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On a class of intersection graphs equivalent to interval graphs and their interval numbers

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ABSTRACT

A graph G is said to be a an interval graph, if for each vertex u of G, one can assign a set A_u which is a finite union of intervals on the real line such that u is adjacent to v in G if and only if $A_u \cap A_v \neq \varnothing$. In this paper, we introduce a class of intersection graphs and show that it is equivalent to the class of interval graphs. We also investigate interval numbers of certain intersection graphs and establish several related results.

Keywords: interval number, interval graphs, intersection graphs

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1. Introduction

In recent years, intersection graph classes have been widely studied beyond classical interval graphs, both from structural and algorithmic perspectives. A prominent direction concerns multiple-interval and t-interval graphs, where each vertex is represented by a bounded union of intervals on the real line. These generalisations have been studied extensively from an algorithmic perspective, including approximation algorithms and hardness results (see, e.g., [7, 2]), as well as in recent structural developments involving interval representations (see, e.g., [1, 12]). In parallel, geometric intersection graph classes have been extensively studied, particularly through representations by curves, segments, and paths in the plane. Notable developments include intersection representations by grid paths and their generalisations, as well as broader frameworks such as H-graphs, which unify many classical graph classes (see, e.g., [4, 5, 6].

In this paper, we investigate a class of intersection graphs equivalent to interval graphs and their interval numbers. All the graphs are simple and finite. The vertex set and edge

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set of a graph G will be denoted by $V(G)$ and $E(G)$, respectively. We write $u \sim_G v$ if uv is an edge in $E(G)$.

Let $\mathbb{R}$ be the set of real numbers. An interval of $\mathbb{R}$ is denoted by

$$[a,b] = \{x \in \mathbb{R} \ : \ a \leq x \leq b\}.$$

We allow $a = b$ and in this scenario $[a,a] = \{a\}$ is just a singleton. A disjoint union of m-intervals, say A, is of the following form

$$A = [a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_m, b_m],$$

where $b_j < a_{j+1}$ for $1 \leq j \leq m-1$. Furthermore, we say $[a_j, b_j]$ is an interval appearing in A. For each integer $k \geq 1$, let

$$\mathcal{P}_k = \{A \subseteq \mathbb{R} \ : \ A \ \text{ is a disjoint union of at most } k\text{-intervals}\}.$$

Note that $\mathcal{P}_{k'} \subseteq \mathcal{P}_k$ if $k' \leq k$. Set $\mathcal{P} = \bigcup_{k \geq 1} \mathcal{P}_k$. Let V be a finite set. For each $u \in V$, let $A_u \in \mathcal{P}$. We allow $A_u = A_v$ for $u \neq v$. The graph $\mathfrak{D}\left(\{A_u\}_{u \in V}\right)$ is defined to be the graph with vertex set V and for all $u, v \in V$, u is adjacent to v if and only if $A_u \cap A_v \neq \varnothing$. If $A_u \in \mathcal{P}_k$ for all $u \in V$, then $\mathfrak{D}\left(\{A_u\}_{u \in V}\right)$ is called a k-interval graph. A graph G is said to have a k-interval representation if there exists $\mathcal{A} = \{A_u\}_{u \in V(G)} \subseteq \mathcal{P}_k$ such that $G = \mathfrak{D}(\mathcal{A})$.

Given a graph G, the interval number $i(G)$ is defined to be the smallest positive integer k such that G has a k-interval representation, i.e.,

$$i(G) = \min\left\{k \in \mathbb{N} \ : \ G = \mathfrak{D}(\{A_u\}_{u \in V(G)}) \ \text{ for some } \ \{A_u\}_{u \in V(G)} \subseteq \mathcal{P}_k\right\}.$$

The interval number was introduced by Trotter and Harary [14] in 1979. Later, Scheinerman and West [13] constructed planar graphs with interval number at least 3 and proved the following theorem.

Theorem 1.1. *If G is planar, then $i(G) \leq 3$.*

However, there is a flaw in the proof and this is pointed out by Guégan et al. [9]. An alternative proof of Theorem 1.1 is given by Guégan et al. [10] recently. We refer readers to [3] and [8] for interval numbers of other classes of planar graphs and general graphs.

In this paper, we will consider intersection graph, which we will describe shortly. Let $\mathbb{N}$ be the set of positive integers and let

$$2_0^{\mathbb{N}} = \{A \subseteq \mathbb{N} \ : \ A \ \text{ is finite}\}.$$

Let V be a finite set. For each $u \in V$, let $B_u \in 2_0^{\mathbb{N}}$. We allow $B_u = B_v$ for $u \neq v$. The graph $\mathfrak{F}\left(\{B_u\}_{u \in V}\right)$ is defined to be the graph with vertex set V and for all $u, v \in V$, u is adjacent to v if and only if $B_u \cap B_v \neq \varnothing$. The graph $\mathfrak{F}\left(\{B_u\}_{u \in V}\right)$ is called an intersection graph. The class of all the intersection graphs is denoted by $\mathcal{T}_1$. Basically, a graph G is in $\mathcal{T}_1$ if there exists $\mathcal{B} = \{B_u\}_{u \in V(G)} \subseteq 2_0^{\mathbb{N}}$ such that $G = \mathfrak{F}(\mathcal{B})$.

For each $B \in 2_0^{\mathbb{N}}$ and $B \neq \varnothing$, we define $g(B) = \{X_1, X_2, \ldots, X_l\}$ to be the partition of B such that

(i) $B = \bigcup_{i=1}^{l} X_i$,

(ii) $X_i \cap X_j = \varnothing$ for $i \neq j$,

(iii) for each i, the set X_i consists of consecutive integers, i.e., $X_i = \{a_i, a_i+1, \ldots, a_i+s_i\}$,

(iv) for each i $(1 \leq i \leq l-1)$, $a < b$ for all $a \in X_i$ and $b \in X_{i+1}$,

(v) for each i $(1 \leq i \leq l-1)$, $X_i \cup X_{i+1}$ is not a set consists of consecutive integers.

Conditions (i) and (ii) are definition for partition, whereas conditions (iii), (iv) and (v) are used to show that $g(B)$ is unique. By conditions (iv) and (v), if $a_i + s_i$ is the largest integer in X_i and a_{i+1} is the smallest integer in X_{i+1}, then $a_i + s_i + 1 < a_{i+1}$. Note that $|g(B)|= l$ is the number of elements of the partition of B satisfying conditions (i) to (v). Each X_i is called a consecutive part of the partition $g(B)$. If $X_i = \{a_i, a_i + 1, \ldots, a_i + s_i\}$, then a_i is called the initial point of the consecutive part X_i and $a_i + s_i$ is called the terminal point of the consecutive part X_i. An element in X_i is called a middle point if it is not an initial point or a terminal point. For convenience if $B = \varnothing$, we set $g(B) = \varnothing$. Note that if $B = \{1, 2, 4, 5, 7, 9, 10, 11, 15, 16, 17, 18\}$, then $g(B) = \{\{1, 2\}, \{4, 5\}, \{7\}, \{9, 10, 11\}, \{15, 16, 17, 18\}\}$. Furthermore, $\{1, 2\}$ is a consecutive part of $g(B)$ and 1 is the initial point of $\{1, 2\}$ whereas 2 is the terminal point of $\{1, 2\}$. Note that 7 is the initial point and also the terminal point of the consecutive part $\{7\}$.

For $k \geq 1$, let

$$2_0^{\mathbb{N}}(k) = \{A \in 2_0^{\mathbb{N}} \ : \ |g(A)| \leq k\}.$$

Let

$$\mathcal{T}_1(k) = \left\{\mathfrak{F}(\mathcal{B}) \ : \ \mathcal{B} = \{B_u\}_{u \in V} \subseteq 2_0^{\mathbb{N}}(k) \ \text{ for some finite set } V\right\}.$$

Note that $\mathcal{T}_1 = \bigcup_{k \geq 1} \mathcal{T}_1(k)$ and $\mathcal{T}_1(k') \subseteq \mathcal{T}_1(k)$ if $k' \leq k$.

In Section 2, we showed that interval graphs are equivalent to intersection graph (Theorem 2.8). In Section 3, we introduce the class of 1-intersection graphs, $\mathcal{T}_2$ and define the $\mathcal{T}_2$-interval number, $i_{\mathcal{T}_2}(G)$. We determine all the graphs with $i_{\mathcal{T}_2}(G) = 1$ (Theorem 3.10). In Section 4, we showed if G is planar, then $i_{\mathcal{T}_2}(G) \leq 3$ (Theorem 4.8). Finally, in Section 5, we introduce the minimum size number and use it to characterize the triangle free graphs (Theorem 5.5).

2. Intersection graph

Recall that we allow an interval to be of the form $[a, a] = \{a\}$. We shall show that if G has a k-interval representation, i.e., $G = \mathfrak{D}(\{A_u\}_{u \in V(G)})$ for some $\{A_u\}_{u \in V(G)} \subseteq \mathcal{P}_k$, then all the A_u can be chosen to be singleton free (all intervals appearing in A_u are not singleton) (Lemma 2.2). In this section, we also will prove Theorem 2.8 which states that a graph has a k-interval representation if and only if it is in $\mathcal{T}_1(k)$.

Lemma 2.1. *Let Q be the union of t intervals. If $a \notin Q$, then there exists an $\epsilon > 0$ such that*

$$[a - \epsilon, a + \epsilon] \cap Q = \varnothing.$$

Proof. Let $Q = \bigcup_{i=1}^{t}[c_i, d_i]$. Since $a \notin [c_i, d_i]$, there exists an $\epsilon_i > 0$ such that $[a - \epsilon_i, a + \epsilon_i] \cap [c_i, d_i] = \varnothing$. Let $\epsilon = \min_{1 \le i \le t} \epsilon_i$. Then, $[a - \epsilon, a + \epsilon] \cap Q = \varnothing$. □

Let $\mathcal{A} = \{A_u\}_{u \in V} \subseteq \mathcal{P}_k$. Note that for each $u \in V$,

$$A_u = [a_{u1}, b_{u1}] \cup [a_{u2}, b_{u2}] \cup \cdots \cup [a_{uk_u}, b_{uk_u}],$$

where $k_u \le k$ and $b_{uj} < a_{u(j+1)}$ for $1 \le j \le k_u - 1$. Suppose there exists a $u_0 \in V$ with $c = a_{u_0 j_0} = b_{u_0 j_0}$. Let Q be the union of all the intervals $[a_{uj}, b_{uj}]$ for which $c \notin [a_{uj}, b_{uj}]$. By Lemma 2.1, we can replace $[a_{u_0 j_0}, b_{u_0 j_0}] = \{c\}$ in A_{u_0} with $[c - \epsilon, c + \epsilon]$ and $[c - \epsilon, c + \epsilon] \cap Q = \varnothing$. Let

$$\begin{aligned} A'_{u_0} = &[a_{u_0 1}, b_{u_0 1}] \cup \cdots \cup [a_{u_0 (j_0 - 1)}, b_{u_0 (j_0 - 1)}] \cup [c - \epsilon, c + \epsilon] \cup \\ &[a_{u_0 (j_0 + 1)}, b_{u_0 (j_0 + 1)}] \cup \cdots \cup [a_{u_0 k_{u_0}}, b_{u_0 k_{u_0}}]. \end{aligned}$$

Note that $\mathfrak{D}(\mathcal{A}) = \mathfrak{D}(\mathcal{A}')$ where $\mathcal{A}' = \{A_u\}_{u \in V \setminus \{u_0\}} \cup \{A'_{u_0}\}$. Therefore, given an k-interval graph $\mathfrak{D}(\{A_u\}_{u \in V})$, by applying Lemma 2.1 repeatedly, we may obtain the same graph $\mathfrak{D}(\{A'_u\}_{u \in V}) = \mathfrak{D}(\{A_u\}_{u \in V})$ and each interval in A'_u is not a singleton for all $u \in V$. Hence, we have proved the following lemma.

Lemma 2.2. *Let $\mathfrak{D}(\{A_u\}_{u \in V})$ be a k-interval graph. There exists $\{A'_u\}_{u \in V}$ such that*

(a) *for all $u \in V$, if $A_u \in \mathcal{P}_k$, then $A'_u \in \mathcal{P}_k$,*

(b) *for all $u \in V$, all intervals appearing in A'_u are not singleton,*

(c) $\mathfrak{D}(\{A'_u\}_{u \in V}) = \mathfrak{D}(\{A_u\}_{u \in V})$.

Now, let V be a finite set and $\mathcal{B} = \{B_u\}_{u \in V} \subseteq 2_0^{\mathbb{N}}(k)$. Then, $|g(B_u)| \le k$ for all $u \in V$. Let $g(B_u) = \{X_{u1}, X_{u2}, \ldots, X_{uk_u}\}$. Then, $k_u \le k$. For each i, let $X_{ui} = \{a_{ui}, a_{ui} + 1, \ldots, a_{ui} + s_{ui}\}$. Now, for each $u \in V$, set

$$A_u = [a_{u1}, a_{u1} + s_{u1}] \cup [a_{u2}, a_{u2} + s_{u2}] \cup \cdots \cup [a_{uk_u}, a_{uk_u} + s_{uk_u}].$$

Note that A_u is a disjoint union of k_u intervals and $A_u \in \mathcal{P}_k$. Furthermore, $A_u \cap A_v \neq \varnothing$ if and only if $B_u \cap B_v \neq \varnothing$. This implies that $\mathfrak{D}(\mathcal{A}) = \mathfrak{F}(\mathcal{B})$ where $\mathcal{A} = \{A_u\}_{u \in V}$. The following lemma then follows from Lemma 2.2.

Lemma 2.3. *If $G \in \mathcal{T}_1(k)$, then there exists $\mathcal{A} = \{A_u\}_{u \in V(G)} \subseteq \mathcal{P}_k$ such that $G = \mathfrak{D}(\mathcal{A})$. Furthermore, all intervals appearing in each A_u are not singleton.*

The next two lemmas explain what will happen to $g(B)$ if an element is added or removed from B.

Lemma 2.4. *Let $B \in 2_0^{\mathbb{N}}$ be non-empty and $c \in \mathbb{N} \setminus B$. Then the followings hold:*

(a) *if $c + 1$ is not the initial point of a consecutive part of $g(B)$ and $c - 1$ is not the terminal point of a consecutive part of $g(B)$, then $|g(B \cup \{c\})| = |g(B)| + 1$,*

(b) *if either $c + 1$ is the initial point of a consecutive part of $g(B)$ or $c - 1$ is the terminal point of a consecutive part of $g(B)$ but not both, then $|g(B \cup \{c\})| = |g(B)|$,*

(c) *if $c+1$ is the initial point of a consecutive part of $g(B)$ and $c-1$ is the terminal point of a consecutive part of $g(B)$, then $|g(B\cup\{c\})|=|g(B)|-1$.*

Proof. (a) Note that $\{c\}$ is a consecutive part of $g(B\cup\{c\})$ and each consecutive part of $g(B)$ is also a consecutive part of $g(B\cup\{c\})$. Hence, $|g(B\cup\{c\})|=|g(B)|+1$.

