 in $L(G)$. Since e is adjacent to both e_1 and e_2 in $L(G)^2$, in this case, there exist vertices y_1 and y_2 of $L(G)$ such that y_1 is adjacent to both e_1 and e whereas y_2 is adjacent to both e_2 and e in $L(G)$. The following claims will enable us to conclude that Case III is not possible.

Claim 4. The vertex y_1 is adjacent to none of e_2, x_2 and x_3 in $L(G)$.

Proof of Claim 4. Suppose to the contrary that y_1 is adjacent to e_2 in $L(G)$. By Lemma 2.8(a), e_1 and e_2 are not adjacent in $L(G)$. Therefore, $L(G)$ contains a $K_{1,3}$ as an induced subgraph with $\{y_1\}$ and $\{e_2, e_1, e\}$ as parts of the bipartition, which is in contradiction to Lemma 2.1.

Also, if y_1 is adjacent to x_i in $L(G)$ ($i = 2$ or 3), then by Claim 1 and Lemma 2.8(b), $L(G)$ would contain a $K_{1,3}$ as an induced subgraph with $\{y_1\}$ and $\{x_i, e_1, e\}$ as parts of the bipartition, which is again in contradiction to Lemma 2.1. □

Claim 5. The vertex y_2 is adjacent to none of e_1, x_3 and x_4 in $L(G)$.

Proof of Claim 5. This follows from Claim 4 by symmetry. □

Claim 6. The vertex y_1 is not adjacent to y_2 in $L(G)$.

Proof of Claim 6. Suppose that y_1 is adjacent to y_2 in $L(G)$. There are two possibilities: either y_1 is adjacent to x_4 in $L(G)$ or it is not.

First, suppose that y_1 is adjacent to x_4 in $L(G)$. Then by Claims 4 and 5 as well as parts (b) and (c) of Lemma 2.8, we have the following:

- $y_1y_2x_2x_3x_4y_1$ is an induced cycle of length 5 in $L(G)$, provided y_2 is adjacent to x_2 in $L(G)$;
- $y_1y_2e_2x_2x_3x_4y_1$ is an induced cycle of length 6 in $L(G)$, provided y_2 is not adjacent to x_2 in $L(G)$.

Both of the above lead to contradictions, since each of them, together with Lemma 2.6, implies that G has a cycle of length greater than 4.

Now, suppose that y_1 is not adjacent to x_4 in $L(G)$. Then by Claims 4 and 5 as well as parts (a), (b) and (c) of Lemma 2.8, we have the following:

- $y_1y_2x_2x_3x_4e_1y_1$ is an induced cycle of length 6 in $L(G)$, provided y_2 is adjacent to x_2 in $L(G)$;

- $y_1y_2e_2x_2x_3x_4e_1y_1$ is an induced cycle of length 7 in $L(G)$, provided y_2 is not adjacent to x_2 in $L(G)$.

Again, both lead to contradictions, since each of them, together with Lemma 2.6, implies that G has a cycle of length ≥ 6.

Since both possibilities lead to a contradiction, Claim 6 holds true. □

Claim 7. The vertex y_1 is not adjacent to x_4 in $L(G)$.

Proof of Claim 7. Suppose that y_1 is adjacent to x_4 in $L(G)$. There are two possibilities: either y_2 is adjacent to x_2 in $L(G)$ or it is not.

First, suppose that y_2 is adjacent to x_2 in $L(G)$. Then by Claims 1, 4, 5 and 6 as well as Lemma 2.8(c), $y_1ey_2x_2x_3x_4y_1$ is an induced cycle of length 6 in $L(G)$. This, together with Lemma 2.6, implies that G has a cycle of length 6, a contradiction.

Now, suppose that y_2 is not adjacent to x_2 in $L(G)$. Then by Claims 1, 4, 5 and 6 as well as parts (b) and (c) of Lemma 2.8, $y_1ey_2e_2x_2x_3x_4y_1$ is an induced cycle of length 7 in $L(G)$. By Lemma 2.6, this implies that G has a cycle of length 7, again a contradiction.

Since both possibilities lead to a contradiction, Claim 7 holds true. □

Now, there are three possibilities: neither of y_1 and y_2 is adjacent to x_1 in $L(G)$; only one of y_1 and y_2 is adjacent to x_1 in $L(G)$; or both y_1 and y_2 are adjacent to x_1 in $L(G)$.

