ARS COMBINATORIA 168 (2026) 49–56

Ars Combinatoria
www.combinatorialpress.com/ars

Combinatorial Press

Note on bipartite domination in outerplanar graphs

Atsuhiro Nakamoto*

ABSTRACT

For a graph $G = (V(G), E(G))$, a subset $S \subset V(G)$ is a *bipartite dominating set* if every vertex in $G - S$ is adjacent to a vertex in S, and if the subgraph of G induced by S is bipartite. The *bipartite domination number* of G, denoted by $\gamma_{bip}(G)$, is the minimum cardinality of all bipartite dominating sets of G. Xi and Yue [4] claimed that for every 2-connected outerplanar n-vertex graph G, $\gamma_{bip}(G) \leq \lceil \frac{n}{3} \rceil$, and that this bound is sharp. In this paper, correcting the result, we prove that $\gamma_{bip}(G) \leq \lceil \frac{3}{8} n \rceil$, where this bound is sharp.

Keywords: outerplanar graph, bipartite domination, independent domination

2020 Mathematics Subject Classification: 05C69, 05C10.

1. Introduction

For a graph $G = (V(G), E(G))$, a subset $S \subset V(G)$ is a *dominating set* if every vertex in $G - S$ is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of all dominating sets in G. There is a significant amount of research on domination in graphs, as referenced in [3].

Bachstein et al. [1] gave a concept of "bipartite domination", as follows. For a graph G, a subset $S \subset V(G)$ is a *bipartite dominating set* if S is a dominating set of G such that the subgraph of G induced by S, denoted by $G[S]$, is bipartite. We note that every graph G admits a bipartite dominating set, since we can take a maximal independent set in G as a bipartite dominating set. The *bipartite domination number* of G, denoted by $\gamma_{bip}(G)$, is the minimum cardinality of all bipartite dominating sets in G. By definition, it is evident that $\gamma_{bip}(G) \geq \gamma(G)$ for all graphs G.

An *outerplanar* graph G is one with a planar embedding such that all vertices are

* Corresponding author.

Received 25 Jan 2026; Revised 17 Jul 2026; Accepted 23 Jul 2026; Published Online 11 Aug 2026.

DOI: 10.61091/ars168-04

incident to the boundary of the infinite face. Xi and Yue studied the bipartite domination on outerplanar graphs [4]. They first pointed out that an n-vertex outerplanar graph with cut vertices can have a large bipartite domination number close to n. Then, restricting graphs to be 2-connected, they presented the following:

Theorem 1.1 (Xi and Yue [4]). *Let G be a 2-connected n-vertex outerplanar graph. Then*

$$\gamma_{bip}(G) \leq \left\lceil \frac{n}{3} \right\rceil,$$

where the estimation is sharp.

In this paper, correcting Theorem 1.1, we prove the following:

Theorem 1.2. *Let G be a 2-connected n-vertex outerplanar graph. Then*

$$\gamma_{bip}(G) \leq \left\lceil \frac{3}{8}n \right\rceil,$$

where the estimation is sharp for any positive integer $n \equiv 0 \bmod 8$.

Let $\mathcal{G}_n$ be the set of all 2-connected n-vertex outerplanar graphs, where $n \geq 3$. It is easy to see that $\gamma(G) \leq \lceil \frac{n}{3} \rceil$ for any $G \in \mathcal{G}_n$, Moreover, it was proved in [2] that $i(G) \leq \frac{2n+1}{5}$ for any $G \in \mathcal{G}_n$, where $i(G)$ denotes the *independent domination number* of a graph G, that is, the minimum cardinality of a dominating set S which is also an independent set in G. Since the above two results are best possible and since $\gamma(G) \leq \gamma_{bip}(G) \leq i(G)$ for all graphs G, it follows that:

$$\left\lceil \frac{n}{3} \right\rceil \leq \max_{G \in \mathcal{G}_n} \gamma_{bip}(G) \leq \frac{2n+1}{5}.$$

Then Theorem 1.2 asserts that $\max_{G \in \mathcal{G}_n} \gamma_{bip}(G) = \lceil \frac{3}{8}n \rceil$.

2. Proof of Theorem 1.2

We first establish the sharpness of Theorem 1.2, as follows. A *k-vertex* means a vertex of degree k, and a *k-cycle* a cycle of length k.