(b) Suppose $c+1$ is the initial point of a consecutive part of $g(B)$ and $c-1$ is not the terminal point of a consecutive part of $g(B)$. This implies that $c-1\notin B$. Note that if $c+1$ is the initial point of a consecutive part of $g(B)$, say X, then $X\cup\{c\}$ is also a consecutive part of $g(B\cup\{c\})$ for $c-1\notin B$. Each consecutive part of $g(B)$ that do not contain $c+1$ is also a consecutive part of $g(B\cup\{c\})$. Hence, $|g(B\cup\{c\})|=|g(B)|$. Similarly, if $c-1$ is the terminal point of a consecutive part of $g(B)$ and $c+1$ is not the initial point of a consecutive part of $g(B)$, then $|g(B\cup\{c\})|=|g(B)|$.

(c) If $c+1$ is the initial point of a consecutive part of $g(B)$, say X_1, and $c-1$ is the terminal point of a consecutive part of $g(B)$, say X_2, then $X_1\cup X_2\cup\{c\}$ is a consecutive part of $g(B\cup\{c\})$. Each consecutive part of $g(B)$ that do not contain $c+1$ or $c-1$ is also a consecutive part of $g(B\cup\{c\})$. Hence, $|g(B\cup\{c\})|=|g(B)|-1$. □

Lemma 2.5. *Let $B\in 2^{\mathbb{N}}_0$ be non-empty and $c\in B$. Then the following hold:*

(a) *if c is the initial point or the terminal point of a consecutive part of $g(B)$, then $|g(B\setminus\{c\})|=|g(B)|$,*

(b) *if c is a middle point of a consecutive part of $g(B)$, then $|g(B\setminus\{c\})|=|g(B)|+1$.*

Proof. (a) Suppose c is the initial point of a consecutive part of $g(B)$, say X. Then $X=\{c,c+1,\dots,c+m\}$ for some non-negative integer m. Note that $X\setminus\{c\}$ is a consecutive part of $g(B\setminus\{c\})$. Each consecutive part of $g(B)$ that do not contain c is also a consecutive part of $g(B\setminus\{c\})$. Hence, $|g(B\setminus\{c\})|=|g(B)|$. Similarly, the same conclusion holds if c is the terminal point of a consecutive part of $g(B)$.

(b) Let X be the consecutive part of $g(B)$ that contains c. Since c is a middle point of X, we have $X=\{c-m_1,\dots,c-1,c,c+1,\dots,c+m_2\}$ for some positive integers m_1,m_2. Note that $\{c-m_1,\dots,c-1\}$ and $\{c+1,\dots,c+m_2\}$ are consecutive parts of $g(B\setminus\{c\})$. Each consecutive part of $g(B)$ that do not contain c is also a consecutive part of $g(B\setminus\{c\})$. Hence, $|g(B\setminus\{c\})|=|g(B)|+1$. □

For each $B\in 2^{\mathbb{N}}_0$, $B\neq\varnothing$ and any integer p, we set

$$B+p=\{b+p\ :\ b\in B\}.$$

Lemma 2.6. *Let $B_1,B_2\in 2^{\mathbb{N}}_0$ be non-empty and p be an integer. Then $B_1\cap B_2=\varnothing$ if and only if $(B_1+p)\cap(B_2+p)=\varnothing$. Furthermore, if $B_1\cap B_2\neq\varnothing$, then $(B_1+p)\cap(B_2+p)=(B_1\cap B_2)+p$.*

Proof. Suppose $B_1\cap B_2\neq\varnothing$. Then, $((B_1\cap B_2)+p)\subseteq(B_1+p)\cap(B_2+p)$. Let $y\in(B_1+p)\cap(B_2+p)$. Then, $y=b_1+p=b_2+b$ for some $b_1\in B_1$ and $b_2\in B_2$. This implies that $b_1=b_2\in B_1\cap B_2$. Hence, $y\in(B_1\cap B_2)+p$ and $(B_1+p)\cap(B_2+p)=(B_1\cap B_2)+p$.

If $(B_1+p)\cap(B_2+p)\neq\varnothing$, then by the previous paragraph, $B_1\cap B_2=((B_1+p)\cap(B_2+p))-p\neq\varnothing$. This completes the proof of the lemma. $\square$

Lemma 2.7. *Let $[a,b]$ be an interval in $\mathbb{R}$ and $c\in\mathbb{R}$ such that $a<c\leq b$. If $[z_1,z_2]$ is an interval in $\mathbb{R}$ such that $[z_1,z_2]\cap[a,b]\neq\varnothing$ and $z_1\geq c$, then $[z_1,z_2]\cap[c,b]\neq\varnothing$.*

Proof. Since $[z_1,z_2]\cap[a,b]\neq\varnothing$, there is a $y\in[z_1,z_2]\cap[a,b]$. This implies that $y\geq z_1\geq c$. Since $y\leq b$, we have $y\in[c,b]$. Hence, $y\in[z_1,z_2]\cap[c,b]$ and $[z_1,z_2]\cap[c,b]\neq\varnothing$. $\square$

Given a graph G and a vertex $u\in V(G)$, the graph obtained from G by removing u and all edges incident to it is denoted by $G-u$. If $S\subseteq V(G)$, then $G-S$ is the graph obtained by removing all vertices in S and all edges incident to them from G. We write $G[S]$ to denote the induced subgraph of G with vertex set S.

For each $A\subseteq\mathcal{P}$ and

$$A=[x_1,y_1]\cup[x_2,y_2]\cup\cdots\cup[x_m,y_m],$$

where $y_i<x_{i+1}$ for $1\leq i\leq m-1$, we set $\min A=x_1$.

Theorem 2.8. *A graph G has a k-interval representation if and only if $G\in\mathcal{T}_1(k)$.*

Proof. By Lemma 2.3, it is sufficient to show that if $G=\mathfrak{D}(\mathcal{A})$ for some $\mathcal{A}=\{A_u\}_{u\in V(G)}\subseteq\mathcal{P}_k$, then $G\in\mathcal{T}_1(k)$. Let us first introduce some notations. For each $u\in V(G)$, let $t(A_u)$ be the number of disjoint intervals in A_u. Since $A_u\in\mathcal{P}_k$, $t(A_u)\leq k$ for all $u\in V(G)$. We shall prove the following statement.

Statement: Let $G=\mathfrak{D}(\mathcal{A})$ for some $\mathcal{A}=\{A_u\}_{u\in V(G)}\subseteq\mathcal{P}_k$. If $c=\min\{\min A_u : u\in V(G)\}$, then there exists $\mathcal{B}=\{B_u\}_{u\in V(G)}\subseteq 2_0^{\mathbb{N}}(k)$ such that $G=\mathfrak{F}(\mathcal{B})\in\mathcal{T}_1(k)$, $|g(B_u)|\leq t(A_u)$ for all $u\in V(G)$ and $1\in B_v$ for all $v\in V(G)$ satisfying $\min A_v=c$.

We shall prove the statement by induction on $T=\sum\limits_{u\in V(G)}t(A_u)$. Note that $T\geq|V(G)|$ and equality holds if and only if $t(A_u)=1$ for all u.

If $T=1$, then G consists of a single vertex, say u_0. Set $B_{u_0}=\{1\}$. Then, the statement holds. So, we may assume that $T\geq 2$. Assume that the statement holds for any graph $G'=\mathfrak{D}(\{A'_u\}_{u\in V(G')})$ with $\{A'_u\}_{u\in V(G')}\subseteq\mathcal{P}_k$ satisfying $\sum\limits_{u\in V(G')}t(A'_u)<T$.

For each $u\in V(G)$, let

$$A_u=[x_{u1},y_{u1}]\cup[x_{u2},y_{u2}]\cup\cdots\cup[x_{uk_u},y_{uk_u}],$$

where $y_{ui}<x_{u(i+1)}$ for $1\leq i\leq k_u-1$. Then, $t(A_u)=k_u\leq k$ and $\min A_u=x_{u1}$.

Let

$$W(\{A_u\}_{u\in V(G)})=\bigcup_{u\in V(G)}\{x_{u1},y_{u1},x_{u2},y_{u2},\ldots,x_{uk_u},y_{uk_u}\}.$$

If $|W(\{A_u\}_{u\in V(G)})|=1$, then $k_u=1$ and $x_{u1}=y_{u1}=c$ for all $u\in V(G)$. This implies that $G=\mathfrak{D}(\mathcal{A})$ is the complete graph K_n where $n=|V(G)|$. Let $B_u=\{1\}$ for all u

and $\mathcal{B} = \{B_u\}_{u\in V(G)}$. Then, $\mathcal{B} \subseteq 2_0^{\mathbb{N}}(1)$ and $\mathfrak{F}(\mathcal{B})$ is the complete graph K_n. Hence, $G = \mathfrak{F}(\mathcal{B}) \in \mathcal{T}_1(1)$, $|g(B_u)|= t(A_u) = 1$ for all $u \in V(G)$ and $1 \in B_v$ for all v satisfying $x_v = c$.

Suppose $w_0 = |W(\{A_u\}_{u\in V(G)})|> 1$. We shall assume that the statement holds for any graph $G' = \mathfrak{D}(\{A'_u\}_{u\in V(G')})$ with $\{A'_u\}_{u\in V(G')} \subseteq \mathcal{P}_k$ satisfying $\sum\limits_{u\in V(G')} t(A'_u) = T$ and $1 \leq |W(\{A'_u\}_{u\in V(G')})|< w_0$. Note that $c = \min\{x_{u1} \ : \ u \in V(G)\}$. Let $u_1, u_2, \dots, u_r$ be the only vertices in $V(G)$ with $x_{u_j1} = c$. By relabelling if necessary we may assume that $y_{u_11} \leq y_{u_21} \leq \cdots \leq y_{u_r1}$.

Case 1. Suppose $x_{u_j1} = y_{u_j1}$ for all $1 \leq j \leq p$ where $1 \leq p \leq r$ and $c = x_{u_j1} < y_{u_j1}$ for $p+1 \leq j \leq r$.

Suppose $r = p$. We may assume that $t(A_{u_j}) = k_{u_j} = 1$ for $1 \leq j \leq q$ and $t(A_{u_j}) = k_{u_j} \geq 2$ for $q+1 \leq j \leq p$. For $q+1 \leq j \leq p$, let $A'_{u_j} = A_{u_j} \setminus \{c\}$. Then,

$$A'_{u_j} = [x_{u_j2}, y_{u_j2}] \cup [x_{u_j3}, y_{u_j3}] \cup \cdots \cup [x_{u_jk_{u_j}}, y_{u_jk_{u_j}}],$$

and $t(A'_{u_j}) = k_{u_j} - 1$. For other $u \in V(G) \setminus \{u_1, u_2, \dots, u_p\}$, we set $A'_u = A_u$. Now, let $G' = \mathfrak{D}(\{A'_u\}_{u\in V(G')})$. Note that G' is a subgraph of G, $V(G') = V(G) \setminus \{u_1, u_2, \dots, u_q\}$ and $\sum\limits_{u\in V(G')} t(A'_u) = T-p < T$. By induction, there exists $\mathcal{B}'' = \{B''_u\}_{u\in V(G')} \subseteq 2_0^{\mathbb{N}}(k)$ such that $G' = \mathfrak{F}(\mathcal{B}'') \in \mathcal{T}_1(k)$, $|g(B''_u)|\leq t(A'_u)$ for all $u \in V(G')$ and $1 \in B''_v$ for all v satisfying $\min A'_v = c'$ where $c' = \min\{\min A'_u \ : \ u \in V(G')\}$. Now, we need to 'add back' vertices $u_1, u_2, \dots, u_q$ to G' to become G. Note that for $q+1 \leq j \leq p$, $|g(B''_{u_j})|\leq k_{u_j} - 1$. Set $B'_u = B''_u + 1$ for all $u \in V(G')$. Clearly, $|g(B'_u)|= |g(B''_u)|$. By Lemma 2.6, $G' = \mathfrak{F}(\{B'_u\}_{u\in V(G')})$. Now, we set $B_{u_j} = \{1\}$ for $1 \leq j \leq q$, $B_{u_j} = \{1\} \cup B'_{u_j}$ for $q+1 \leq j \leq p$ and $B_u = B'_u$ for other u in $V(G) \setminus \{u_1, u_2, \dots, u_p\}$. Then, $|g(B_{u_j})|= t(A_{u_j}) = 1$ for $1 \leq j \leq q$. By Lemma 2.4, $|g(B_{u_j})|\leq |g(B'_{u_j})|+1 \leq (k_{u_j} - 1) + 1 = t(A_{u_j})$ for $q+1 \leq j \leq p$. For other $u \in V(G) \setminus \{u_1, u_2, \dots, u_p\}$, $|g(B_u)|= |g(B'_u)|\leq t(A'_u) = t(A_u)$. Now, $1 \in B_{u_j}$ for all $1 \leq j \leq p$, $\mathcal{B} = \{B_u\}_{u\in V(G)} \subseteq 2_0^{\mathbb{N}}(k)$ and $G = \mathfrak{F}(\mathcal{B}) \in \mathcal{T}_1(k)$. Hence, the statement holds for the graph G.

