H-Decompositions of Generalized Johnson Graphs

Braxton Carrigan, Aaron Clark, and James Hammer

Abstract

A family of graphs, called Generalized Johnson graphs, are an abstraction of both Kneser and Johnson graphs. Given the symmetric nature of Generalized Johnson graphs, we provide various decompositions of these graphs and show non-trivial instances of the inability to decompose such graphs into triples.

1 Introduction

The Johnson and Kneser graphs are two well known families of graphs on the same set of vertices defined by the same two positive integer parameters n and k. Each graph is on a set of $\binom{n}{k}$ vertices corresponding to the k-subsets of a set of size n. The edges of each are defined slightly differently: in the case of the Kneser graph, two vertices are adjacent if the subsets corresponding to the vertices have no elements in common, whereas in Johnson graphs, two vertices are adjacent if the subsets corresponding to the vertices have exactly $k-1$ elements in common. While others have provided generalizations to Kneser graphs, we wish to explore a generalization which includes both Kneser and Johnson graphs.

Definition 1.1. *A **generalized Johnson graph**, denoted as $J(n,k,i)$, is defined by three non-negative integers, n, k, and i where the vertex set is all the k-subsets of a set of size n and two k-subsets, S and T, are adjacent if and only if $|S \cap T| = i$.*

To avoid the degenerate and edge-less graphs, we will assume

$$n > k > i \geq n - k > 0. \qquad (1)$$

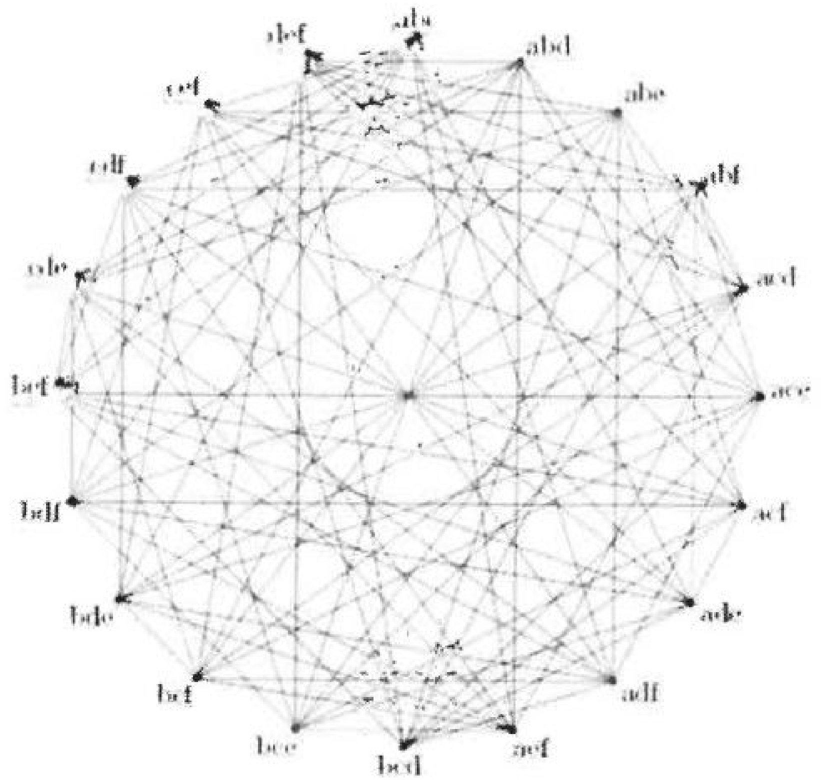
We make the following observations and provide one example to orient the reader to the family of generalized Johnson graphs.

Observation 1. *$J(n,k,i)$ is vertex and edge transitive.*

Observation 2. $J(n,k,i) = J(n,n-k,n-k-i)$

Observation 3. *$J(n,1,0)$ is the complete graph K_n, $J(n,k,0)$ is the Kneser graph $KG_{n,k}$, and $J(n,k,k-1)$ is the Johnson graph $J(n,k)$.*

Example 1.2. $J(6,3,1)$



In this paper we will be partitioning generalized Johnson graphs into smaller graphs, more specifically we are looking for H-decompositions.

Definition 1.3. *An H-**decomposition** of a graph G is a partition of the edges of G such that each part forms a graph isomorphic to H.*

Since we will be concerned with H-decompositions of generalized Johnson graphs, the following combinatorial observations concerning generalized Johnson graphs will be useful.