Theorem 2.1. *For $m \geq 1$, there exists a 2-connected outerplanar graph G with $8m$ vertices such that $\gamma_{bip}(G) = 3m$.*

Proof. Let H_i be the graph shown in Figure 1(1), for $i = 1, \ldots, m$, and let G_m be the outerplanar graph with $8m$ vertices obtained from $H_1, \ldots, H_m$ by connecting z_i and x_{i+1}, as in Figure 1(2), where the subscripts are taken modulo m. If we let $S = \bigcup_{i=1}^{m} \{b_i, d_i, z_i\}$ as in Figure 1(3), then S is a bipartite dominating set with $|S| = 3m$.

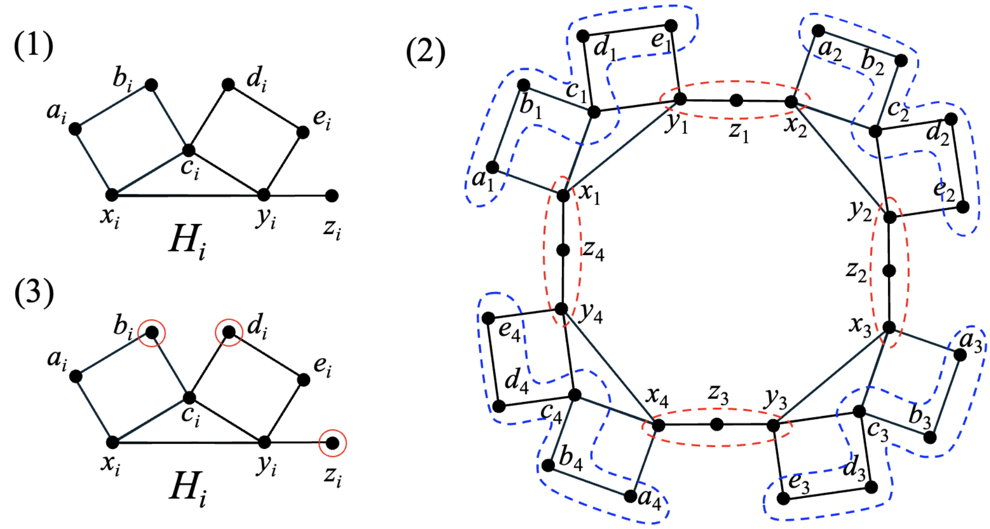

Fig. 1. (1) The graph H_i, (2) Outerplanar graph G_4 consisting of H_1, H_2, H_3, H_4, (3) Dominating set of H_i

Let T be any bipartite dominating set of G_m. Let $X_i = \{x_i, y_{i-1}, z_{i-1}\}$ and let $A_i = \{a_i, b_i, c_i, d_i, e_i\}$, for $i = 1, \ldots, m$. Observe that each X_i has at least one vertex in T. Moreover, each A_i has at least two vertices in T even if $x_i, y_i \in T$, where we note that x_i, y_i, c_i cannot be contained in T simultaneously, since these three vertices induce a 3-cycle. Hence, $|T| \geq 3m$ and consequently, $\gamma_{bip}(G_m) = 3m$. □

Secondly, an upper bound of $\gamma_{bip}(G)$ for a 2-connected outerplanar graph G is provided, and the proof follows the approach outlined in [4]. Recall that $\mathcal{G}_n$ denotes the set of all 2-connected n-vertex outerplanar graphs.

A configuration R in $G \in \mathcal{G}_n$ is *k-reducible* if either of the following holds:

- if G contains R, then $\gamma_{bip}(G) \leq \lceil kn \rceil$, or
- there is an operation for R reducing G into a *smaller* graph $G' \in \mathcal{G}_{n'}$ with $n' \leq n$, i.e., $n' < n$ or $|E(G')| < |E(G)|$, such that $\gamma_{bip}(G') \leq \lceil kn' \rceil$ implies $\gamma_{bip}(G) \leq \lceil kn \rceil$.

Xi and Yue used the following configurations for $k = \frac{1}{3}$, and this argument holds for any $k' \geq \frac{1}{3}$. Hence we use those for $k' = \frac{3}{8}$.

Observe that $G \in \mathcal{G}_n$ has a unique Hamilton cycle C, and we always fix a planar drawing of G such that C is the boundary cycle of the infinite face. An edge $xy \in E(G)$ is a *chord* of C if $x, y \in V(G)$ but $xy \notin E(C)$. A chord xy in G is *short* if x and y have distance at most 3 in C, and *long* otherwise. A short chord xy *has a triangle* (resp., *has a quadrilateral*) if x and y have distance 2 (resp., 3) in C.