First, suppose that neither of y_1 and y_2 is adjacent to x_1 in $L(G)$. Then, by Claims 1, 4, 5 and 6 as well as Lemma 2.8(a), $y_1ey_2e_2x_1e_1y_1$ is an induced cycle of length 6 in $L(G)$. This, together with Lemma 2.6, implies that G has a cycle of length 6, a contradiction.

Next, suppose that only one of y_1 and y_2 is adjacent to x_1 in $L(G)$. Without loss of generality, say y_1 is adjacent to x_1 in $L(G)$. Then, by Claims 1, 4 and 6, $y_1ey_2e_2x_1y_1$ is an induced cycle of length 5 in $L(G)$. Again, by Lemma 2.6, this implies that G has a cycle of length 5, a contradiction.

Finally, suppose that both y_1 and y_2 are adjacent to x_1 in $L(G)$. Then, by Claims 5, 6 and 7, $L(G)$ contains a $K_{1,3}$ as an induced subgraph with $\{x_1\}$ and $\{y_1, y_2, x_4\}$ as parts of the bipartition. But this is not possible by Lemma 2.1.

These contradictions tell us that Case III also does not occur.

Since Cases I, II and III all lead to a contradiction, we conclude that $L(G)^2$ cannot contain the graph H as an induced subgraph. This completes the proof. □

Lemma 2.10. *Let G be a graph containing an induced cycle $C = x_1x_2 \ldots x_nx_1$ of length $n \geq 6$, Then there exist five vertices of C whose induced subgraph in the complement $\overline{G}$ is isomorphic to the graph H shown in Figure 3.*

Proof. Let $C = x_1x_2 \ldots x_nx_1$ be an induced cycle of G with $n \geq 6$. Since C is induced, vertices x_i and x_j are adjacent in G if and only if $j \equiv i \pm 1 \pmod{n}$.

Consider the five vertices $x_1, x_2, x_{n-2}, x_{n-1}, x_n$ of C. Among these vertices, the adjacencies in G occur only between consecutive vertices in the cycle. Consequently, the nonadjacencies in G, which become edges in the complement $\overline{G}$, yield exactly the graph H shown in Figure 3. Hence, $\overline{G}$ contains H as an induced subgraph. □

We conclude this section by providing a proof of Theorem 1.1 (Theorem A) and adding a remark.

Proof of Theorem 1.1 (Theorem A). We show below that neither $L(G)^2$ nor $\overline{L(G)^2}$ contains an induced odd cycle of length at least 5. Then it follows from the strong perfect graph theorem that $L(G)^2$ is perfect; thus completing the proof.

The graph $L(G)^2$ does not contain any induced odd cycle of length $n \geq 5$; otherwise, by Lemma 2.5, $L(G)$ would contain an induced cycle of length $\geq n \geq 5$. Then it follows from Lemma 2.6 that G contains a cycle of length ≥ 5, which contradicts the assumption.

The graph $\overline{L(G)^2}$ does not contain any induced odd cycle of length $n \geq 5$. Indeed, it does not contain any induced cycle of length 5, since the complement of an induced cycle of length 5 is again an induced cycle of length 5. Also, it does not contain any induced cycle of length $n \geq 7$; otherwise, based on Lemma 2.10, $L(G)^2$ would contain the graph H, presented in Figure 3, as an induced subgraph, which is in contradiction to Lemma 2.9. □

It is worthwhile to note that if the graph G contains no cycle of length greater than 4, then it follows immediately that G contains no induced cycle of such length. By [19, Lemma 3.2], one may then deduce that $L(G)^2$ likewise contains no induced cycle of length greater than 4. This yields an alternative argument, distinct from the one presented in our proof of Theorem 1.1 (Theorem A), that $L(G)^2$ contains no induced odd cycle of length ≥ 5.