Observation 4. *Given $J(n,k,i) = G(V,E)$*

- $|V| = \binom{n}{k}$
- *For every $v \in V$, $deg(v) = \binom{k}{i} \cdot \binom{n-k}{k-i}$*
- $|E| = \binom{n}{k} \cdot \binom{k}{i} \cdot \binom{n-k}{k-i} / 2 = \dfrac{n!}{2(k-i)!^2 i!(n-2k-i)!}$

While [4] showed the graph decomposition question is NP-complete whenever H is connected and has three or more edges, the literature is rich with specific families of H-decomposable graphs. A widely studied H-decomposition problem is the construction of Steiner systems, which are specific H-decomposition where $H = K_n$. First solved for triple systems in [1] and [6], recent results such as [5] and [2] have extended the design of H-decompositions to the case where H is a clique or cycle. In this paper we further the study of the existence of decompositions for generalized Johnson graphs.

2 Clique Decomposition of $J(n, k, i)$

Given the literature, it is natural to search for K_m-decomposition of graphs. Furthermore, given Observation 4 we get the following trivial necessary conditions for K_m-decompositions of $J(n, k, i)$.

1. $m - 1$ must divide $\binom{k}{i} \cdot \binom{n-k}{k-i}$
2. $\binom{m}{2}$ must divide $\frac{n!}{2(k-i)!^2 i!(n-2k-i)!}$

Let us first start with an example construction before providing a general decomposition construction in Theorem 2.2.

Example 2.1. *There exists a K_3-decomposition of $J(11, 3, 2)$.*

To design a K_3-decomposition of $J(11, 3, 2)$, first label each edge of the graph with the 2-element subset that is the intersection of its incident vertices. We will call this the edge's intersection label. Now notices that the edges of $J(11, 3, 2)$ are partitioned into $\binom{11}{2}$ disjoint subsets by their intersection label. Furthermore, the subset of edges with the same intersection label induces a K_9 subgraph. Therefore we have a K_9-decomposition of $J(11, 3, 2)$ corresponding to the intersection labels. Furthermore, since $9 \equiv 3 \pmod 6$, each subset of edges with the same label can further be decomposed into K_3's with the Steiner triple system on 9 elements provided by [1].

Example 2.1 works specifically because the edges with the same label induce a complete graph which is known to contain a Steiner triple system. Unfortunately, this is not the case for all Generalized Johnson graphs, but we can still generalize this approach to some degree. In reality the K_9-decomposition in Example 2.1 should be viewed as a $J(9, 1, 0)$-decomposition as a result of the following Theorem.

Theorem 2.2. *There exists a $J(n - i, k - i, 0)$-decomposition of $J(n, k, i)$.*

Proof. As seen in Example 2.1 we can label each edge with the i-element subset that is the intersection of its incident vertices. *Figure 1 illustrates such a labeling system, where highlighted edges are labeled $\{a\}$ from the intersection of incident vertex labels, which induce a $J(5, 2, 0)$ subgraph of* $J(6, 3, 1)$. Pick any i-element subset of the set of size n and call it S. Since each vertex is a k-element subset, there exists $\binom{n-i}{k-i}$ vertices with S as a subset and the edges labelled S induce a Kneser subgraph $J(n-i, k-i, 0)$ of $J(n, k, i)$. Therefore the $\binom{n}{i}$ subsets each induce edge disjoint Kneser subgraphs which form a decomposition of $J(n, k, i)$. □