Suppose $r > p$. By Lemma 2.1, there is an $\epsilon > 0$ such that $[c-\epsilon, c+\epsilon] \cap A_u = \varnothing$ for all $u \in V(G) \setminus \{u_1, u_2, \dots, u_r\}$. The ϵ can be chosen so that $c + \epsilon < y_{u_j1}$ for $p+1 \leq j \leq r$. Now, for $p+1 \leq j \leq r$, let

$$A'_{u_j} = [c+\epsilon, y_{u_j1}] \cup [x_{u_j2}, y_{u_j2}] \cup [x_{u_j3}, y_{u_j3}] \cup \cdots \cup [x_{u_jk_{u_j}}, y_{u_jk_{u_j}}].$$

By relabelling if necessary, we may assume that $t(A_{u_j}) = k_{u_j} = 1$ for $1 \leq j \leq q$ and $t(A_{u_j}) = k_{u_j} \geq 2$ for $q+1 \leq j \leq p$. For $q+1 \leq j \leq p$, let $A'_{u_j} = A_{u_j} \setminus \{c\}$. Then,

$$A'_{u_j} = [x_{u_j2}, y_{u_j2}] \cup [x_{u_j3}, y_{u_j3}] \cup \cdots \cup [x_{u_jk_{u_j}}, y_{u_jk_{u_j}}],$$

and $t(A'_{u_j}) = k_{u_j} - 1$. Finally, for other $u \in V(G)\setminus\{u_1, u_2, \dots, u_r\}$, let $A'_u = A_u$. Note that $t(A'_u) = t(A_u)$ for all $u \in V(G)\setminus\{u_1, u_2, \dots, u_p\}$. Now, let $G' = \mathfrak{D}(\{A'_u\}_{u\in V(G')})$. Note that G' is a subgraph of G, $V(G') = V(G) \setminus \{u_1, u_2, \dots, u_q\}$ and $\sum\limits_{u\in V(G')} t(A'_u) = T - p < T$. By induction, there exists $\mathcal{B}'' = \{B''_u\}_{u\in V(G')} \subseteq 2_0^{\mathbb{N}}(k)$ such that $G' = \mathfrak{F}(\mathcal{B}'') \in \mathcal{T}_1(k)$,

$|g(B''_u)| \le t(A'_u)$ for all $u \in V(G')$ and $1 \in B''_v$ for all v satisfying $\min A'_v = c'$ where $c' = \min\{\min A'_u : u \in V(G')\}$. Note that $c' = c + \epsilon$. Therefore, $1 \in B''_{u_j}$ for $p+1 \le j \le r$. Now, we shall 'add back' vertices $u_1, u_2, \ldots, u_q$ to G' to become G. Note that for $q+1 \le j \le p$, $|g(B''_{u_j})| \le k_{u_j} - 1$. Set $B'_u = B''_u + 1$ for all $u \in V(G')$. Clearly, $|g(B'_u)| = |g(B''_u)|$. By Lemma 2.6, $G' = \mathfrak{F}(\{B'_u\}_{u \in V(G')})$. Now, we set $B_{u_j} = \{1\}$ for $1 \le j \le q$, $B_{u_j} = \{1\} \cup B'_{u_j}$ for $q+1 \le j \le r$ and $B_u = B'_u$ for other u in $V(G) \setminus \{u_1, u_2, \ldots, u_r\}$. Then, $|g(B_{u_j})| = t(A_{u_j}) = 1$ for $1 \le j \le q$. Since $1 \in B''_{u_j}$ for $p+1 \le j \le r$ and $B'_{u_j} = B''_{u_j} + 1$, we have $2 \in B'_{u_j}$ and $1 \notin B'_{u_j}$. By Lemma 2.4, $|g(B_{u_j})| = |g(B'_{u_j})| \le k_{u_j} = t(A_{u_j})$ for $p+1 \le j \le r$. Again, by Lemma 2.4, $|g(B_{u_j})| = |g(B'_{u_j})| \le (k_{u_j} - 1) + 1 = t(A_{u_j})$ for $q+1 \le j \le p$. For other $u \in V(G) \setminus \{u_1, u_2, \ldots, u_r\}$, $|g(B_u)| = |g(B'_u)| \le t(A'_u) = t(A_u)$. Now, $1 \in B_{u_j}$ for all $1 \le j \le r$, $\mathcal{B} = \{B_u\}_{u \in V(G)} \subseteq 2_0^{\mathbb{N}}(k)$ and $G = \mathfrak{F}(\mathcal{B}) \in \mathcal{T}_1(k)$. Hence, the statement holds.

Case 2. Suppose $c = x_{u_j 1} < y_{u_j 1}$ for $1 \le j \le r$.

Assume that there is a vertex u_0 such that $c < x_{u_0} \le y_{u_1 1}$ and $x_{u_0} \le x_u$ for other $u \in V(G) \setminus \{u_1, u_2, \ldots, u_r\}$. For $1 \le j \le r$, let

$$A'_{u_j} = [x_{u_0}, y_{u_j 1}] \cup [x_{u_j 2}, y_{u_j 2}] \cup [x_{u_j 3}, y_{u_j 3}] \cup \cdots \cup [x_{u_j k_{u_j}}, y_{u_j k_{u_j}}],$$

and for other $u \in V(G) \setminus \{u_1, u_2, \ldots, u_r\}$ let $A'_u = A_u$. By Lemma 2.7, $G = \mathfrak{D}(\{A'_u\}_{u \in V(G)})$. Since $t(A'_u) = t(A_u)$ for all $u \in V(G)$, $\sum_{u \in V(G)} t(A'_u) = \sum_{u \in V(G)} t(A_u) = T$. On the other hand, $|W(\{A'_u\}_{u \in V(G)})| = w_0 - 1 < w_0$. By induction, there exists $\mathcal{B} = \{B_u\}_{u \in V(G)} \subseteq 2_0^{\mathbb{N}}(k)$ such that $G = \mathfrak{F}(\mathcal{B}) \in \mathcal{T}_1(k)$, $|g(B_u)| \le t(A_u)$ for all $u \in V(G)$ and $1 \in B_v$ for all v satisfying $\min A'_v = c'$ where $c' = \min\{\min A'_u : u \in V(G)\}$. Note that $c' = x_{u_0}$. So, in particular, $1 \in B_{u_j}$ for all $1 \le j \le r$. Hence, the statement holds.

Suppose for all $u \in V(G) \setminus \{u_1, u_2, \ldots, u_r\}$, $x_u \ge y_{u_1 1}$. For $1 \le j \le r$, let

$$A'_{u_j} = [y_{u_1 1}, y_{u_j 1}] \cup [x_{u_j 2}, y_{u_j 2}] \cup [x_{u_j 3}, y_{u_j 3}] \cup \cdots \cup [x_{u_j k_{u_j}}, y_{u_j k_{u_j}}],$$

and for other $u \in V(G) \setminus \{u_1, u_2, \ldots, u_r\}$ let $A'_u = A_u$. By Lemma 2.7, $G = \mathfrak{D}(\{A'_u\}_{u \in V(G)})$. Since $t(A'_u) = t(A_u)$ for all $u \in V(G)$, $\sum_{u \in V(G)} t(A'_u) = \sum_{u \in V(G)} t(A_u) = T$. On the other hand, $|W(\{A'_u\}_{u \in V(G)})| = w_0 - 1 < w_0$. By induction, there exists $\mathcal{B} = \{B_u\}_{u \in V(G)} \subseteq 2_0^{\mathbb{N}}(k)$ such that $G = \mathfrak{F}(\mathcal{B}) \in \mathcal{T}_1(k)$, $|g(B_u)| \le t(A_u)$ for all $u \in V(G)$ and $1 \in B_v$ for all v satisfying $\min A'_v = c'$ where $c' = \min\{\min A'_u : u \in V(G)\}$. Note that $c' = y_{u_1 1}$. So, in particular, $1 \in B_{u_j}$ for all $1 \le j \le r$. Hence, the statement holds.

This completes the proof of the theorem. □

The $\mathcal{T}_1$-interval number $i_{\mathcal{T}_1}(G)$ is defined as

$$i_{\mathcal{T}_1}(G) = \min\{k \in \mathbb{N} : G \in \mathcal{T}_1(k)\}.$$

The following corollary is an immediate consequence of Theorem 2.8.

Corollary 2.9. *$i(G) = i_{\mathcal{T}_1}(G)$.*

3. $\mathcal{T}_2$-interval number

Recall that the class of all the intersection graphs is denoted by $\mathcal{T}_1$. In this section, we will introduce the class of 1-intersection graphs, $\mathcal{T}_2$. Let V be a finite set. Recall that a graph $G \in \mathcal{T}_1$ if $G = \mathfrak{F}(\{B_u\}_{u\in V})$ for some $\{B_u\}_{u\in V(G)} \subseteq 2_0^{\mathbb{N}}$. Here, $u \sim_G v$ if and only if $B_u \cap B_v \neq \varnothing$. There is no restriction on the number of elements in $B_u \cap B_v$. For the class $\mathcal{T}_2$, we require that $|B_u \cap B_v| \leq 1$. Formally,

$$\begin{aligned}\mathcal{T}_2 = \big\{\mathfrak{F}(\mathcal{B}) \ :\ & \mathcal{B} = \{B_u\}_{u\in V} \subseteq 2_0^{\mathbb{N}} \ \text{ for some finite set } V \text{ and} \\ & |B_u \cap B_v| \leq 1 \text{ for all } u, v \in V \text{ with } u \neq v\big\}.\end{aligned}$$

For each positive integer $k \geq 1$, let

$$\begin{aligned}\mathcal{T}_2(k) = \big\{\mathfrak{F}(\mathcal{B}) \ :\ & \mathcal{B} = \{B_u\}_{u\in V} \subseteq 2_0^{\mathbb{N}}(k) \ \text{ for some finite set } V \text{ and} \\ & |B_u \cap B_v| \leq 1 \text{ for all } u, v \in V \text{ with } u \neq v\big\}.\end{aligned}$$

Note that $\mathcal{T}_2 = \bigcup_{k\geq 1} \mathcal{T}_2(k)$ and $\mathcal{T}_2(k') \subseteq \mathcal{T}_2(k)$ if $k' \leq k$. The following lemma follows from definitions of $\mathcal{T}_1(k)$ and $\mathcal{T}_2(k)$.

Lemma 3.1. *For $k \geq 1$, $\mathcal{T}_2(k) \subseteq \mathcal{T}_1(k)$.*

Lemma 3.2. *$G \in \mathcal{T}_2$ for any graph G.*

Proof. For each uv in $E(G)$, we assign a positive integer c_{uv} to it. The assignment of integers can be done so that $c_{uv} \neq c_{u'v'}$ for distinct edges uv and $u'v'$. Note that $c_{uv} = c_{vu}$ for all $u \sim_G v$. Let $B_u = \{c_{uv} \ :\ v \in N_G(u)\}$ where $N_G(u)$ is the neighbourhood of u. Since all the c_{uv}'s are distinct, we have $B_{u'} \cap B_{v'} = \{c_{u'v'}\}$ for all $u' \sim_G v'$ and $B_{u'} \cap B_{v'} = \varnothing$ for all $u' \not\sim_G v'$. Thus, $G = \mathcal{F}(\{B_u\}_{u\in V(G)})$. Furthermore, $|B_u \cap B_v| \leq 1$ for all $u, v \in V(G)$ implies that $G \in \mathcal{T}_2$. $\square$

The following corollary follows from Lemmas 3.1 and 3.2.

Corollary 3.3. *$\mathcal{T}_1 = \mathcal{T}_2$.*

The $\mathcal{T}_2$-interval number $i_{\mathcal{T}_2}(G)$ is defined as

$$i_{\mathcal{T}_2}(G) = \min\{k \in \mathbb{N} \ :\ G \in \mathcal{T}_2(k)\}.$$

The following lemma follows from Lemma 3.1.

Lemma 3.4. *$i_{\mathcal{T}_1}(G) \leq i_{\mathcal{T}_2}(G)$.*

Note that in general, $i_{\mathcal{T}_2}(G) \neq i_{\mathcal{T}_1}(G)$. See the following example.

Example 3.5. Consider the graph G with the vertex set $V = \{u, v, w, x\}$, and edges given by uv, ux, vx, vw, wx. Let $B_u = \{1\}$, $B_v = \{1, 2\} = B_x$ and $B_w = \{2\}$. Then,

$\mathcal{B} = \{B_z\}_{z\in V} \subseteq 2_0^{\mathbb{N}}(1)$ and $G = \mathfrak{F}(\mathcal{B}) \in \mathcal{T}_1(1)$ (see Figure 1). Since each B_z, $z \in \{u, v, x, w\}$ is a consecutive part, we have $|g(B_z)|= 1$. Thus $i_{\mathcal{T}_1}(G) = 1$.

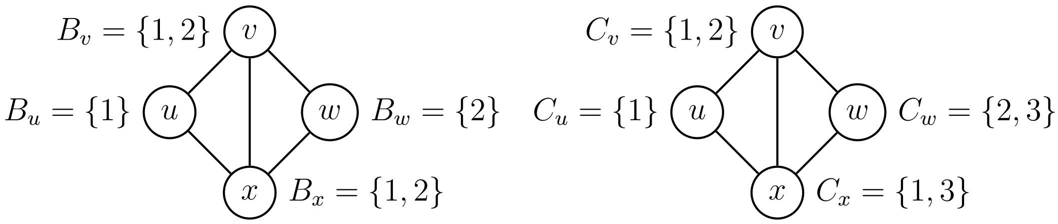

Fig. 1. The graph G as a graph in $\mathcal{T}_1$ and $\mathcal{T}_2$

The graph G can also be realized as a graph in $\mathcal{T}_2$. Let $C_u = \{1\}$, $C_v = \{1,2\}$, $C_w = \{2,3\}$ and $C_x = \{1,3\}$. Then, $|C_{z_1} \cap C_{z_2}|\leq 1$ for all $z_1, z_2 \in V$ with $z_1 \neq z_2$. Furthermore, $g(C_u) = \{C_u\}$, $g(C_v) = \{C_v\}$, $g(C_w) = \{C_w\}$ and $g(C_x) = \{\{1\},\{3\}\}$. Since $|g(C_x)|= 2$ and $|g(C_z)|= 1$ for all $z \in V \setminus \{x\}$, we have $\mathcal{C} = \{C_z\}_{z\in V} \subseteq 2_0^{\mathbb{N}}(2)$ and $G = \mathfrak{F}(\mathcal{C}) \in \mathcal{T}_2(2)$ (see Figure 1). Thus, $i_{\mathcal{T}_2}(G) \leq 2$. We will show that $i_{\mathcal{T}_2}(G) = 2$.

Suppose $i_{\mathcal{T}_2}(G) = 1$. Then, there exists $\{U_z\}_{z\in V} \subseteq 2_0^{\mathbb{N}}(1)$ satisfying $|U_{z_1} \cap U_{z_2}|\leq 1$ for all $z_1, z_2 \in V$ with $z_1 \neq z_2$ such that $G = \mathfrak{F}(\{U_z\}_{z\in V})$. Note that $|U_{z_1} \cap U_{z_2}|= 1$ for all $z_1 \sim_G z_2$.

Case 1. Suppose $U_u \cap U_v = \{a\} = U_u \cap U_x$. This implies that $U_v \cap U_x = \{a\}$. Let $U_v \cap U_w = \{b\}$ and $U_w \cap U_x = \{c\}$. We may assume that $c \geq b$. Since u is not adjacent to w in G, we must have $b, c \neq a$. If $c = b$, then $c \in U_v \cap U_x = \{a\}$ and so $c = a$, a contradiction. Suppose $c > b$. Assume that $b > a$. Since $g(U_x)$ has only one consecutive part (i.e. $|g(U_x)|= 1$), $a, c \in U_x$ and $c > b > a$, we must have $b \in U_x$. This implies that $b \in U_x \cap U_v = \{a\}$ and $b = a$, a contradiction. Assume that $b < a$. Suppose $b < c < a$. Since $g(U_v)$ has only one consecutive part and $a, b \in U_v$, we must have $c \in U_v$. This implies that $c \in U_v \cap U_x = \{a\}$ and so $c = a$, a contradiction. Suppose $b < a < c$. Since $g(U_w)$ has only one consecutive part and $b, c \in U_w$, we must have $a \in U_w$. This implies that u is adjacent to w in G, a contradiction.