The following are six $\frac{1}{3}$-reducible configurations established in [4] (see Figure 2).

(1) Configuration $\mathcal{Z}$: A 4-cycle $xaby$ with a diagonal xb, where xy is a chord of C, $\deg(a) = 2$, $\deg(b) = 3$, $\deg(x) \geq 4$ and $\deg(y) \geq 3$.

(2) Configuration $\mathcal{TX}$: A triangle xab and one edge $by \in E(C)$, where xb is a chord of C, $\deg(a) = 2$, $\deg(b) = 3$, $\deg(x) \geq 3$ and $\deg(y) \geq 2$.

(3) Configuration $\mathcal{TT}$: Two adjacent triangles xab and bcy, where xb and by are chords

of C, $\deg(a) = \deg(c) = 2$, $\deg(b) = 4$, $\deg(x) \geq 3$ and $\deg(y) \geq 3$.

(4) Configuration $\mathcal{TQ}$: A triangle xab and a quadrilateral $bcdy$, where xb and by are chords of C, $\deg(a) = \deg(c) = \deg(d) = 2$, $\deg(b) = 4$, $\deg(x) \geq 3$ and $\deg(y) \geq 3$.

(5) Configuration $\mathcal{XXXX}$: Three consecutive 2-vertices a, b, c in C.

(6) Configuration $\mathcal{QX}$: A quadrilateral $xabc$ and one edge $cy \in E(C)$, where xc is a chord of C, $\deg(a) = \deg(b) = 2$, $\deg(c) = 3$, $\deg(x) \geq 3$ and $\deg(y) \geq 2$.

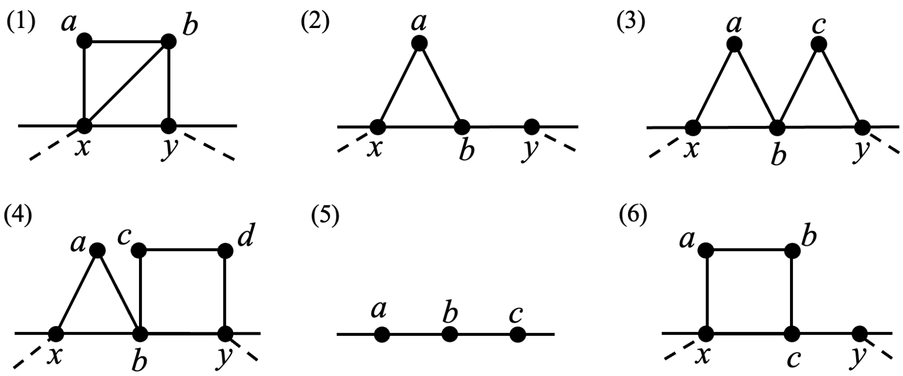

Fig. 2. (1) $\mathcal{Z}$, (2) $\mathcal{TX}$, (3) $\mathcal{TT}$, (4) $\mathcal{TQ}$, (5) $\mathcal{XXXX}$, (6) $\mathcal{QX}$

Lemma 2.2 (Xi and Yue [4]). *The configurations $\mathcal{Z}$, $\mathcal{TX}$, $\mathcal{TT}$, $\mathcal{TQ}$, $\mathcal{XXXX}$, $\mathcal{QX}$ are $\frac{3}{8}$-reducible.*

In the following, we suppose that $G \in \mathcal{G}_n$ has none of the six configurations in Lemma 2.2. A configuration $\mathcal{QQQ}$ consists of three consecutive quadrilaterals $xabc, cdef, fghy$, shown in the left of Figure 3, where $x \neq y$, and $\deg(a) = \deg(b) = \deg(d) = \deg(e) = \deg(g) = \deg(h) = 2$, and $\deg(x) \geq 4$, $\deg(c) = \deg(f) = 4$, $\deg(y) \geq 4$.