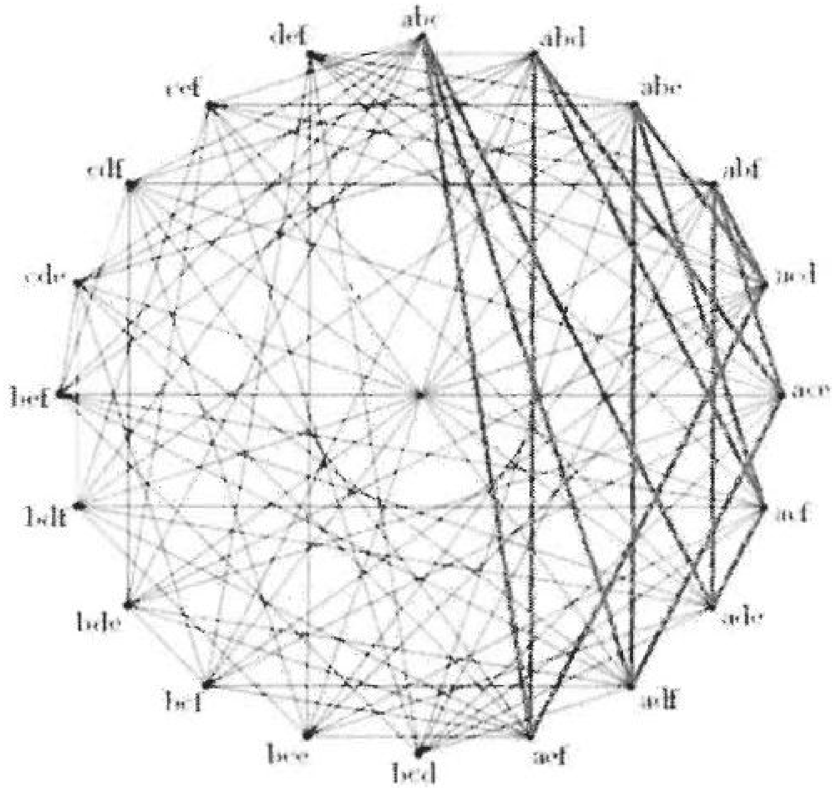


Figure 1: Induced $J(5, 2, 0)$ Subset with the $\{a\}$-labelled edges

Corollary 2.3. *If there exists an H-decomposition of the Kneser graph $J(n-i, k-i, 0)$, then there is an H-decomposition of $J(n, k, i)$.*

Corollary 2.4. *Given a Johnson graph $J(n, k) = J(n, k, k-1)$, there exists a K_{n-k+1}-decomposition of $J(n, k)$.*

Corollary 2.4, which follows immediately from Observation 3 is quite useful given the expansive literature on finding an H-decomposition of complete graphs. For example, we know there is a K_3-decomposition of $J(n, k, k-1)$ whenever $n - k + 1 \equiv 1$ or 3 (mod 6).

In a similar manner, we can apply the cycle decompositions from [2] or other similar decompositions from the Handbook of Combinatorial Design [3]. In comparison, Corollary 2.3 is limited by the relationships between n, k, and i. This leads us to the exploration of decompositions where this process fails.

3 K_3-Decompositions of $J(n,k,i)$

As demonstrated in Example 2.1, we will now focus specifically on K_3-decompositions of $J(n,k,i)$. We will begin exploring the well known Johnson and Kneser families of graphs by considering $k = 2$, in which case $J(n,2,1)$ is the Johnson graph $J(n,2)$, while $J(n,2,0)$ is the Kneser graph $KG_{n,2}$.

Example 3.1. *There exists a K_3-decomposition of $J(n,2)$ for $n \geq 3$.*

This can be shown by way of induction on $n+4$, so it is necessary to have a K_3-decomposition for $n = 3, 4, 5,$ and 6. We leave it to the reader to create these small examples; they can also be extracted from the proof of Theorem 3.2.

Assume there is a K_3-decomposition of $J(n-4,2)$ and let the vertices be all the pairs of the set $[n] = \{s \in \mathbb{Z} \mid 1 \leq s \leq n\}$ and construct a K_3-decomposition of $J(n,2)$ for $n > 6$ in the following manner.

Without loss of generality, let $(1,2)-(1,3)-(1,4)$ be a triple in the decomposition and define $S = \{s \in \mathbb{Z} \mid 5 \leq s \leq n\}$. Now consider the vertices that are **not** *neighbors of any of the three vertices of the assumed triple. This is the set of vertices $N = \{(x,y)|x,y \in S\}$. The induced subgraph on N is isomorphic to $J(n-4,2)$, therefore by induction contains a K_3-decomposition. Hence we will include each such triple in the K_3-decomposition of $J(n,2)$.*

Now define $N_{(x,y)}$ to be the set of neighbors of vertex (x,y). We wish to consider the following intersection of neighbor sets.

$$N_1 = N_{(1,2)} \cap N_{(1,3)} \cap N_{(1,4)} = \{(1,x)|x \in S\}$$

$$N_2 = N_{(1,2)} - (N_{(1,3)} \cap N_{(1,4)}) = \{(2,x)|x \in S\}$$

$$N_3 = N_{(1,3)} - (N_{(1,2)} \cap N_{(1,4)}) = \{(3,x)|x \in S\}$$

$$N_4 = N_{(1,4)} - (N_{(1,2)} \cap N_{(1,3)}) = \{(4,x)|x \in S\}$$

Each induced subgraph on the vertices of N_i is isomorphic to K_{n-4} and each edge of these induced subgraphs will be paired with the vertex of N to form a K_3 in our decomposition. More precisely, an edge $(i,x)-(i,y)$, where $1 \leq i \leq 4$ and $x,y \in S$ is in the triple $(i,x)-(i,y)-(x,y)$ of the decomposition.