Case 2. Suppose $U_u \cap U_v = \{a\}$, $U_u \cap U_x = \{b\}$, $U_v \cap U_x = \{c\}$, $U_x \cap U_w = \{d\}$ and $U_v \cap U_w = \{e\}$. We shall assume that a, b, c, d, e are distinct integers, otherwise, it will go back to Case 1. Since $g(U_x)$ has only one consecutive part, $U_x = \{p, p + 1, \ldots, p + q\}$. Note that $b, c, d \in U_x$ and $a, e \notin U_x$. We may assume that $e > a$. Now, $a, e, c \in U_v$, $b, d \notin U_v$ and $g(U_v)$ has only one consecutive part imply that $a > p+q$ or $e < p$. Suppose $a > p + q$. Since $c \leq p + q$ and $U_v \cap U_x = \{c\}$, $c = p + q$. Now, $b \notin U_v$ implies that $b < c$. We consider U_u. Note that $a, b \in U_u$ and $g(U_u)$ has only one consecutive part imply that $c \in U_u$. Thus, $c \in U_u \cap U_v = \{a\}$ and then $c = a$, a contradiction. Suppose $e < p$. Since $c \geq p$ and $U_v \cap U_x = \{c\}$, $c = p$. Now, $d \notin U_v$ implies that $d > c$. We consider U_w. Note that $d, e \in U_w$ and $g(U_w)$ has only one consecutive part imply that $c \in U_w$. Thus, $c \in U_w \cap U_v = \{e\}$ and so $c = e$, a contradiction.

Hence, we conclude that $i_{\mathcal{T}_2}(G) = 2$ and $i_{\mathcal{T}_1}(G) < i_{\mathcal{T}_2}(G)$.

Now, we shall characterize graph G with $i_{\mathcal{T}_2}(G) = 1$. Given any finite set V and

$\{B_u\}_{u\in V} \subseteq 2_0^{\mathbb{N}}$, let

$$L(\{B_u\}_{u\in V}) = \bigcup_{u\in V} B_u.$$

Lemma 3.6. *Let $G = \mathfrak{F}(\{B_u\}_{u\in V(G)})$ for some $\{B_u\}_{u\in V(G)} \subseteq 2_0^{\mathbb{N}}$. Then the following statements hold:*

(a) *Let x be the smallest positive integer in $L(\{B_u\}_{u\in V(G)})$, $p \leq x$ be a positive integer and $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(a) = a - p + 1$ for all $a \in \mathbb{N}$. If $C_u = f(B_u) = \{f(a) \ : \ a \in B_u\}$ for all $u \in V(G)$, then $G = \mathfrak{F}(\{C_u\}_{u\in V(G)})$ and $|g(C_u)| = |g(B_u)|$ and $|C_u \cap C_v| = |B_u \cap B_v|$ for all $u, v \in V(G)$ with $u \neq v$. In particular, there exists $\{C_u\}_{u\in V(G)} \subseteq 2_0^{\mathbb{N}}$ with $G = \mathfrak{F}(\{C_u\}_{u\in V(G)})$ and $1 \in L(\{C_u\}_{u\in V(G)})$.*

(b) *Suppose $c, d \in L(\{B_u\}_{u\in V(G)})$ with $c+1 < d$ and $c+1, c+2, \ldots, d-1 \notin L(\{B_u\}_{u\in V(G)})$. Let $f : \mathbb{N} \to \mathbb{N}$ be defined by*

$$f(a) = \begin{cases} a, & \text{for } 1 \leq a \leq d-1; \\ a - d + c + 1, & \text{for } a \geq d. \end{cases}$$

If $C_u = f(B_u) = \{f(a) \ : \ a \in B_u\}$ for all $u \in V(G)$, then $G = \mathfrak{F}(\{C_u\}_{u\in V(G)})$ and $c, c+1 \in L(\{C_u\}_{u\in V(G)})$. Furthermore, for all $u, v \in V(G)$ with $u \neq v$, $|g(C_u)| = |g(B_u)|$ if $\{c, d\} \nsubseteq B_u$ and $|g(C_u)| = |g(B_u)| - 1$ if $\{c, d\} \subseteq B_u$.

(c) *Suppose $c \in B_{u_0}$ for exactly one $u_0 \in V(G)$. Let $C_{u_0} = B_{u_0} \setminus \{c\}$ and $C_u = B_u$ for all $u \in V(G) \setminus \{c\}$. Then $G = \mathfrak{F}(\{C_u\}_{u\in V(G)})$.*

Proof. (a) It follows from Lemma 2.6 by noting that $C_u = B_u - p + 1$. Furthermore, if $g(B_u) = \{X_1, X_2, \ldots, X_m\}$, then $g(C_u) = \{X_1 - p + 1, X_2 - p + 1, \ldots, X_m - p + 1\}$. Thus, $|g(C_u)| = |g(B_u)|$. By taking $p = x$, we see that $1 \in L(\{C_u\}_{u\in V(G)})$.

(b) Note that the restriction of f to $\{1, 2, \ldots, c\} \cup (\mathbb{N} \setminus \{1, 2, \ldots, d-1\})$ is injective. This implies that $f(B_u) \cap f(B_v) = f(B_u \cap B_v)$ for all $u, v \in V(G)$. Thus, $|C_u \cap C_v| = |B_u \cap B_v|$. Therefore, $G = \mathfrak{F}(\{C_u\}_{u\in V(G)})$ and $c, c+1 \in L(\{C_u\}_{u\in V(G)})$. Now, $c+1, c+2, \ldots, d-1 \notin L(\{B_u\}_{u\in V(G)})$ imply that

(i) if $d \in B_{u_1}$ for some $u_1 \in V(G)$, then d is the initial point of a consecutive part of $g(B_{u_1})$;

(ii) if $c \in B_{u_2}$ for some $u_2 \in V(G)$, then c is the terminal initial point of a consecutive part of $g(B_{u_2})$.

Therefore, for each $u \in V(G)$, either

(1) $g(B_u) = \{X_1, X_2, \ldots, X_m\}$ and $c, d \notin B_u$, or

(2) $g(B_u) = \{X_1, X_2, \ldots, X_{i_0}, X_{i_0+1}, \ldots, X_m\}$, $c \notin B_u$ and d is the initial point in X_{i_0+1}, or

(3) $g(B_u) = \{X_1, X_2, \ldots, X_{i_0}, X_{i_0+1}, \ldots, X_m\}$, $d \notin B_u$ and c is the terminal point in X_{i_0}, or

(4) $g(B_u) = \{X_1, X_2, \ldots, X_{i_0}, X_{i_0+1}, \ldots, X_m\}$, c is the terminal point in X_{i_0} and d is the initial point in X_{i_0+1}.

If (1), (2) or (3) holds, then $|g(C_u)|= |g(B_u)|$. If (4) holds, then $|g(C_u)|= |g(B_u)|-1$. In fact,

$$g(C_u) = \{X_1, X_2, \dots, X_{i_0-1}, (X_{i_0} \cup X'_{i_0+1}), X'_{i_0+2} \dots, X'_m\},$$

where $X'_i = X_i - d + c + 1$. Hence, $|g(C_u)|\leq |g(B_u)|$ for all $u \in V(G)$.

(c) Since c is not an element in $B_u \cap B_v$ for all $u, v \in V(G)$ with $u \neq v$, we have $C_u \cap C_v = B_u \cap B_v$. Hence, $G = \mathfrak{F}(\{C_u\}_{u\in V(G)})$. □

Lemma 3.7. *Let $G = \mathfrak{F}(\{B_u\}_{u\in V(G)})$ for some $\{B_u\}_{u\in V(G)} \subseteq 2^{\mathbb{N}}_0$. Then there exists $\{C_u\}_{u\in V(G)} \subseteq 2^{\mathbb{N}}_0$ such that $G = \mathfrak{F}(\{C_u\}_{u\in V(G)})$ and $|g(C_u)|\leq |g(B_u)|$ and $|C_u \cap C_v|= |B_u \cap B_v|$ for all $u, v \in V(G)$ with $u \neq v$. Furthermore, $L(\{C_u\}_{u\in V(G)}) = \{1, 2, \dots, m\}$ and each $i \in \{1, 2, \dots, m\}$ is contained in at least two C_u's.*

Proof. Suppose $c \in B_{u_0}$ for exactly one $u_0 \in V(G)$. By part (c) of Lemma 3.6, $G = \mathfrak{F}(\{C_u\}_{u\in V(G)})$ where $C_{u_0} = B_{u_0} \setminus \{c\}$ and $C_u = B_u$ for all $u \in V(G) \setminus \{c\}$. By Lemma 2.5, $|g(C_{u_0})|= |g(B_{u_0})|+1$ if c is a middle point of a consecutive part of $g(B_0)$ and $|g(C_{u_0})|= |g(B_{u_0})|$, otherwise. If $|g(C_{u_0})|= |g(B_{u_0})|+1$, then $c-1, c+1 \in C_{u_0}$. By taking

$$f(a) = \begin{cases} a, & \text{for } 1 \leq a \leq c; \\ a-1, & \text{for } a \geq c+1, \end{cases}$$

in part (b) of Lemma 3.6, we can close the gap. In fact, if $C'_u = f(C_u) = \{f(a) : a \in C_u\}$ for all $u \in V(G)$, then $G = \mathfrak{F}(\{C'_u\}_{u\in V(G)})$ and $c, c+1 \in L(\{C'_u\}_{u\in V(G)})$. Furthermore, for all $u, v \in V(G)$ with $u \neq v$, $|g(C'_u)|= |g(C_u)|= |g(B_u)|$ for $u \neq u_0$ and $|g(C'_{u_0})|= |g(C_{u_0})|-1 = |g(B_{u_0})|$. Similarly, we can close the gap for the case $|g(C_{u_0})|= |g(B_{u_0})|$.

If there is a gap like in the hypothesis of part (b) of Lemma 3.6, we can close the gap like in the conclusion of part (b) the lemma. If $1 \notin L(\{B_u\}_{u\in V(G)})$, we can apply part (a) of Lemma 3.6. Therefore, after finite number of steps, we will obtain $\{C_u\}_{u\in V(G)} \subseteq 2^{\mathbb{N}}_0$ for which $L(\{C_u\}_{u\in V(G)}) = \{1, 2, \dots, m\}$ and other conclusions of the lemma also hold. □

Given a graph G, we shall label its edges with an element in $\mathbb{N}$. If an edge uv is labelled with $a \in \mathbb{N}$, we set $E_{uv} = \{a\}$. Note that $E_{uv} = E_{vu}$ as $uv = vu$ is the same edge. The set $\{E_{uv}\}_{u\sim_G v}$ is called an edge labelling of G. For each $u \in V(G)$, let $N_G(u) = \{v \in V(G) : v \sim_G u\}$ be the neighbourhood of u and

$$C_u = \bigcup_{v\in N_G(u)} E_{uv}.$$

The edge labelling $\{E_{uv}\}_{u\sim_G v}$ is said to be $\mathcal{T}_2$-compatible if

$$C_u \cap C_v = \begin{cases} E_{uv}, & \text{for all } u \sim_G v; \\ \varnothing, & \text{otherwise.} \end{cases}$$

Lemma 3.8. *Let G be a graph.*

(a) *Suppose* $G = \mathfrak{F}(\{B_u\}_{u\in V(G)})$ *for some* $\{B_u\}_{u\in V(G)} \subseteq 2_0^{\mathbb{N}}$ *and* $|B_u \cap B_v| \leq 1$ *for all* $u, v \in V(G)$ *with* $u \neq v$. *Let* $E_{uv} = B_u \cap B_v$ *for all* $u \sim_G v$. *Then,* $\{E_{uv}\}_{u\sim_G v}$ *is a* $\mathcal{T}_2$*-compatible edge labelling of* G.

(b) *Suppose* $\{E_{uv}\}_{u\sim_G v}$ *is a* $\mathcal{T}_2$*-compatible edge labelling of* G. *If* $C_u = \bigcup_{v\in N_G(u)} E_{uv}$ *for all* $u \in V(G)$, *then* $G = \mathfrak{F}(\{C_u\}_{u\in V(G)}) \in \mathcal{T}_2$.

Proof. Note that E_{uv} is an one element set for all $u \sim_G v$. Let $C_u = \bigcup_{v\in N_G(u)} E_{uv}$. Since $E_{uv} \subseteq B_u$ for all $u \sim_G v$, we have $C_u \subseteq B_u$. Therefore, $C_u \cap C_v \subseteq B_u \cap B_v$ for all $u, v \in V(G)$. This means if $u \nsim_G v$, then $C_u \cap C_v = \varnothing$. On the other hand, if $u \sim_G v$, then $E_{uv} = B_u \cap B_v \neq \varnothing$. Now $E_{uv} \subseteq C_u$ and $E_{uv} = E_{vu} \subseteq C_v$ imply that $C_u \cap C_v = E_{uv}$. Hence, $\{E_{uv}\}_{u\sim_G v}$ is $\mathcal{T}_2$-compatible.

Suppose $\{E_{uv}\}_{u\sim_G v}$ is $\mathcal{T}_2$-compatible. Then $\{C_u\}_{u\in V(G)} \subseteq 2_0^{\mathbb{N}}$ and $|C_u \cap C_v| \leq 1$ for all $u, v \in V(G)$ with $u \neq v$. By $\mathcal{T}_2$-compatibility, $C_u \cap C_v \neq \varnothing$ if and only if u is adjacent to v. Hence, $G = \mathfrak{F}(\{C_u\}_{u\in V(G)}) \in \mathcal{T}_2$. □

The following corollary is a consequence of Lemma 3.8.

Corollary 3.9. $G \in \mathcal{T}_2$ *if and only if there exists a* $\mathcal{T}_2$*-compatible edge labelling of* G.

For $1 \leq i \leq n$, $1 \leq j \leq m_i$ let H_i and Q_{ij} be complete graphs. Let G' be the union of these complete graphs, i.e.,

$$G' = \bigcup_{1\leq i\leq n} H_i \cup \bigcup_{\substack{1\leq i\leq n,\\ 1\leq j\leq m_i}} Q_{ij}.$$

Let $x_i \in V(H_i)$. Since $V(H_i) \cap V(H_{i'}) = \varnothing$ for $i \neq i'$, we have $x_i \neq x_{i'}$. For each $u \in V(Q_{ij})$ add an edge $x_i u$ to the graph G'. For each $u \in V(H_{i+1})$ add an edge $x_i u$ to the graph G'. Let G denotes the resulting graph. Note that G is connected, $G[\{x_i\} \cup V(Q_{ij})]$ and $G[\{x_i\} \cup V(H_{i+1})]$ are complete graphs. Furthermore,

$$E(G) = E(G') \cup \bigcup_{\substack{1\leq i\leq n,\\ 1\leq j\leq m_i}} \{x_i u \ : \ u \in V(Q_{ij})\} \cup \bigcup_{1\leq i\leq n-1} \{x_i u \ : \ u \in V(H_{i+1})\}.$$

The class of graphs formed this way is denoted by $\mathcal{H}$.

Theorem 3.10. *Let* G *be a connected graph. Then,* $i_{\mathcal{T}_2}(G) = 1$ *if and only if* $G \in \mathcal{H}$.