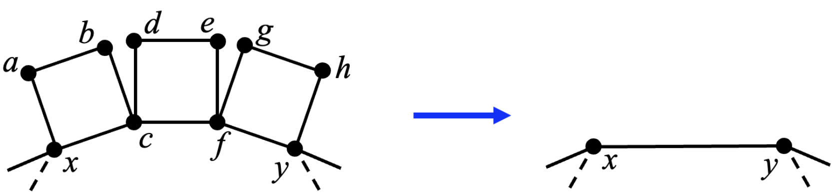

Fig. 3. Configuration $\mathcal{QQQ}$ and a reduction to obtain G'

Lemma 2.3. *The configuration $\mathcal{QQQ}$ is $\frac{3}{8}$-reducible.*

Proof. Suppose that $G \in \mathcal{G}_n$ contains $\mathcal{QQQ}$, and let G' be the graph obtained from G by removing the eight vertices a, b, c, d, e, f, g, h and adding an edge xy, as shown in Figure 3. If this operation makes multiple edges between x and y, we replace the multiple edges with a single edge xy. Since $\deg(x) \geq 4$ and $\deg(y) \geq 4$ by the definition, we have $G' \in \mathcal{G}_{n-8}$.

Suppose that G' has a bipartite dominating set S' with $|S'|\leq \lceil\frac{3}{8}(n-8)\rceil$. If $x, y \notin S'$, then $S = S' \cup \{a, d, g\}$ is a dominating set of G with $|S|\leq \lceil\frac{3}{8}n\rceil$. If $x \in S'$ or $y \in S'$, say the former by symmetry, then $S = S' \cup \{b, e, h\}$ is one with $|S|\leq \lceil\frac{3}{8}n\rceil$, where we note that S dominates y in G even if S' does not dominate y in $G' - xy$. Moreover, $G[S]$ is bipartite in both cases. Therefore, $\gamma_{bip}(G) \leq \lceil\frac{3}{8}n\rceil$. □

Let $\mathcal{P}$ be the configuration consisting of two quadrilaterals $xabc$ and $cdey$ with a long chord xy, as shown in Figure 4(1), such that $\deg(a) = \deg(b) = \deg(d) = \deg(e) = 2$, $\deg(c) = 4$, and $\deg(x) \geq 4$ and $\deg(y) \geq 4$, where x, y are the *feet* of $\mathcal{P}$.

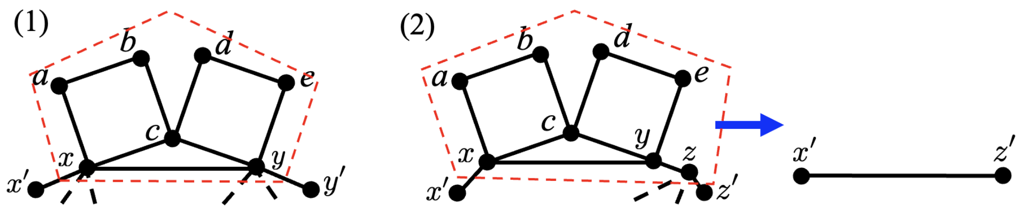

Fig. 4. (1) Configuration $\mathcal{P}$, (2) Configuration $\tilde{\mathcal{P}}$ and its removal

Lemma 2.4. *Suppose that $G \in \mathcal{G}_n$ has $\mathcal{P}$ with feet x, y. Let $E \subset E(G) \setminus E(C)$ be the set of edges incident to x or y except cx, cy, xy, and let $G' \in \mathcal{G}_n$ be the one obtained from G by removing E. If G' has a bipartite dominating set with cardinality at most $\lceil\frac{3}{8}n\rceil$, then so does G.*

Proof. We may suppose $E \neq \emptyset$, for otherwise, the lemma trivially holds. By assumption, G' has a bipartite dominating set S' with $S' \leq \lceil\frac{3}{8}n\rceil$. If S' is also a bipartite dominating set of G, then we are done. Hence, we may suppose that the addition of E to G' creates an odd cycle in $G[S']$ through x or y. Let $x' \neq a$ and $y' \neq e$ be the neighbors of x and y in C, respectively.

We first suppose that an odd cycle arisen in $G[S']$ passes through one of x and y, say x. In this case, $x \in S'$ and there is an edge $xp \in E$ with some $p \in S'$ in G. If we let $T' = \{a, b, c, d, e\} \cap S'$, then $|T'|\geq 2$ in G', since S' cannot contain c, x, y in G' simultaneously. Let $S = S' \setminus \{T' \cup \{x\}\} \cup \{b, e\}$. Observe that x is dominated by $p \in S$. If S dominates x' as well, then S is a bipartite dominating set of G with $|S|< |S'|$. On the other hand, if S does not dominate x', then $\tilde{S} = S \cup \{x'\}$ is a required bipartite dominating set with $|\tilde{S}|\leq |S'|$.