Now we will show the remaining triples in Figure 2, where the assumed triple $(1,2)-(1,3)-(1,4)$ is shown in bold.

1. *$(1,i)-(1,x)-(i,x)$ for $2 \leq i \leq 4$ and $x \in S$ (solid)*

2. *$(i,j)-(i,x)-(j,x)$ for $2 \leq i < j \leq 4$ and $x \in S$ (dashed)*

3. *$(i,j)-(i,k)-(i,l)$ for $2 \leq i \leq 4$ and $\{j,k,l\} = S - \{i\}$ (dotted)*

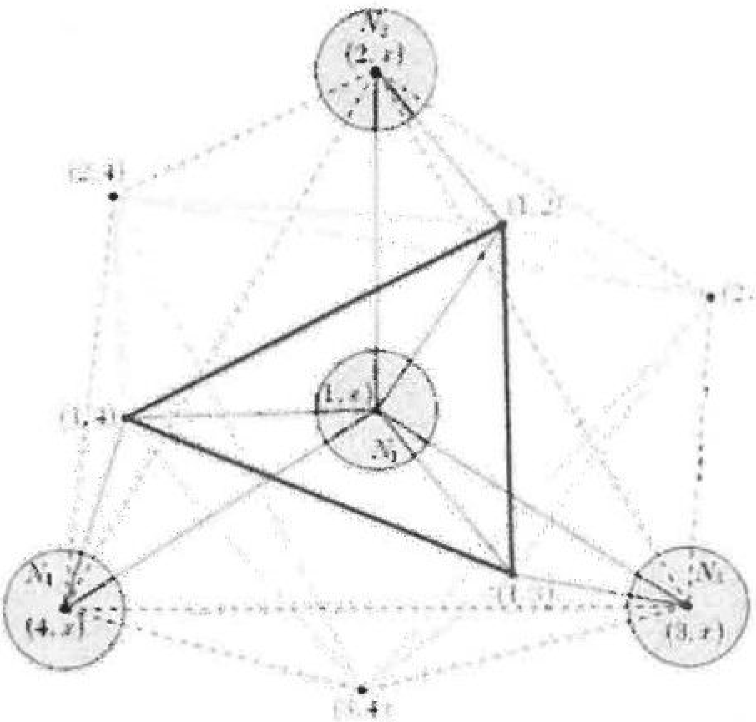


Figure 2: Remaining Triples in K_3-decomposition

Let us now generalize the result of Example 3.1

Theorem 3.2. *If $k = 2i$, then there exists a K_3-decomposition of $J(n,k,i)$.*

Proof. Assign each vertex of $J(n,k,i)$ a unique k-element subset from a set of size n, where $\{u\}$ is the k-element subset assigned to the vertex $u \in J(n,k,i)$ so that any edge $(u,v) \in J(n,k,i)$, satisfies the property that $|\{u\} \cap \{v\}| = i$, as is used to define generalized Johnson graphs. Therefore $|\{u\} \cup \{v\} - \{u\} \cap \{v\}| = k$. Let w be the vertex corresponding to the k-element subset $\{u\} \cup \{v\} - \{u\} \cap \{v\}$ and define $t_{u,v} = \{(u,v),(u,w),(v,w)\}$ to be a K_3 subgraph of $J(n,k,i)$ containing edge (u,v).

We see that $\{u\} = \{w\} \cup \{v\} - \{w\} \cap \{v\}$ and $\{v\} = \{u\} \cup \{w\} - \{u\} \cap \{w\}$, Hence $t_{u,v}$ is unique and $t_{u,v} = t_{u,w} = t_{v,w}$. Therefore $\bigcup_{(x,y) \in J(n,k,i)} t_{x,y}$ is a K_3-decomposition of $J(n,k,i)$ □

This result naturally raises the question of the conditions under which $J(n,k,i)$ admits a K_3-decomposition. As the next Theorem shows, the answer is not immediate.