Proof. Suppose $G \in \mathcal{H}$. Now, we shall construct a $\mathcal{T}_2$-compatible edge labelling for G.

(1) Label all edges in H_1 with 1.

(2) For $2 \leq i \leq n$, label all edges in H_i with $m_1 + m_2 + \cdots + m_{i-1} + i$.

(3) For $1 \leq i \leq n-1$, label all edges in $\{x_i u \ : \ u \in V(H_{i+1})\}$ with $m_1 + m_2 + \cdots + m_i + i + 1$.

(4) For $1 \leq i \leq n$ and $1 \leq j \leq m_i$, label all edges in Q_{ij} with $m_1+m_2+\cdots+m_{i-1}+i+j$.

(5) For $1 \leq i \leq n$ and $1 \leq j \leq m_i$, label all edges in $\{x_i u \ : \ u \in V(Q_{ij})\}$ with $m_1 + m_2 + \cdots + m_{i-1} + i + j$.

For each $u \in V(G)$, let $C_u = \bigcup_{v \in N_G(u)} E_{uv}$. We shall show that $\{E_{uv}\}_{u \sim_G v}$ is $\mathcal{T}_2$-compatible. Suppose $C_{u_1} \cap C_{v_1} \neq \varnothing$ for some none adjacent vertices u_1, v_1. Let $y \in C_{u_1} \cap C_{v_1}$. Then $E_{u_1 z_1} = E_{v_1 z_2} = \{y\}$ for some vertices z_1 and z_2 such that $u_1 \sim_G z_1$ and $v_1 \sim_G z_2$. If $u_1 z_1$ is an edge in H_i, then $y = m_1 + m_2 + \cdots + m_{i-1} + i$. Therefore, $v_1 z_2$ is an edge in H_i or it is an edge in $\{x_{i-1} u \ : \ u \in V(H_i)\}$. In either case, u_1 is adjacent to v_1, a contradiction. If $u_1 z_1$ is an edge in Q_{ij}, then $y = m_1 + m_2 + \cdots + m_{i-1} + i + j$. Therefore, $v_1 z_2$ is an edge in Q_{ij} or it is an edge in $\{x_i u \ : \ u \in V(Q_{ij})\}$. In either case, u_1 is adjacent to v_1, a contradiction. Similarly, if $u_1 z_1$ is an edge in $\{x_i u \ : \ u \in V(H_{i+1})\}$ or it is an edge in $\{x_i u \ : \ u \in V(Q_{ij})\}$, we also will obtain the conclusion that u_1 is adjacent to v_1. Hence, $C_u \cap C_v = \varnothing$ for u not adjacent to v.

Suppose u is adjacent to v. Clearly, $E_{uv} \subseteq C_u \cap C_v$. Suppose $E_{uv} \subsetneq C_u \cap C_v$. Let $y \in (C_u \cap C_v) \setminus E_{uv}$. Then $E_{uz_1} = E_{vz_2} = \{y\}$ for some vertices z_1 and z_2 such that $u \sim_G z_1$ and $v \sim_G z_2$. Let $E_{uv} = \{y_0\}$. Then, $y \neq y_0$. If uz_1 is an edge in H_i, then $y = m_1 + m_2 + \cdots + m_{i-1} + i$. Therefore, vz_2 is an edge in H_i or it is an edge in $\{x_{i-1} z \ : \ z \in V(H_i)\}$. If $v \in V(H_i)$, then uv is labelled with y. Thus, $y_0 = y$, a contradiction. So, $v = x_{i-1}$ and $z_2 \in V(H_i)$. However, x_{i-1} is adjacent to all vertices in H_i. By construction, $x_{i-1} u$ is labelled with y. This implies that $y_0 = y$, a contradiction. If uz_1 is an edge in Q_{ij}, then $y = m_1 + m_2 + \cdots + m_{i-1} + i + j$. Therefore, vz_2 is an edge in Q_{ij} or it is an edge in $\{x_i z \ : \ z \in V(Q_{ij})\}$. In either case, uv is labelled with y too, a contradiction. Similarly, if uz_1 is an edge in $\{x_i z \ : \ z \in V(H_{i+1})\}$ or it is an edge in $\{x_i z \ : \ z \in V(Q_{ij})\}$, we also will obtain the conclusion that uv is labelled with y. Hence, $E_{uv} = C_u \cap C_v$ for u adjacent to v.

We have shown that $\{E_{uv}\}_{u \sim_G v}$ is $\mathcal{T}_2$-compatible. By part (b) of Lemma 3.8, $G = \mathfrak{F}(\{C_u\}_{u \in V(G)})$. Now, for all $1 \leq i \leq n$ and $1 \leq j \leq m_i$,

$$
\begin{aligned}
C_{x_i} =& \{m_1 + m_2 + \cdots + m_{i-1} + i, m_1 + m_2 + \cdots + m_{i-1} + i + 1, \ldots, \\
& m_1 + m_2 + \cdots + m_{i-1} + i + m_i, m_1 + m_2 + \cdots + m_{i-1} + m_i + i + 1\}; \\
C_u =& \{m_1 + m_2 + \cdots + m_{i-1} + i\}, \qquad \text{for all } u \in H_i \setminus \{x_i\}; \\
C_u =& \{m_1 + m_2 + \cdots + m_{i-1} + i + j\}, \qquad \text{for all } u \in Q_{ij}.
\end{aligned}
$$

Hence, $|g(C_u)| = 1$ for all $u \in V(G)$ and $i_{\mathcal{T}_2}(G) = 1$.

Suppose $i_{\mathcal{T}_2}(G) = 1$. Then there exists $\{B_u\}_{u \in V(G)} \subseteq 2_0^{\mathbb{N}}(1)$ and $|B_u \cap B_v| \leq 1$ for all $u, v \in V(G)$ with $u \neq v$ such that $G = \mathfrak{F}(\{B_u\}_{u \in V(G)})$. By Lemma 3.7, we may assume that $L(\{B_u\}_{u \in V(G)}) = \{1, 2, \ldots, l\}$ and each $i \in \{1, 2, \ldots, l\}$ is contained in at least two B_u's. Let $Y_i = \{u \in V(G) \ : \ i \in B_u\}$ for $1 \leq i \leq l$. Note that $H_1 = G[Y_1]$ is a complete graph. If $G = H_1$, then $G \in \mathcal{H}$. Suppose $G \neq H_1$. Since G is connected, there is a $x_1 \in V(H_1)$ with $|B_{x_1}| \geq 2$. Let $B_{x_1} = \{1, 2, \ldots, m_1 + 2\}$. Note that $B_u = \{1\}$ for other $u \in V(H_1) \setminus \{x_1\}$, otherwise, we have $|B_u \cap B_{x_1}| \geq 2$ or $|g(B_u)| \geq 2$. For $1 \leq j \leq m_1$, set $Q_{1j} = G[Y_{j+1} \setminus \{x_1\}]$. Let $H_2 = G[Y_{m_1+2} \setminus \{x_1\}]$. If $m_1 + 2 = l$, then we see that $G \in \mathcal{H}$. Suppose $l > m_1 + 2$. Since G is connected, there is a $x_2 \in V(H_2)$ with $|B_{x_2}| \geq 2$. Let $B_{x_2} = \{m_1 + 2, m_1 + 3, \ldots, m_1 + m_2 + 3\}$. Note that $B_u = \{m_1 + 2\}$ for other

$u \in V(H_2) \setminus \{x_2\}$, otherwise, we have $|B_u \cap B_{x_1}| \geq 2$, $|B_u \cap B_{x_2}| \geq 2$ or $|g(B_u)| \geq 2$. For $1 \leq j \leq m_2$, set $Q_{2j} = G[Y_{j+m_1+2} \setminus \{x_2\}]$. Let $H_3 = G[Y_{m_1+m_2+3} \setminus \{x_2\}]$. If $m_1+m_2+3 = l$, then we see that $G \in \mathcal{H}$. If $l > m_1 + m_2 + 3$, then we continue this process. Eventually, we will have $G \in \mathcal{H}$.

This completes the proof of the theorem. $\square$

4. Planar graph

In this section, we will prove Theorem 4.8. We begin with the following lemmas.

Lemma 4.1. *Suppose $\{E_{uv}\}_{u \sim_G v}$ is a $\mathcal{T}_2$-compatible edge labelling of G. If $E_{u_1v_1} = E_{u_2v_2} = \{c\}$ and $u_1 \neq u_2$, then u_1 is adjacent to u_2 and $E_{u_1u_2} = \{c\}$.*

Proof. For each $u \in V(G)$, let $C_u = \bigcup_{v \in N_G(u)} E_{uv}$. Now $c \in C_{u_1} \cap C_{u_2}$ implies that $C_{u_1} \cap C_{u_2} \neq \varnothing$. Thus, u_1 and u_2 are adjacent. By $\mathcal{T}_2$-compatibility, $\{c\} = C_{u_1} \cap C_{u_2} = E_{u_1u_2}$. $\square$

Lemma 4.2. *Suppose $\{E_{uv}\}_{u \sim_G v}$ is a $\mathcal{T}_2$-compatible edge labelling of G. If H is a subgraph of G, then $\{E_{uv}\}_{u \sim_H v}$ is a $\mathcal{T}_2$-compatible edge labelling of H.*

Proof. For each $u \in V(G)$, let $B_u = \bigcup_{v \in N_G(u)} E_{uv}$. For each $u \in V(H)$, let $C_u = \bigcup_{v \in N_H(u)} E_{uv}$. Since $u \sim_H v$ implies that $u \sim_G v$, $C_u \subseteq B_u$. Therefore $C_u \cap C_v \subseteq B_u \cap B_v$ for all $u, v \in V(H)$ with $u \neq v$. So, if $B_u \cap B_v = \varnothing$, then $C_u \cap C_v = \varnothing$.

Next, if $u \sim_H v$, then $u \sim_G v$. This means $B_u \cap B_v = E_{uv}$. Since u, v are vertices in H, $E_{uv} \subseteq C_u \cap C_v$. Hence, $C_u \cap C_v = E_{uv}$ and $\{E_{uv}\}_{u \sim_H v}$ is a $\mathcal{T}_2$-compatible edge labelling of H. $\square$

Lemma 4.3. *Let G' be a graph and $u_0, v_0 \in V(G')$ such that $u_0 \neq v_0$. Suppose*

(a) *$G' = \mathfrak{F}(\{B_u\}_{u \in V(G')})$ for some $\{B_u\}_{u \in V(G')} \subseteq 2_0^{\mathbb{N}}$ and $|B_u \cap B_v| \leq 1$ for all $u, v \in V(G')$ with $u \neq v$;*

(b) *$c \in B_{u_0} \cap B_{v_0}$ and $c \notin B_u$ for all $u \in V(G') \setminus \{u_0, v_0\}$.*

Let G be a graph obtained from G' by deleting the edge u_0v_0. Then, there exists $\{C_u\}_{u \in V(G)} \subseteq 2_0^{\mathbb{N}}$ such that $G = \mathfrak{F}(\{C_u\}_{u \in V(G)})$, $|C_u \cap C_v| \leq 1$ for all $u, v \in V(G)$ with $u \neq v$, $|g(C_u)| = |g(B_u)|$ for all $u \in V(G)$.

Proof. Let $D_u = B_u$ for all $u \in V(G)$, $D_{u_0} = B_{u_0} \setminus \{c\}$ and $D_{v_0} = B_{v_0} \setminus \{c\}$. Then, $D_u \cap D_v = B_u \cap B_v$ for all $u, v \in V(G) \setminus \{u_0, v_0\}$. Let $v \notin \{u_0, v_0\}$. Clearly, $D_{u_0} \cap D_v \subseteq B_{u_0} \cap B_v$. If $u_0 \not\sim_G v$, then $u_0 \not\sim_{G'} v$ and $B_{u_0} \cap B_v = \varnothing$. This means $D_{u_0} \cap D_v = \varnothing$. If $u_0 \sim_G v$, then $u_0 \sim_{G'} v$ and $B_{u_0} \cap B_v = \{d\}$ for some $d \in \mathbb{N}$. Since $d \neq c$, we have $d \in D_{u_0} = B_{u_0} \setminus \{c\}$. Hence, $D_{u_0} \cap D_v = \{d\}$. In either case, $D_{u_0} \cap D_v = B_{u_0} \cap B_v$. Similarly, $D_{v_0} \cap D_v = B_{v_0} \cap B_v$ for $v \notin \{u_0, v_0\}$. Finally, we have $D_{u_0} \cap D_{v_0} = \varnothing$. Thus, $G = \mathfrak{F}(\{D_u\}_{u \in V(G)})$. Furthermore, $|g(D_u)| = |g(B_u)|$ for all $u \in V(G) \setminus \{u_0, v_0\}$. Let

$x \in \{u_0, v_0\}$. Note that if c is the initial point or the terminal point of a consecutive part of $g(B_x)$, then $|g(D_x)|= |g(B_x)|$.

Let $\{x, y\} = \{u_0, v_0\}$. Suppose c is the initial point of a consecutive part of $g(B_x)$ and c is a middle point of a consecutive part of $g(B_y)$. Then, $|g(D_x)|= |g(B_x)|$ and $|g(D_y)|= |g(B_y)|+1$ and $c-1, c+1 \in D_y$. By part (b) of Lemma 3.6, there exists $\{C_u\}_{u\in V(G)} \subseteq 2_0^{\mathbb{N}}$ such that $G = \mathfrak{F}(\{C_u\}_{u\in V(G)})$ and $|C_u \cap C_v|\leq 1$ for all $u, v \in V(G)$ with $u \neq v$. Furthermore, $|g(C_u)|= |g(D_u)|= |g(B_u)|$ for all $u \in V(G) \setminus \{y\}$ and $|g(C_y)|= |g(D_y)|-1 = |g(B_y)|$.

Similarly, the same conclusions hold if

(i) c is the terminal point of a consecutive part of $g(B_x)$ and c is a middle point of a consecutive part of $g(B_y)$; or

(ii) c is a middle point of a consecutive part of $g(B_x)$ and c is a middle point of a consecutive part of $g(B_y)$.