3. An application of Theorem 1.1 (Theorem A)

Let G be a graph. A *strong edge-coloring* of G is an assignment of colors to the edges of G in such a way that no two edges of G with a common endpoint or with an edge that is adjacent to both of them receive the same color. The *strong chromatic index* of G, denoted by ${}_S\chi'(G)$, is the minimum number of colors that allow a strong edge-coloring of G. Many open problems that involve the strong edge-coloring of graphs are known to exist and among them, it appears that the problems concerning upper bounds for ${}_S\chi'(G)$ have generated considerable interest. In what follows, we point out some known conjectures about upper bounds for ${}_S\chi'(G)$.

Suppose that G has maximum degree Δ, that is, Δ is the largest degree among vertices of G. Perhaps the most important conjecture related to upper bounds for ${}_S\chi'(G)$ was proposed by Erdős and Nešetřil (see [7]). They conjectured that

$$ {}_S\chi'(G) \leq \begin{cases} \frac{5}{4}\Delta^2 & \text{if } \Delta \text{ is even,} \\ \frac{1}{4}(5\Delta^2 - 2\Delta + 1) & \text{if } \Delta \text{ is odd.} \end{cases} $$

Note that Δ^2 is less than or equal to both of the above-mentioned upper bounds. Faudree et al. [8] have in fact suggested Δ^2 as a sharper upper bound for ${}_S\chi'(G)$ when G is bipartite. This was confirmed for the case $\Delta = 3$ by Steger and Yu [20]. Brualdi and Massey [2] proposed a stronger version of the second conjecture by replacing the upper bound Δ^2 with $\Delta_1\Delta_2$, where Δ_1 and Δ_2 are the maximum degrees among vertices in the two parts of the bipartition, respectively. Another generalization of the conjecture by Faudree et al. [8] was given by Mahdian [14]. In order to state this conjecture, let C_5 denote the cycle of length 5. A graph is called C_5*-free* if it does not contain C_5 as an induced cycle. Note that bipartite graphs and 4-chordal graphs are both C_5-free. Mahdian [14] conjectured that if G is C_5-free, then ${}_S\chi'(G) \leq \Delta^2$. These conjectures are still wide open and we refer the reader to [16, 17, 18, 20] for some partial answers to them.

Theorem 3.1 (Theorem B) below provides a partial affirmative answer to the conjecture of Faudree et al. [8] and the related conjectures mentioned above. The bound $S\chi'(G) \leq \Delta^2$ for bipartite graphs was previously proved by Brualdi and Massey [2, Theorem 2.6] under the assumption that every cycle length of G is divisible by 4. Since graphs with no cycle of length greater than 4 satisfy this condition, Theorem 3.1 (Theorem B) applies to a subclass of the graphs considered in [2]. While the earlier result is therefore more general, our proof derives the bound from the fact that $L(G)^2$ is perfect (Theorem 1.1 (Theorem A)), offering a structural perspective in this restricted setting.

Theorem 3.1 (Theorem B)**.** *Let G be a bipartite graph with maximum degree Δ. If G does not contain any cycle of length greater than 4, then ${}_S\chi'(G) \leq \Delta^2$.*

We conclude this paper with a short proof of Theorem 3.1 (Theorem B). The proof uses Theorem 1.1 (Theorem A) and a few notions that we introduce as follows. A *matching* of a graph G is a set of edges of G, no two of which share a common vertex. An *induced matching* M of G is a matching such that no two edges of M are joined by an edge of G. A set of vertices of G is called *independent* if no two of them are adjacent.

Proof of Theorem 3.1 (Theorem B). Note that a color class in a strong edge coloring of G is an induced matching of G. By [3, Remark 1], every induced matching in G is an independent set of vertices in $L(G)^2$, and conversely. This implies that ${}_S\chi'(G) = \chi(L(G)^2)$. By Theorem 1.1 (Theorem A), $L(G)^2$ is perfect, that is, $\chi(L(G)^2) = \omega(L(G)^2)$. Hence, we get ${}_S\chi'(G) = \omega(L(G)^2)$. But, by [9, Theorem 1], we have $\omega(L(G)^2) \leq \Delta^2$, and thus, we obtain that ${}_S\chi'(G) \leq \Delta^2$, as required. □

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References

[1] L. W. Beineke. Characterizations of derived graphs. *Journal of Combinatorial Theory*, 9(2):129–135, 1970. https://doi.org/10.1016/S0021-9800(70)80019-9.