Observation 5. *$KG_{5,2}$ is the Petersen Graph, which has exactly six perfect matchings that form two 1-factorizations of the graph. Furthermore, the Petersen Graph only has one perfect matching, up to symmetry.*

Theorem 3.3. *$KG_{7,2} = J(7,2,0)$ is not K_3-decomposable*

Proof. By way of contradiction, we assume a K_3-decomposition exists. Since $KG_{7,2}$ is vertex and edge transitive (Observation 1), we can consider the triples of a specific vertex. So, without loss of generality, let that vertex be $(1,2)$. Now the induced subgraph on the neighbors of $(1,2)$ is $KG_{5,2}$ on the element set $\{3,4,5,6,7\}$; we will call this the $KG_{5,2}$ neighborhood of $(1,2)$. Thus any K_3 containing $(1,2)$ would contain exactly one edge from the $KG_{5,2}$ neighborhood of $(1,2)$, hence the five triples containing $(1,2)$ form a perfect matching in the $KG_{5,2}$ neighborhood of $(1,2)$. Now by Observation 5 we can arbitrarily select a perfect matching of the $KG_{5,2}$ neighborhood of $(1,2)$.

Alternatively, we consider an arbitrary K_3 in the decomposition. Therefore, without loss of generality, assume $(1,2)-(3,4)-(5,6)$ is a triple of the decomposition. Then there are exactly two perfect matchings in the $KG_{5,2}$ neighborhood of $(1,2)$ containing edge $(3,4)-(5,6)$; these are shown in Figure 3 as the set of four dashed edge and the set of four dotted edges. The grey edges in Figure 3 represent edges that cannot be in a triple with $(1,2)$ in the decomposition of $KG_{7,2}$ containing the assumed triple.