This completes the proof of the lemma. □

Lemma 4.4. *Let G' be a graph and $u_0 \in V(G')$. Let G be a graph obtained from G' by deleting the vertex u_0 and all edges in incident to it. Then,*

$$i_{\mathcal{T}_2}(G) \leq i_{\mathcal{T}_2}(G') \leq i_{\mathcal{T}_2}(G) + 1.$$

Proof. Firstly, we will prove $i_{\mathcal{T}_2}(G) \leq i_{\mathcal{T}_2}(G')$. Let $i_{\mathcal{T}_2}(G') = k$. Then $G' = \mathfrak{F}(\{B_u\}_{u\in V(G')})$ for some $\{B_u\}_{u\in V(G')} \subseteq 2_0^{\mathbb{N}}(k)$ and $|B_u \cap B_v|\leq 1$ for all $u, v \in V(G')$ with $u \neq v$. By Lemma 3.7, we may assume that $L(\{B_u\}_{u\in V(G')}) = \{1, 2, \ldots, m\}$ and each $i \in \{1, 2, \ldots, m\}$ is contained in at least two B_u's. Let $E_{uv} = B_u \cap B_v$ for all $u \sim_{G'} v$. By part (a) of Lemma 3.8, $\{E_{uv}\}_{u\sim_{G'} v}$ is a $\mathcal{T}_2$-compatible edge labelling of G'. Since $E_{uv} \subseteq B_u$ for all $u \sim_{G'} v$, $\bigcup_{v\in N_{G'}(u)} E_{uv} \subseteq B_u$. On the other hand, let $y \in B_u$. There is a $v' \in V(G')$ with $y \in B_{v'}$. Therefore, u is adjacent to v' and $y \in B_u \cap B_{v'} = E_{uv'}$. Hence, $y \in \bigcup_{v\in N_{G'}(u)} E_{uv}$ and $B_u = \bigcup_{v\in N_{G'}(u)} E_{uv}$. Note that $|g(B_u)|\leq k$ for all $u \in V(G')$.

Suppose $c \in B_{u_0} \cap B_{v_0}$ and $c \notin B_u$ for all $u \in V(G') \setminus \{u_0, v_0\}$. Let G'_1 be the graph obtained from G' by removing the edge u_0v_0 forom G'. By Lemma 4.3, there exists $\{D_u\}_{u\in V(G)} \subseteq 2_0^{\mathbb{N}}$ such that $G = \mathfrak{F}(\{D_u\}_{u\in V(G)})$, $|D_u \cap D_v|\leq 1$ for all $u, v \in V(G)$ with $u \neq v$, $|g(D_u)|= |g(B_u)|$ for all $u \in V(G)$. Thus, $i_{\mathcal{T}_2}(G'_1) \leq i_{\mathcal{T}_2}(G')$. We can apply Lemma 4.3 again to remove an edge connecting a vertex in G'_1 to u_0 whenever condition (b) of Lemma 4.3 is satisfied. Therefore, we may as well assume that condition (b) of Lemma 4.3 is not satisfied for the original graph G'. To be precise, let $v_1, v_2, \ldots, v_l$ be the only vertices in G' adjacent to u_0. Then, for each i, there exists an edge $u'v'$ in G' such that $u'v' \neq u_0v_i$ and $E_{u'v'} = E_{u_0v_i}$.

By Lemma 4.2, $\{E_{uv}\}_{u\sim_G v}$ is a $\mathcal{T}_2$-compatible edge labelling of G. Note that $v_1, v_2, \ldots, v_l$ are vertices of G. Let $C_u = \bigcup_{v\in N_G(u)} E_{uv}$. Then, $C_u = B_u$ for all $u \in V(G)\setminus\{v_1, v_2, \ldots, v_l\}$ and $C_{v_i} \subseteq B_{v_i}$ for all i. Therefore, $|g(C_u)|= |g(B_u)|$ for all $u \in V(G) \setminus \{v_1, v_2, \ldots, v_l\}$. Let $i \in \{1, 2, \ldots, l\}$ be fixed and $E_{u_0v_i} = \{c\}$. Note that either $C_{v_i} = B_{v_i}$ or $C_{v_i} = B_{v_i} \setminus \{c\}$. We shall show that the latter cannot occur.

Recall that there exists an edge $u'v'$ in G' such that $u'v' \neq u_0v_i$ and $E_{u'v'} = E_{u_0v_i}$. Note that u' and v' cannot be both equal to u_0. So, by Lemma 4.1, u_0 is adjacent to u' or v'. Suppose u_0 is adjacent to u'. If $u' = v_i$, then $v' \neq u_0$, for $u'v'$ and u_0v_i are different edges. So, u_0 is adjacent to v' and $v' \neq v_i$. Similarly, if u_0 is adjacent to v' and $v' = v_i$, we will get u_0 is adjacent to u' and $u' \neq v_i$. Therefore, we may as well assume that u_0 is adjacent to u' and $u' \neq v_i$. This implies that $u' = v_j$ for some $j \neq i$. So, $E_{u_0v_i} = \{c\} = E_{u_0v_j}$. Again, by Lemma 4.1, v_i is adjacent to v_j and $E_{v_iv_j} = \{c\}$. This implies that $c \in C_{v_i}$ and $C_{v_i} = B_{v_i}$. Hence, $C_{v_i} = B_{v_i}$ for all i and $|g(C_u)|= |g(B_u)|$ for all $u \in V(G)$. This means $i_{\mathcal{T}_2}(G) \leq k = i_{\mathcal{T}_2}(G')$.

Now, we shall prove $i_{\mathcal{T}_2}(G') \leq i_{\mathcal{T}_2}(G) + 1$. Let $i_{\mathcal{T}_2}(G) = k$. Then, $G = \mathfrak{F}(\{B_u\}_{u\in V(G)})$ for some $\{B_u\}_{u\in V(G)} \subseteq 2_0^{\mathbb{N}}(k)$ and $|B_u \cap B_v|\leq 1$ for all $u, v \in V(G)$ with $u \neq v$. Let $\max L(\{B_u\}_{u\in V(G)}) = m$. Then, $x \notin L(\{B_u\}_{u\in V(G)})$ for any integer $x \geq m + 1$. Let $v_1, v_2, \ldots, v_l$ be the only vertices in G' adjacent to u_0. Set $C_{u_0} = \{m+1, m+2, \ldots, m+l\}$, $C_{v_i} = B_{v_i} \cup \{m + i\}$ for all i and $C_u = B_u$ for all $u \in V(G') \setminus \{u_0, v_1, v_2, \ldots, v_l\}$. Then, $G' = \mathfrak{F}(\{C_u\}_{u\in V(G')})$. By Lemma 2.4, $|g(C_{v_i})|\leq |g(B_{v_i})|+1$ for all i. Clearly, $|g(C_{u_0})|= 1$ and $|g(C_u)|\leq |g(B_u)|$ for other $u \in V(G') \setminus \{u_0, v_1, v_2, \ldots, v_l\}$. Hence, $i_{\mathcal{T}_2}(G') \leq k + 1 = i_{\mathcal{T}_2}(G) + 1$. □

Lemma 4.5. *Let $\{E_{uv}\}_{u\sim_G v}$ be an edge labelling of G. Suppose that if $u_1v_1, u_2v_2, \ldots, u_mv_m$ are distinct edges in G with $E_{u_1v_1} = E_{u_2v_2} = \cdots = E_{u_mv_m}$,then*

(a) *$m = 1$, or*

(b) *$m = 3$ and the three edges u_1v_1, u_2v_2 and u_3v_3 form a triangle (e.g. $u_1 = u_2$ and $G[\{u_1, v_1, v_2\}]$ is a triangle).*

Then, $\{E_{uv}\}_{u\sim_G v}$ is $\mathcal{T}_2$-compatible.

Proof. Let $C_u = \bigcup_{v\in N_G(u)} E_{uv}$. Suppose $u \not\sim_G v$. If $C_u \cap C_v \neq \varnothing$, then there exist $z_1, z_2 \in V(G)$ and $y \in \mathbb{N}$ such that $E_{uz_1} = \{y\} = E_{vz_2}$. By condition (b), the edges uz_1 and vz_2 are parts of the edges of a triangle. So, u is adjacent to v as uv must be the remaining edge of the triangle. This contradicts the assumption $u \not\sim_G v$. Hence, $C_u \cap C_v = \varnothing$.

Suppose $u \sim_G v$. Note that $E_{uv} \subseteq C_u \cap C_v$. If $E_{uv} \subsetneq C_u \cap C_v$, then there exist $z_1, z_2 \in V(G)$ and $y \in \mathbb{N}$ such that $y \notin E_{uv}$ and $E_{uz_1} = \{y\} = E_{vz_2}$. By condition (b), the edges uz_1 and vz_2 are parts of the edges of a triangle. So, uv must be the remaining edge of the triangle. This implies that $E_{uv} = \{y\}$, a contradiction. Hence, $C_u \cap C_v = E_{uv}$ and $\{E_{uv}\}_{u\sim_G v}$ is $\mathcal{T}_2$-compatible. □

A triangulation is a planar graph for which every face is a triangle. Note that if a graph is a triangulation, then there are exactly 3 vertices on the outer boundary of the planar graph and the subgraph induced by the three vertices is a triangle. We shall call the the outer boundary the outer triangle. All other triangles in the graph will be called inner triangle. A triangle in a planar graph is said to be non-empty if its interior contains at least one vertex. A 4-connected triangulation is a triangulation whose only non-empty triangle is the outer triangle.

Let G be a 4-connected triangulation with at least 4 vertices. Let x, y, z be the vertices on the outer triangle $\triangle_{\text{out}}$. Note that there is a unique vertex u_x in the interior of $\triangle_{\text{out}}$ such that u_x is adjacent to y and z. In fact, if there are more than one vertices in the interior of $\triangle_{\text{out}}$ that are adjacent to y and z, then there is a non-empty inner triangle in which yz is an edge of this triangle. We shall call the unique vertex u_x the vertex opposing x. The vertex u_y opposing y is defined to be the unique vertex in the interior of $\triangle_{\text{out}}$ that is adjacent to x and z. Finally, the vertex u_z opposing z is defined similarly. We need the following lemma (see also [10, Fig. 1] for an illustration).

Lemma 4.6. [11, Lemma 3.1] *Let G be a 4-connected triangulation with outer triangle $\triangle_{out}$ and $V(\triangle_{out}) = \{x, y, z\}$. Let u_x, u_y and u_z be the vertices opposing to x, y and z, respectively. Then the inner edges of G can be partitioned into three forests F_x, F_y, F_z such that*

(a) *F_x is a Hamiltonian path of $G - \{y, z\}$ going from x to u_x,*

(b) *F_y is a spanning tree of $G - \{x, z\}$,*

(c) *F_z is a spanning forest of $G - \{y\}$ consisting of two trees, one containing x and one containing z.*

In particular, when $G = K_4$ the complete graph with 4 vertices, then $u_x = u_y = u_z$, $F_x = G[\{x, u_x\}]$ is just the path xu_x, $F_y = G[\{y, u_x\}]$ is just the path yu_x and $F_z = G[\{x\}] \cup G[\{z, u_x\}]$ is the union of two graphs, one with a single vertex x and the other just the path zu_x.

Remark 4.7. We note here that by conditions (b) and (c) of Lemma 4.6, xu_y is an edge of F_x and zu_y is an edge of F_z.

Theorem 4.8. *If G is a planar graph, then $i_{\mathcal{T}_2}(G) \leq 3$.*

Proof. Firstly, we will construct a triangulation G' in which G is a subgraph of G'. If G' has an inner face which is not a triangle, then we put a vertex in the middle of the face and add edges connecting all the vertices on the boundary of the face to this newly added vertex. This process will partition this face into smaller faces in which each of the newly faces is a triangle. If the outer face is not a triangle, then we add a vertex in the outer face and add edges connecting all the vertices on the outer boundary of the graph to this newly added vertex. Let G' denote the resulting graph. It is not hard to see that G' is a triangulation. By Lemma 4.4, $i_{\mathcal{T}_2}(G) \leq i_{\mathcal{T}_2}(G')$. Hence, we may as well assume that the original graph G is a triangulation.

Now, $\{E_{uv}\}_{u \sim_G v}$ will be called a partial graph labelling of G if each E_{uv} is an one element set or an empty set. Let $C_u = D_u \cup \bigcup_{v \in N_G(u)} E_{uv}$. At the beginning all the E_{uv}'s and D_u's are empty set. We will add a label to an edge one step at a time. In the next step, we will choose an unlabelled edge and label it. If the chosen edge is u_1v_1, then $E_{u_1v_1}$ will be changed from empty set to one element set. During this process, we might also change the labels of the edges that are labelled earlier. This means if $E_{u_2v_2} = \{c\}$, then it may be changed to $E_{u_2v_2} = \{d\}$ for some $d \neq c$. So, $\{E_{uv}\}_{u \sim_G v}$ and all the C_u's are changing throughout the process. At the end of each step, we require that

(i) $\{E_{uv}\}_{u\sim_G v}$ satisfies the condition of Lemma 4.5,

(ii) $|g(C_u)|\leq 3$ for all $u \in V(G)$,

(iii) if $u_1u_2u_3u_1$ is an empty triangle and all the three edges are labelled, then either $E_{u_1u_2} = E_{u_1u_3} = E_{u_2u_3}$ or there is an edge $u_{i_0}u_{j_0}$ such that $E_{u_{i_0}u_{j_0}}$ is unique, i.e., $E_{uv} \neq E_{u_{i_0}u_{j_0}}$ for all other edges $uv \in E(G) \setminus \{u_{i_0}u_{j_0}\}$.

We also require conditions (iv) and (v) which we will state later. An element c is said to be special in C_u if it is the terminal point of a consecutive part of $g(C_u)$ and if $c \in C_v$ for some $v \neq u$, then c is the initial point of a consecutive part of $g(C_v)$. An element c is said to be extra special in C_u if it is the initial point of a consecutive part of $g(C_u)$ and if $c \in C_v$ for some $v \neq u$, then c is the initial point of a consecutive part of $g(C_v)$. We are ready to state conditions (iv) and (v):

(iv) for each $u \in V(G)$, if C_u is non-empty, then it contains a special element,

(v) if $u_1u_2u_3u_1$ is an empty triangle and $E_{u_1u_2} = E_{u_1u_3} = E_{u_2u_3} \neq \varnothing$, then there is a vertex u_{i_0} such that $C_{u_{i_0}}$ contains an extra special element.

At the end, when all the edges have been labelled, i.e., $E_{uv} \neq \varnothing$ for all $u \sim_G v$, we see that $\{E_{uv}\}_{u\sim_G v}$ is a $\mathcal{T}_2$-compatible (Lemma 4.5). By part (b) of Lemma 3.8, $G = \mathfrak{F}(\{C_u\}_{u\in V(G)})$. Thus, by condition (ii), $i_{\mathcal{T}_2}(G) \leq 3$. Conditions (iii), (iv) and (v) are needed for the purpose of induction. We note here that D_u could be empty set at the end. The reason D_u is not empty is because we need to have special and extra special elements for conditions (iv) and (v).

We shall prove by induction on $|V(G)|$ that there is a graph labelling $\{E_{uv}\}_{u\sim_G v}$ of G and $\{D_u\}_{u\in V(G)}$ that satisfy conditions (i), (ii), (iii), (iv) and (v). Since G is a triangulation, $|V(G)|\geq 3$.