Secondly suppose that odd cycles through x and those through y appear. After dealing with odd cycles in G through x as above, we deal with those through y, and consider whether y' should be contained in S instead of y. We omit a detailed disposition on y. □

Let $\tilde{\mathcal{P}}$ be the configuration obtained from $\mathcal{P}$ by adding the conditions $\deg(x) = \deg(y) = 4$ and introducing one more vertex z adjacent to y in C, as shown in Figure 4(2), where $x' \neq a$ and $z' \neq y$ are the neighbors of x and z in C respectively, and x', x, y, z, z' are all distinct.

Lemma 2.5. *A configuration $\tilde{\mathcal{P}}$ is $\frac{3}{8}$-reducible.*

Proof. Suppose that $G \in \mathcal{G}_n$ contains $\tilde{\mathcal{P}}$, but G has none of $\mathcal{Z}$, $\mathcal{TX}$, $\mathcal{TT}$, $\mathcal{TQ}$, $\mathcal{XXXX}$, $\mathcal{QX}$ and $\mathcal{QQQ}$. If $x'z' \in E(C)$ (as shown in Figure 5(1)), then G contains either $\mathcal{TX}$ or $\mathcal{XXXX}$, a contradiction. Hence, if we let G' be the graph obtained from G by removing a, b, c, d, e, x, y, z and adding an edge $x'z'$ (as shown in Figure 4(2)), then we have $G' \in \mathcal{G}_{n-8}$ with $n - 8 \geq 3$.

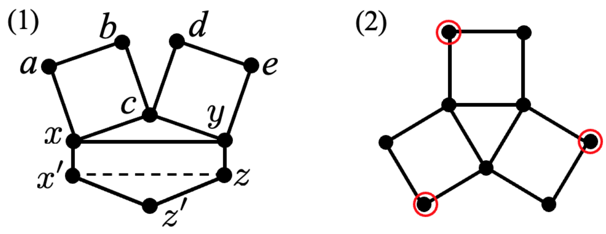

Fig. 5. (1) Case when $x'z' \in E(C)$, (2) Minimum graph with short chords

Suppose that G' has a bipartite dominating set S' with $|S'| \leq \lceil \frac{3}{8}(n-8) \rceil$. Depending on whether x' and z' are contained in S', we have the following cases (see Figure 6):

(1) If $x' \notin S'$ and $z' \notin S'$, then let $S = S' \cup \{a, d, y\}$.

(2) If $x' \in S'$ and $z' \notin S'$, then let $S = S' \cup \{b, e, z'\}$ if z' is not dominated by S' in $G' - x'z'$. We note that $G[S]$ has no odd cycle in G, since z' is not dominated by S' by the assumption. If z' has a neighbor in S', then let $S = S' \cup \{b, e, y\}$.

(3) If $z' \in S'$, then let $S = S' \cup \{x, b, e\}$.

Then $G'[S']$ is bipartite, and thus S is a required dominating set of G with $|S| \leq \lceil \frac{3}{8}n \rceil$. $\square$

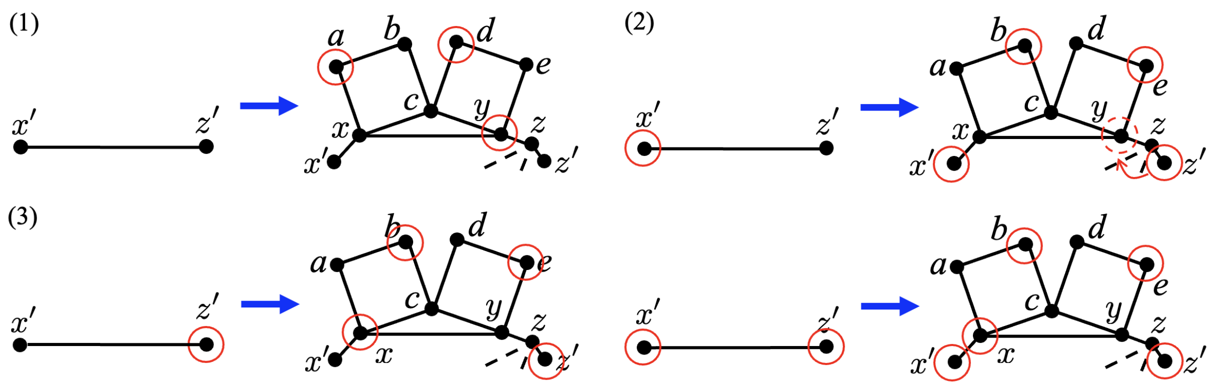

Fig. 6. Construction of a bipartite dominating set S in G from S'

Now we prove Theorem 1.2.