[2] R. A. Brualdi and J. J. Quinn Massey. Incidence and strong edge colorings of graphs. *Discrete Mathematics*, 122(1–3):51–58, 1993. https://doi.org/10.1016/0012-365X(93)90286-3.

[3] K. Cameron. Induced matchings in intersection graphs. *Discrete Mathematics*, 278(1–3):1–9, 2004. https://doi.org/10.1016/j.disc.2003.05.001.

[4] M. Chudnovsky, M. Pilipczuk, M. Pilipczuk, and S. Thomassé. On the maximum weight independent set problem in graphs without induced cycles of length at least five. *SIAM Journal on Discrete Mathematics*, 34(2):1472–1483, 2020. https://doi.org/10.1137/19M1249473.

[5] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas. The strong perfect graph theorem. *Annals of Mathematics*, 164(1):51–229, 2006. https://doi.org/10.4007/annals.2006.164.51.

[6] G. A. Dirac. On rigid circuit graphs. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 25:71–76, 1961. https://doi.org/10.1007/BF02992776.

[7] P. Erdős and J. Nešetřil. Problems. In G. Halász and V. T. Sós, editors, *Irregularities of Partitions*. Volume 8, Algorithms and Combinatorics: Study and Research Texts, pages 162–163. Springer-Verlag, Berlin and New York, 1989.

[8] R. J. Faudree, A. Gyárfás, R. H. Schelp, and Z. Tuza. Induced matchings in bipartite graphs. *Discrete Mathematics*, 78(1–2):83–87, 1989. https://doi.org/10.1016/0012-365X(89)90163-5.

[9] R. J. Faudree, R. H. Schelp, A. Gyárfás, and Z. Tuza. The strong chromatic index of graphs. *Ars Combinatoria*, 29B:205–211, 1990. Twelfth British Combinatorial Conference, Norwich, 1989.

[10] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman and Co., San Francisco, CA, 1979. https://doi.org/10.5555/574848.

[11] F. Harary. *Graph Theory*. Addison-Wesley Publishing Co., Reading, MA, 1969.

[12] M. Hatzel and S. Wiederrecht. On perfect linegraph squares. In *Graph-Theoretic Concepts in Computer Science*. Volume 11159, Lecture Notes in Computer Science, pages 252–265. Springer, Cham, 2018. https://doi.org/10.1007/978-3-030-00256-5_21.

[13] M. Kubale, editor. *Graph Colorings*, volume 352 of *Contemporary Mathematics*. American Mathematical Society, Providence, RI, 2004. https://doi.org/10.1090/conm/352.

[14] M. Mahdian. The strong chromatic index of C_4-free graphs. *Random Structures & Algorithms*, 17(3–4):357–375, 2000. https://doi.org/10.1002/1098-2418(200010/12)17:3/4<357::AID-RSA9>3.0.CO;2-Y.

[15] C. McDiarmid and B. Reed. Channel assignment and weighted coloring. *Networks*, 36(2):114–117, 2000. https://doi.org/10.1002/1097-0037(200009)36:2<114::AID-NET6>3.0.CO;2-G.

[16] K. Nakprasit. A note on the strong chromatic index of bipartite graphs. *Discrete Mathematics*, 308(16):3726–3728, 2008. https://doi.org/10.1016/j.disc.2007.07.034.

[17] J. J. Quinn and A. T. Benjamin. Strong chromatic index of subset graphs. *Journal of Graph Theory*, 24(3):267–273, 1997. https://doi.org/10.1002/(SICI)1097-0118(199703)23:3<267::AID-JGT8>3.0.CO;2-N.

[18] J. J. Quinn and E. L. Sundberg. Strong chromatic index in subset graphs. *Ars Combinatoria*, 49:155–159, 1998.

[19] R. Scheidweiler and S. Wiederrecht. On chordal graph and line graph squares. *Discrete Applied Mathematics*, 243:239–247, 2018. https://doi.org/10.1016/j.dam.2018.02.013.

[20] A. Steger and M.-L. Yu. On induced matchings. *Discrete Mathematics*, 120(1–3):291–295, 1993. https://doi.org/10.1016/0012-365X(93)90590-P.

[21] D. B. West. *Introduction to Graph Theory*. Prentice Hall, Inc., Upper Saddle River, NJ, 1996.

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