Suppose $|V(G)|= 3$. Then, G is a triangle with vertices x, y, z. Now, let $E_{xy} = \{1\}$, $E_{yz} = \{2\}$ and $E_{zx} = \{3\}$. Then, $C_x = \{1, 3\}$, $C_y = \{1, 2\}$ and $C_z = \{2, 3\}$. Note that $g(C_x) = \{\{1\}, \{3\}\}$, $g(C_y) = \{\{1, 2\}\}$ and $g(C_z) = \{\{2, 3\}\}$. Therefore, $|g(C_x)|= 2$ and $|g(C_y)|= 1 = |g(C_z)|$. Next, $xyzx$ is the only empty triangle and E_{xz} is unique. Thus, conditions (i), (ii) and (iii) are satisfied. Now, 1 is a special element in C_x, 2 is a special element in C_y and 3 is a special element in C_z. Hence, conditions (iii), (iv) and (v) are satisfied. Here all the D_u's are empty set.

Suppose $|V(G)|\geq 4$. Assume that for any graph G' with $|V(G')|< |V(G)|$, there exist a graph labelling $\{E_{uv}\}_{u\sim_{G'} v}$ of G' and $\{D_u\}_{u\in V(G')}$ that satisfy conditions (i), (ii), (iii), (iv) and (v).

Let $\triangle_0 = G[\{x, y, z\}]$ be a non-empty triangle in G with inclusion-minimal interior among all non-empty triangles in G. Let $S \subseteq V(G)$ be the set of all vertices in the interior of $\triangle_0$. By the choice of inclusion-minimality, $G[S \cup \{x, y, z\}]$ is a 4-connected triangulation. Let $G' = G - S$. Then, G' is a triangulation and $|V(G')|< |V(G)|$. We remark here that if G is 4-connected, then $G' = G[\{x, y, z\}]$ is the outer triangle of G.

By induction, there exist a graph labelling $\{E_{uv}\}_{u\sim_{G'} v}$ of G' and $\{D_u\}_{u\in V(G')}$ that satisfy conditions (i), (ii), (iii), (iv) and (v). For $u \sim_G v$ but $u \not\sim_{G'} v$, we set $E_{uv} = \varnothing$. Now, $\{E_{uv}\}_{u\sim_G v}$ is a partial graph labelling of G. In fact, all edges are labelled except those edges in the interior of $\triangle_0$. We shall proceed to label those edges based on the decomposition of edges in Lemma 4.6. Let the inner edges of $G_1 = G[S \cup \{x, y, z\}]$ be partitioned into three forests F_x, F_y, F_z as in Lemma 4.6.

Now, $xyzx$ is an empty triangle in G'. By condition (iii), either all edges of $\triangle_0$ are labelled with the same element or there is an uniquely labelled edge of $\triangle_0$ in G'. By condition (v), if all edges are labelled with the same element, then C_p has an extra special element for some $p \in \{x, y, z\}$. We distinguish these two cases. In fact, by relabelling if necessary, we may assume that either

- E_{xz} is unique in G', or
- $E_{xy} = E_{yz} = E_{xz}$ and C_x contains an extra special element.

Case 1. E_{xz} is unique in G'.

Case 1.1. Suppose $|S|= 1$. This means $G[S \cup \{x, y, z\}] = K_4$ and $u_x = u_y = u_z = w$. Since $y \in V(G')$, there is a special element, say d_0 in C_y. Now, change all the labels of the labelled edges according to the following rules: if $c < d_0$, then c remains the same; and if $c \geq d_0$, then c is changed to $c+1$. After that label the edge yw with d_0, i.e., $E_{yw} = \{d_0\}$. When we change d_0 to $d_0 + 1$, the element d_0 was removed from C_y, however, after we labelled the edge yw, d_0 is added back in C_y. Furthermore, $|g(C_y)|$ remains the same before and after the changes. In fact, if $\{a_1, a_2, \ldots, a_l = d_0\}$ was a consecutive part of $g(C_y)$ before the changes, then $\{a_1, a_2, \ldots, a_l, d_0 + 1\}$ is a consecutive part of $g(C_y)$ after the changes. If X was a consecutive part of $g(C_y)$ before the changes and the terminal point of X was smaller than a_1, then X is a consecutive part of $g(C_y)$ after the changes. If X was a consecutive part of $g(C_y)$ before the changes and the initial point of X was greater than $a_l = d_0$, then $X+1$ is a consecutive part of $g(C_y)$ after the changes. The same is true for other $g(C_u)$, $u \in V(G') \setminus \{y\}$ because either $d_0 \notin C_u$ or d_0 is an initial point of a consecutive part of $g(C_u)$. Note that if b was a special element of C_u for some $u \in V(G)$ and $b < d_0$ before the changes, then b is a special element of C_u after the changes. Also, if b was a special element of C_u for some $u \in V(G)$ and $b \geq d_0$ before the changes, then $b+1$ is a special element of C_u after the changes. Similarly, if C_u contained an extra special element, then after the changes, it still contains an extra special element. Note that if $E_{xz} = \{a_0\}$ and $a_0 < d_0$ before the changes, then $E_{xz} = \{a_0\}$ after the changes and it is still unique, and if $E_{xz} = \{a_0\}$ and $a_0 \geq d_0$ before the changes, then $E_{xz} = \{a_0 + 1\}$ after the changes and it is still unique. We set $E_{xw} = E_{zw} = E_{xz}$. At the moment $|g(C_w)|= 2$. Now, we choose an integer $b_0 \in \mathbb{N}$ such that $b_0 \notin \bigcup_{uv\in E(G)} E_{uv}$ and set $D_w = \{b_0\}$. Then, $|g(C_w)|\leq 3$ and all other $|g(C_u)|$ remains the same. Note that b_0 is both a special and an extra special element in C_w. The empty triangles in G' satisfy condition (iii). Now, there are three empty triangles, $xwzx$, $xywx$ and $yzwy$ in the interior of $\triangle_0$. For the empty triangle $yzwy$, $E_{yw} = \{d_0\}$ is unique whereas for the empty triangle $xywx$, $E_{yw} = \{d_0\}$ is unique. For the triangle $xwzx$, $E_{xz} = E_{xw} = E_{zw}$ and b_0 is both a special and an extra special element in C_w. Thus, conditions (iii) and (v) are satisfied. In fact, it is not hard to see that conditions (i) to (v) are satisfied.

Case 1.2. Suppose $|S|\geq 2$. Choose a $n_0 \in \mathbb{N}$ such that for all $c \geq n_0$, $c \notin \bigcup_{uv\in E(G')} E_{uv}$. In fact, we may choose n_0 to be an integer greater than all other integers in $\bigcup_{uv\in E(G')} E_{uv}$. Now, F_x is a Hamiltonian path of $G_1 - \{y, z\}$ going from x to u_x. Recall that xu_y is an edge in F_x (Remark 4.7). Therefore, the Hamiltonian path is from x to u_y and then from u_y to u_x. Let the path from $u_y = v_1$ to $u_x = v_s$ be $v_1v_2 \dots v_s$. Note that $S = \{u_y = v_1, v_2, \dots, v_s = u_x\}$ and $s \geq 2$. Now, for $1 \leq i \leq s-1$, we set $E_{v_iv_{i+1}} = \{n_0+i\}$. At this moment, conditions (i) to (v) are satisfied. In fact, $|g(C_u)|= 1$ for all $u \in S$ and $|g(C_u)|\leq 3$ for all $u \in V(G')$. Furthermore, $n_0 + i$ is a special element of C_{v_i} and $n_0 + 1$ is an extra special element of $C_{v_1} = C_{u_y}$.

Now, we shall label the edges in F_y. We may consider y to be the root of the spanning tree F_y. Now, let $u \in V(F_y) \setminus \{y\}$ and w be the unique vertex adjacent to u on the path from u to y. Suppose $w = y$. Since $y \in V(G')$, there is a special element say d_0 in C_y. Now, change all the labels of the labelled edges according to the following rules: if $c < d_0$, then c remains the same; and if $c \geq d_0$, then c is changed to $c+1$. After that label the edge yu with d_0, i.e., $E_{yu} = \{d_0\}$. When we change d_0 to $d_0 + 1$, the element d_0 was removed from C_y, however, after we labelled the edge yu, d_0 is added back in C_y. Furthermore, $|g(C_y)|$ remains the same before and after. Note that if b was a special element of C_u for some $u \in V(G)$ and $b < d_0$ before the changes, then b is also a special element of C_u after the changes. Also, if b is a special element of C_u for some $u \in V(G)$ and $b \geq d_0$ before the changes, then $b + 1$ is also a special element of C_u after the changes. Similarly, if C_u contained an extra special element before the changes, then after the changes, it still contains an extra special element. It is not hard to see that conditions (i) to (v) are satisfied.

Suppose $w \neq y$. Then, $w \in S$. Note that C_w contains a special element, say d_1 (from the labelling of the edges of the part of the Hamiltonian path F_x, from u_y to u_x). Now, change all the labels of the labelled edges according to the following rules: if $c < d_1$, then c remains the same; and if $c \geq d_1$, then c is changed to $c + 1$. After that label the edge wu with d_1. When we change d_1 to d_1+1, the element d_1 was removed from C_w, however, after we labelled the edge wu, d_1 is added back in C_w. Furthermore, $|g(C_w)|$ remains the same before and after. Note that for each $u \in V(G)$, if C_u contained an special element before the changes, then after the changes, it still contains an special element. The same is true for extra special element. We remark here that the special element and extra special element in C_{u_y} could be the same at the beginning. In fact, let's say d_1 is the special element and extra special element in C_{u_y}. After the changes, $d_1 + 1$ becomes the special element in C_{u_y} whereas d_1 becomes the extra special element. Again, conditions (i) to (v) are satisfied.

Now, we shall label the edges in F_z. Recall that F_z consists of two trees, one containing x and one containing z. We may consider x to be the root of the tree T_x in F_z that contains x. Now, let $u \in V(T_x) \setminus \{x\}$ and w be the unique vertex adjacent to u on the path from u to x. Suppose $w = x$. There is a special element say d_2 in C_x. Now, change all the labels of the labelled edges according to the following rules: if $c < d_2$, then c remains the same; and if $c \geq d_2$, then c is changed to $c + 1$. After that label the edge xu with d_2. When we change d_2 to $d_2 + 1$, the element d_0 was removed from C_x, however, after we labelled the edge xu, d_2 is added back in C_x. Furthermore, $|g(C_x)|$ remains the same before and after. Note that for each $u \in V(G)$, if C_u contained an special element before the changes, then after the changes, it still contains an special element. The same is true for extra special element. It is not hard to see that conditions (i) to (v) are satisfied.

Suppose $w \neq x$. Then, $w \in S$. Note that C_w contains a special element, say d_3 (from the labelling of the edges of the part of the Hamiltonian path F_x, from u_y to u_x). Now, change all the labels of the labelled edges according to the following rules: if $c < d_3$, then c remains the same; and if $c \geq d_3$, then c is changed to $c+1$. After that label the edge wu with d_3. As before, we see that conditions (i) to (v) are satisfied.

We do the same for the tree T_z in F_z that contains z except that we do not label the edge $u_y z$. After we have done all the labelling except for the edges xu_y and zu_y, conditions (i) to (v) are satisfied. Now, only the edges xu_y and zu_y are not labelled. Now, we set $E_{xz} = E_{xu_y} = E_{zu_y}$.

Note that E_{uv} is unique for all edges uv in the interior of G_1 except for edges xz, xu_y and zw where $E_{xz} = E_{xu_y} = E_{zu_y}$. Recall that C_{u_y} contains a special and an extra special element. Note that if $u \in S \setminus \{u_y\}$, then $g(C_u)$ contains at most three consecutive parts: one part arises from the labelling of edges of the Hamiltonian path F_x; one part arises from the labelling of edges of the spanning tree F_y and one part arises from the labelling of edges of the tree T_x or T_z. The labelling of edges of F_y will contribute a single element to $g(C_u)$. The same is true for labelling of edges of T_x or T_z. If $v_i v_{i+1} v_{i+2}$ is the part of the Hamiltonian path with $v_{i+1} = u$ and $E_{v_i v_{i+1}} = \{c_1\}$, $E_{v_{i+1} v_{i+2}} = \{c_2\}$, then $\{c_1, c_1+1, \ldots, c_2\} \subseteq g(C_u)$. If $u = u_y = v_1$, $E_{v_1 v_2} = \{c_4\}$ and c_3 is the extra special element in C_{u_y}, then $\{c_3, c_3+1, \ldots, c_4\} \subseteq g(C_{u_y})$ (it is possible that $c_3 = c_4$). The labelling of edges of F_y will contribute a single element to $g(C_{u_y})$ and the third element is from the edges xu_y and zu_y (since $E_{xu_y} = E_{zu_y}$). It is not hard to see that conditions (i) to (v) are satisfied.

Case 2. $E_{xy} = E_{yz} = E_{xz}$ and C_x contains an extra special element.

Case 2.1. Suppose $|S|= 1$. This means $G[S \cup \{x, y, z\}] = K_4$ and $u_x = u_y = u_z$. Let d_0 be the extra special element in C_x. Now, change all the labels of the labelled edges according to the following rules: if $c < d_0$, then c remains the same; and if $c \geq d_0$, then c is changed to $c+1$. After that label the edge xu_y with d_0, i.e., $E_{xu_y} = \{d_0\}$. It is not hard to see that conditions (i) to (v) are satisfied.

Since $y \in V(G')$, there is a special element, say d_1 in C_y. Now, change all the labels of the labelled edges according to the following rules: if $c < d_1$, then c remains the same; and if $c \geq d_1$, then c is changed to $c+1$. After that label the edge yu_x with d_1, i.e., $E_{yu_x} = \{d_1\}$. Again, conditions (i) to (v) are satisfied.

Since $z \in V(G')$, there is a special element, say d_2 in C_z. Now, change all the labels of the labelled edges according to the following rules: if $c < d_2$, then c remains the same; and if $c \geq d_2$, then c is changed to $c+1$. After that label the edge zu_y with d_2, i.e., $E_{zu_y} = \{d_2\}$. Again, conditions (i) to (v) are satisfied.

Case 2.2. Suppose $|S|\geq 2$. Let d_0 be the extra special element in C_x. Now, change all the labels of the labelled edges according to the following rules: if $c < d_0$, then c remains the same; and if $c \geq d_0$, then c is changed to $c+s$. Now, F_x is a Hamiltonian path of $G_1 - \{y, z\}$ going from x to u_x. Recall that xu_y is an edge in F_x (Remark 4.7). Therefore, the Hamiltonian path is from x to u_y and then from u_y to u_x. Let the Hamiltonian path in reverse direction from $u_x = v_1$ to $x = v_{s+1}$ be $v_1 v_2 \ldots v_{s+1}$. Note that $S = \{u_x = v_1, v_2, \ldots, v_s = u_y\}$ and $s \geq 2$. Now, for $1 \leq i \leq s$, we set $E_{v_i v_{i+1}} = \{d_0+i-1\}$.