Proof of Theorem 1.2. Let $G \in \mathcal{G}_n$ be a minimal couterexample of Theorem 1.2. Hence G has none of the $\frac{3}{8}$-reducible configurations $\mathcal{Z}$, $\mathcal{TX}$, $\mathcal{TT}$, $\mathcal{TQ}$, $\mathcal{XXXX}$, $\mathcal{QX}$, $\mathcal{QQQ}$, $\tilde{\mathcal{P}}$. (For, if G contains one of them, then $\gamma_{bip}(G) \leq \lceil \frac{3}{8}n \rceil$ or we can take a smaller 2-connected outerplanar graph G'. Since G' satisfies the conclusion by the minimality, so does G, by the property of a $\frac{3}{8}$-reducible configuration, a contradiction.) Let C be a unique Hamiltonian cycle of G.

We first suppose that G has no long chord of C. If G has no short chord either, then G is an n-cycle, which clearly satisfies $\gamma_{bip}(G) \leq \lceil \frac{n}{3} \rceil \leq \lceil \frac{3}{8}n \rceil$. So G has short chords ℓ, but we observe that ℓ has no triangle. (For, a short chord $\ell = xb$ has a triangle xab. Then either $\deg(b) = 3$ or $\deg(b) \geq 4$. In the former case, G contains either $\mathcal{TX}$ or $\mathcal{Z}$, a contradiction. In the latter case, another short chord is incident to b, and hence G contains either $\mathcal{TT}$, $\mathcal{TQ}$ or $\mathcal{Z}$, a contradiction.) Therefore, we may suppose that every short chord has a quadrilateral. Since G contains neither $\mathcal{QX}$ nor $\mathcal{QQQ}$, G must be isomorphic to the one in Figure 5(2), which has nine vertices with a bipartite dominating set with cardinality 3. This contradicts that G is a counterexample.

Secondly we suppose that $G \in \mathcal{G}_n$ has a *long* chord xy. Suppose that xy is chosen in G to be *outermost*, that is, at least one of L and L', say L, has no long chord joining two vertices in L, except xy, where L and L' are the distinct paths in C joining x and y such that $L \cup L' = C$. In this case, L is the *C-path cut by xy*.

Now we show the following claim:

Claim. The subgraph of G induced by $V(L)$ is isomorphic to $\mathcal{P}$ with feet x, y.

Proof. Since xy is an outermost long chord of C in G, the C-path L cut by xy has at least five vertices including x, y. If L has no short chord, then L has at least three consecutive 2-vertices, contrary to G not having $\mathcal{XXXX}$. So L has short chords. Since all short chords have quadrilaterals but G contains neither $\mathcal{QX}$ nor $\mathcal{QQQ}$, L has exactly two short chords each of which has a quadrilateral. Therefore, $G[V(L)] = \mathcal{P}$. □

We apply Lemma 2.4 to the configuration $\mathcal{P}$ with feet x, y which is found by the Claim. Then we may assume that $\deg(x) = \deg(y) = 4$ in $\mathcal{P}$, with the vertices of $\mathcal{P}$ as in Figure 4(1). Let $x' \neq a$ be the neighbor of x in C. Let $z \neq e$ be the neighbor of y in C, and let $z' \neq y$ be the neighbor of z in C, as in the left of Figure 4(2). Since xy is a long chord of C, the vertices x', x, y, z, z' are all distinct, and hence G has the configuration $\tilde{\mathcal{P}}$. This contradicts the assumption on G.