Before the changes and the labelling of edges of the Hamiltonian path, d_0 was the initial point of a consecutive part of of $g(C_x)$. After the changes and labelling of edges of the Hamiltonian path, $d_0 + s - 1$ is added to $g(C_x)$ (from $E_{xu_y} = \{d_0 + s - 1\}$). So, $\{d_0 + s - 1, d_0 + s, \dots\}$ is a consecutive part of $g(C_x)$ and $|g(C_x)|$ remains the same. However, C_x has no extra special element anymore. If $b \neq d_0$ was a special element of C_x and $b < d_0$ before the changes, then b is still a special element of C_x after the changes. If d_0 was also special in C_x before the changes, then d_0 was the initial point and also the terminal point of a consecutive part of $g(C_x)$. So, the consecutive part in C_x that contained d_0 before the changes was $\{d_0\}$ a set consists of one element. After the changes and labelling of edges of the Hamiltonian path, $\{d_0 + s - 1, d_0 + s\}$ is a consecutive part of $g(C_x)$. Therefore, C_x still contains a special element, which is $d_0 + s$.

Now, we shall label the edges in F_y. We may consider y to be the root of the spanning tree F_y. Now, let $u \in V(F_y) \setminus \{y\}$ and w be the unique vertex adjacent to u on the path from u to y. Suppose $w = y$. Since $y \in V(G')$, there is a special element say d_1 in C_y. Now, change all the labels of the labelled edges according to the following rules: if $c < d_1$, then c remains the same; and if $c \geq d_1$, then c is changed to $c + 1$. After that label the edge yu with d_1, i.e., $E_{yu} = \{d_0\}$. It is not hard to see that conditions (i) to (v) are satisfied.

Suppose $w \neq y$. Then, $w \in S$. Note that C_w contains a special element, say d_2 (from the labelling of the edges of the Hamiltonian path F_x in reverse direction, from u_x to x). Now, change all the labels of the labelled edges according to the following rules: if $c < d_2$, then c remains the same; and if $c \geq d_2$, then c is changed to $c + 1$. After that label the edge wu with d_2. Again, conditions (i) to (v) are satisfied.

We shall label the edges in F_z using similar process. Note that E_{uv} is unique for all edges uv in the interior of G_1. Furthermore, if $u \in S$, then $g(C_u)$ contains at most three consecutive parts: one part arises from the labelling of edges of the Hamiltonian path F_x in reverse order; one part arises from the labelling of edges of the spanning tree F_y and one part arises from the labelling of edges of the tree T_x or T_z. It is not hard to see that conditions (i) to (v) are satisfied.

This completes the proof of the theorem. $\square$

The following corollary (which is Theorem 1.1) then follows from Corollary 2.9, Lemma 3.4 and Theorem 4.8.

Corollary 4.9. *If G is planar, then $i(G) \leq 3$.*

5. Minimum size number

By Lemmas 3.2 and 3.7, $G = \mathfrak{F}(\{B_u\}_{u \in V(G)})$ for some $\{B_u\}_{u \in V(G)} \subseteq 2_0^{\mathbb{N}}$ such that $L(\{B_u\}_{u \in V(G)}) = \{1, 2, \dots, m\}$ and each $i \in \{1, 2, \dots, m\}$ is contained in at least two B_u's. This motivates us to consider the following two parameters:

$$m_{\mathcal{T}_1}(G) = \min\left\{\left|L(\{B_u\}_{u \in V(G)})\right| \ : \ G = \mathfrak{F}(\{B_u\}_{u \in V(G)}) \ \text{ for some } \ \{B_u\}_{u \in V(G)} \subseteq 2_0^{\mathbb{N}}\right\};$$

$$m_{\mathcal{T}_2}(G) = \min\Big\{\big|L(\{B_u\}_{u\in V(G)})\big| \ : \ G = \mathfrak{F}(\{B_u\}_{u\in V(G)}) \ \text{ for some } \ \{B_u\}_{u\in V(G)} \subseteq 2_0^{\mathbb{N}} \\ \text{and } \ |B_u \cap B_v| \le 1 \ \text{ for all } \ u,v \in V(G) \ \text{ with } \ u \neq v\Big\}.$$

The two parameters $m_{\mathcal{T}_1}(G)$ and $m_{\mathcal{T}_2}(G)$ will be called minimum $\mathcal{T}_1$-size number and minimum $\mathcal{T}_2$-size number, respectively. The following lemma is obvious.

Lemma 5.1. *$m_{\mathcal{T}_1}(G) \le m_{\mathcal{T}_2}(G)$.*

We note here that in general $m_{\mathcal{T}_1}(G) \neq m_{\mathcal{T}_2}(G)$. In fact, in Example 3.5, it can be seen that $m_{\mathcal{T}_1}(G) = 2$ and $m_{\mathcal{T}_2}(G) = 3$.

Lemma 5.2. *The following statements are equivalent:*

(a) *G is a complete graph.*

(b) *$m_{\mathcal{T}_2}(G) = 1$.*

(c) *$m_{\mathcal{T}_1}(G) = 1$.*

Proof. ((a) $\Rightarrow$ (b)) Suppose G is a complete graph. We set $B_u = \{1\}$ for all $u \in V(G)$. Then, $G = \mathfrak{F}(\{B_u\}_{u\in V(G)})$ and $|B_u \cap B_v| = 1$ for all $u, v \in V(G)$ with $u \neq v$. Hence, $L(\{B_u\}_{u\in V(G)}) = \{1\}$ and $m_{\mathcal{T}_2}(G) = 1$.

((b) $\Rightarrow$ (c)) It follows from Lemma 5.1.

((c) $\Rightarrow$ (a)) Since $m_{\mathcal{T}_1}(G) = 1$, there is a $\{B_u\}_{u\in V(G)} \subseteq 2_0^{\mathbb{N}}$ such that $G = \mathfrak{F}(\{B_u\}_{u\in V(G)})$ and $B_u = B_v$ for all $u, v \in V(G)$. Therefore, G is a complete graph. $\square$

Lemma 5.3. *$m_{\mathcal{T}_2}(G) \le |E(G)|$.*

Proof. Label each of the edges of G with an element in $\mathbb{N}$. All the edges are labelled differently, i.e., $E_{uv} \neq E_{u'v'}$ for distinct edges uv and $u'v'$. By Lemma 4.5, $\{E_{uv}\}_{u\sim_G v}$ is $\mathcal{T}_2$-compatible. By part (b) of Lemma 3.8, $G = \mathfrak{F}(\{C_u\}_{u\in V(G)}) \in \mathcal{T}_2$ where $C_u = \bigcup_{v\in N_G(u)} E_{uv}$ for all $u \in V(G)$. Note that $L(\{C_u\}_{u\in V(G)}) = \bigcup_{u\sim_G v} E_{uv}$. Hence, $|L(\{C_u\}_{u\in V(G)})| = |E(G)|$ and $m_{\mathcal{T}_2}(G) \le |E(G)|$. $\square$

Lemma 5.4. *If G is not K_3-free, then $m_{\mathcal{T}_2}(G) \le |E(G)| - 2$.*

Proof. If G is not K_3-free, then there are three vertices x, y, z such that $G[\{x, y, z\}] = K_3$ is a triangle. Label the three edges with 1. Next, label the rest of the edges with different elements in $\mathbb{N}\backslash\{1\}$, i.e., $E_{uv} \neq E_{u'v'}$ for distinct edges uv and $u'v'$ in $E(G)\backslash\{xy, yz, zx\}$. By Lemma 4.5, $\{E_{uv}\}_{u\sim_G v}$ is $\mathcal{T}_2$-compatible. By part (b) of Lemma 3.8, $G = \mathfrak{F}(\{C_u\}_{u\in V(G)}) \in \mathcal{T}_2$ where $C_u = \bigcup_{v\in N_G(u)} E_{uv}$ for all $u \in V(G)$. Note that $L(\{C_u\}_{u\in V(G)}) = \bigcup_{u\sim_G v} E_{uv}$. So, $|L(\{C_u\}_{u\in V(G)})| = |E(G)| - 2$ and $m_{\mathcal{T}_2}(G) \le |E(G)| - 2$. $\square$

Theorem 5.5. *The following statements are equivalent:*

(a) *G is K_3-free.*

(b) $m_{\mathcal{T}_1}(G) = |E(G)|$.

(c) $m_{\mathcal{T}_2}(G) = |E(G)|$.

Proof. ((a) $\Rightarrow$ (b)) Suppose G is K_3-free. By Lemmas 5.1 and 5.3, $m_{\mathcal{T}_1}(G) \leq |E(G)|$. Suppose $m_{\mathcal{T}_1}(G) < |E(G)|$. Then, there exists a $\{B_u\}_{u\in V(G)} \subseteq 2_0^{\mathbb{N}}$ such that $G = \mathfrak{F}(\{B_u\}_{u\in V(G)})$ and $\left|L(\{B_u\}_{u\in V(G)})\right| = m_{\mathcal{T}_1}(G)$. If $L(\{B_u\}_{u\in V(G)})$ contains an element c such that c is contained only in B_{u_0} for a particular vertex u_0, then we set $C_{u_0} = B_{u_0} \setminus \{c\}$ and $C_u = B_u$ for all $u \in V(G)$. Note that $C_u \cap C_v = B_u \cap B_v$ for all $u, v \in V(G)$ as $c \notin B_v$ for all other $v \in V(G) \setminus \{u_0\}$. Therefore, $G = \mathfrak{F}(\{C_u\}_{u\in V(G)})$ and $\left|L(\{C_u\}_{u\in V(G)})\right| < \left|L(\{B_u\}_{u\in V(G)})\right| = m_{\mathcal{T}_1}(G)$, a contradiction. Hence, we may assume that each $c \in L(\{B_u\}_{u\in V(G)})$ is contained in at least two different B_u's.

Let $E_{uv} = B_u \cap B_v$ for all $u \sim_G v$. Note that $|E_{uv}|$ may be greater than 1. Let $c \in L(\{B_u\}_{u\in V(G)})$. There are distinct vertices u_1 and u_2 such that $c \in B_{u_1} \cap B_{u_2} = E_{u_1u_2}$. Thus, $L(\{B_u\}_{u\in V(G)}) \subseteq \bigcup_{u\sim_G v} E_{uv}$. Clearly, $\bigcup_{u\sim_G v} E_{uv} \subseteq L(\{B_u\}_{u\in V(G)})$. Hence, $L(\{B_u\}_{u\in V(G)}) = \bigcup_{u\sim_G v} E_{uv}$.

Suppose $E_{u_1v_1} \cap E_{u_2v_2} \neq \varnothing$ for some distinct edges u_1v_1 and u_2v_2. We may assume that u_1, v_1 and u_2 are distinct vertices. Let $c \in E_{u_1v_1} \cap E_{u_2v_2}$. Then, $c \in B_{u_1} \cap B_{v_1} \cap B_{u_2}$. So, $G[\{u_1, v_1, u_2\}] = K_3$, a contradiction. So, we may assume that $E_{uv} \cap E_{u'v'} = \varnothing$ for distinct edges uv and $u'v'$.

Now,

$$m_{\mathcal{T}_1}(G) = \left|L(\{B_u\}_{u\in V(G)})\right| = \left|\bigcup_{u\sim_G v} E_{uv}\right| = \sum_{u\sim_G v} |E_{uv}| \geq |E(G)|,$$

a contradiction. Hence, $m_{\mathcal{T}_1}(G) = |E(G)|$.

((b) $\Rightarrow$ (c)) It follows from Lemmas 5.1 and 5.3.

((c) $\Rightarrow$ (d)) It follows from Lemma 5.4. □

Conflict of Interest

The authors have no relevant financial or non-financial interests to disclose.

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References

[1] V. Ardévol Martínez, R. Rizzi, A. Saffidine, F. Sikora, and S. Vialette. Generalizing roberts' characterization of unit interval graphs. In R. Královič and A. Kučera, editors, *49th International Symposium on Mathematical Foundations of Computer Science (MFCS 2024)*, volume 306 of *Leibniz International Proceedings in Informatics (LIPIcs)*, 12:1–12:15,

Dagstuhl, Germany. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2024. http://doi.org/10.4230/LIPIcs.MFCS.2024.12.

[2] V. Ardévol Martínez, R. Rizzi, F. Sikora, and S. Vialette. Recognizing unit multiple interval graphs is hard. *Discrete Applied Mathematics*, 360:258–274, 2025. http://doi.org/10.1016/j.dam.2024.09.011.

[3] J. Balogh and A. Pluhár. A sharp edge bound on the interval number of a graph. *Journal of Graph Theory*, 32(2):153–159, 1999. http://doi.org/10.1002/(SICI)1097-0118(199910)32:2<153::AID-JGT5>3.0.CO;2-P.

[4] S. Chaplick. Intersection graphs of non-crossing paths. *Discrete Mathematics*, 346(8):113498, 2023. http://doi.org/10.1016/j.disc.2023.113498.

[5] S. Chaplick, M. Töpfer, J. Voborník, and P. Zeman. On H-topological intersection graphs. *Algorithmica*, 83(11):3281–3318, 2021. http://doi.org/10.1007/s00453-021-00846-3.

[6] S. Chaplick and T. Ueckerdt. Planar graphs as VPG-graphs. In W. Didimo and M. Patrignani, editors, *Graph Drawing: 20th International Symposium, GD 2012*, volume 7704 of *Lecture Notes in Computer Science*, pages 174–186, Berlin, Heidelberg. Springer, 2013. http://doi.org/10.1007/978-3-642-36763-2_16.

[7] C. Chekuri and T. Inamdar. Algorithms for intersection graphs for t-intervals and t-pseudodisks. *Theory of Computing*, 18(18):1–19, 2022. http://doi.org/10.4086/toc.2022.v018a018.

[8] A. B. de Queiroz, V. Garnero, and P. Ochem. On interval representations of graphs. *Discrete Applied Mathematics*, 202:30–36, 2016. http://doi.org/10.1016/j.dam.2015.08.022.

[9] G. Guégan, K. Knauer, J. Rollin, and T. Ueckerdt. The interval number of a planar graph is at most three, 2018. 10.48550/arXiv.1805.02947. arXiv: 1805.02947 [math.CO].

[10] G. Guégan, K. Knauer, J. Rollin, and T. Ueckerdt. The interval number of a planar graph is at most three. *Journal of Combinatorial Theory, Series B*, 146:61–67, 2021. http://doi.org/10.1016/j.jctb.2020.07.006.

[11] K. Knauer and T. Ueckerdt. Decomposing 4-connected planar triangulations into two trees and one path. *Journal of Combinatorial Theory, Series B*, 134:88–109, 2019. http://doi.org/10.1016/j.jctb.2018.05.006.

[12] J. Matoušek. Near-optimal separators in string graphs. *Combinatorics, Probability and Computing*, 23(1):135–139, 2014. http://doi.org/10.1017/S0963548313000400.

[13] E. R. Scheinerman and D. B. West. The interval number of a planar graph: three intervals suffice. *Journal of Combinatorial Theory, Series B*, 35(3):224–239, 1983. http://doi.org/10.1016/0095-8956(83)90050-3.

[14] J. Trotter William T. and F. Harary. On double and multiple interval graphs. *Journal of Graph Theory*, 3(3):205–211, 1979. http://doi.org/10.1002/jgt.3190030302.

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