Therefore, every $G \in \mathcal{G}_n$ satisfies that $\gamma_{bip}(G) \leq \lceil \frac{3}{8}n \rceil$. □

3. Conclusion

We would like to give a remark on what was wrong in Theorem 1.1 claimed in [4]. They established the six $\frac{1}{3}$-reducible configurations $\mathcal{Z}$, $\mathcal{TX}$, $\mathcal{TT}$, $\mathcal{TQ}$, $\mathcal{XXXX}$ and $\mathcal{QX}$, distinguishing short and long chords of a Hamiltonian cycle C of a 2-connected outerplanar graph G, and fixed the graph D_m without those configurations, which is obtained from a $3m$-cycle $a_1b_1c_1a_2b_2c_2 \cdots a_mb_mc_m$ with short chords a_ia_{i+1} for each i and several long chords, where the subscripts are taken modulo m. Then $S = \{b_1, \ldots, b_m\}$ is a bipartite dominating set of D_m with $|S| = \frac{|V(D_m)|}{3} = m$. Figure 7(1) shows D_{12} and its bipartite dominating set with $m = 12$ vertices.

However, a 2-connected outerplanar graph G might contain a quadrilateral $xabc$ with a short chord xc and a long chord incident to c (shown in Figure 7(2)), since this configuration is not $\mathcal{QX}$ by the degree condition of c. Then using these configurations, we can construct the 2-connected outerplanar graph G_m with $m = 8n$, shown in Figure 1(2), such that $\gamma_{bip}(G) = \frac{3n}{8} > \frac{n}{3}$ for $n \equiv 0 \bmod 8$. Consequently, we establish Theorem 1.2, giving a sharp bound $\gamma_{bip}(G) \leq \lceil \frac{3}{8}n \rceil$ for a 2-connected n-vertex outerplanar graph G.

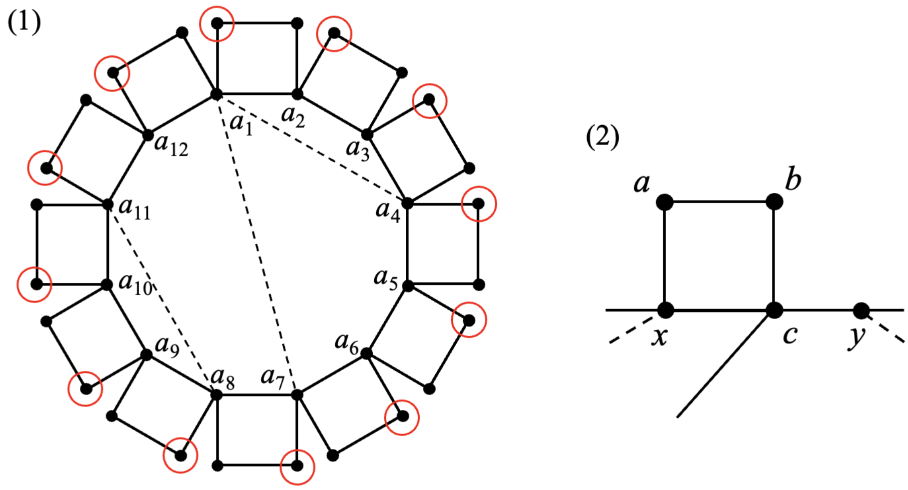

Fig. 7. (1) 2-Connected outerplanar graph D_m with a bipartite domination number m, (2) $\mathcal{QX}$ with a long chord incident to c

References

[1] A. Bachstein, W. Goddard, and M. A. Henning. Bipartite domination in graphs. *Mathematica Pannonica*, 28(S2):118–126, 2022. https://doi.org/10.1556/314.2022.00015.

[2] W. Goddard and M. A. Henning. Independent domination in outerplanar graphs. *Discrete Applied Mathematics*, 325:52–57, 2023. https://doi.org/10.1016/j.dam.2022.10.003.

[3] T. W. Haynes, S. T. Hedetniemi, and M. A. Henning. *Domination in Graphs: Core Concepts*. Springer Monographs in Mathematics. Springer, Cham, 1st edition, 2023. https://doi.org/10.1007/978-3-031-09496-5.

[4] C. Xi and J. Yue. Bipartite domination in outerplanar graphs. In Y. Chen, X. Gao, X. Sun, and A. Zhang, editors, *Computing and Combinatorics*, volume 15161 of *Lecture Notes in Computer Science*, pages 314–325, Singapore. Springer, 2025. https://doi.org/10.1007/978-981-96-1090-7_26. Proceedings of the 30th International Conference on Computing and Combinatorics (COCOON 2024).

Atsuhiro Nakamoto
Faculty of Environment and Information Sciences, Yokohama National University
Yokohama 240-8501, Japan
E-mail nakamoto@ynu.ac.jp