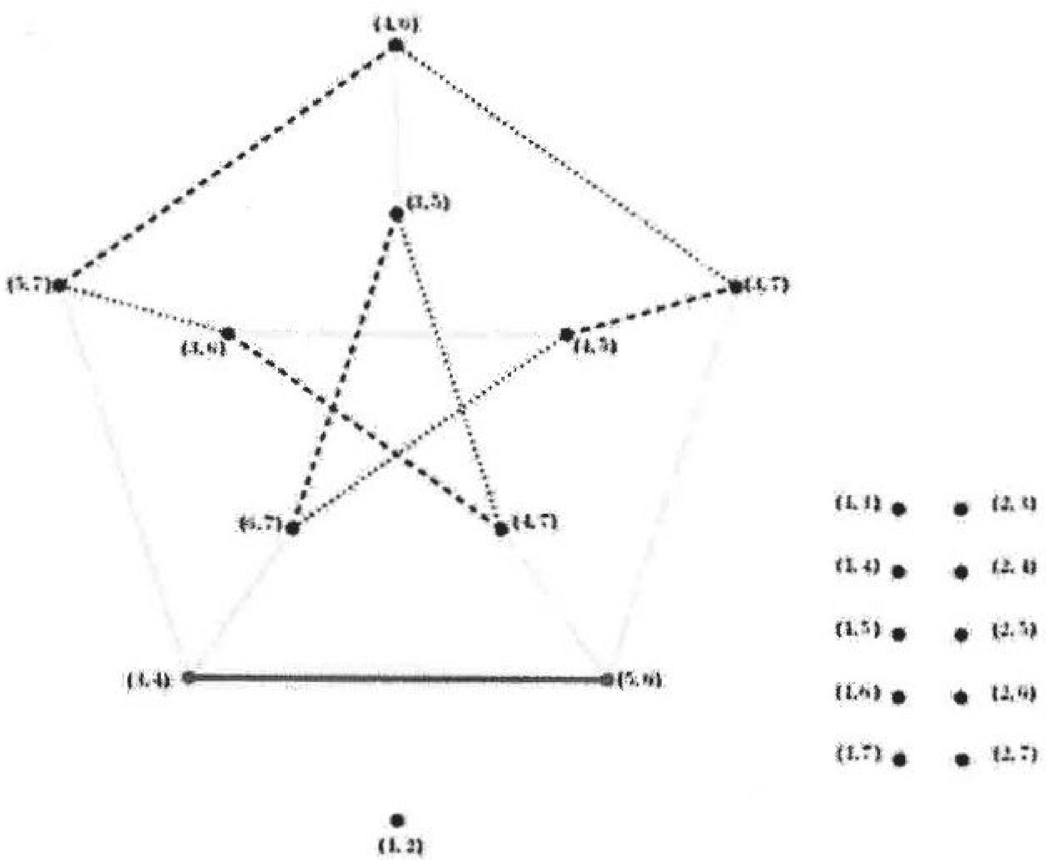


Figure 3: Perfect Matchings in the $KG_{5,2}$ neighborhood of $(1,2)$

Since the vertex $(1,2)$ was arbitrarily chosen, this argument applies to any vertex. Hence we can assume an arbitrary triple is in the assumed K_3-

decomposition and select a perfect matching of the $KG_{5,2}$ neighborhood of each vertex that contains the other edge from the triple. Furthermore the selection of these three perfect matchings is arbitrary as well. This is shown in Figure 4, where we assume the triple $T = (1,2) - (3,4) - (5,6)$ is in the decomposition and have chosen the solid edges as the three perfect matchings. (*Alternatively, selecting any of the sets of dotted edges associated with the vertex and edge pair would produce an isomorphic subgraph.*) Therefore, by Observation 1, these three selections are now arbitrary. The grey edges in Figure 4 represent the edges in the $KG_{5,2}$ neighborhoods of these vertices which cannot be in a K_3 with any of the three vertices of T.

The figure is organized so that the $KG(5,2)$ neighborhood of a vertex in T is surrounding that vertex on the same half-plane bounded by the edge not incident to it in T.

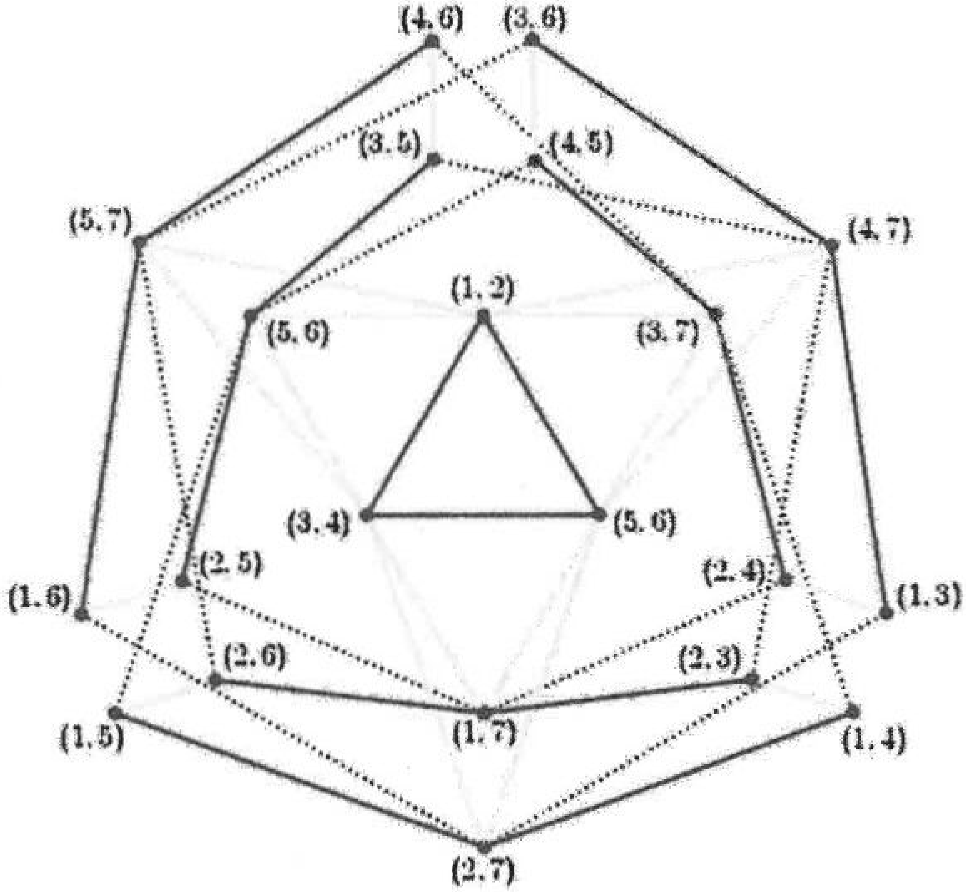
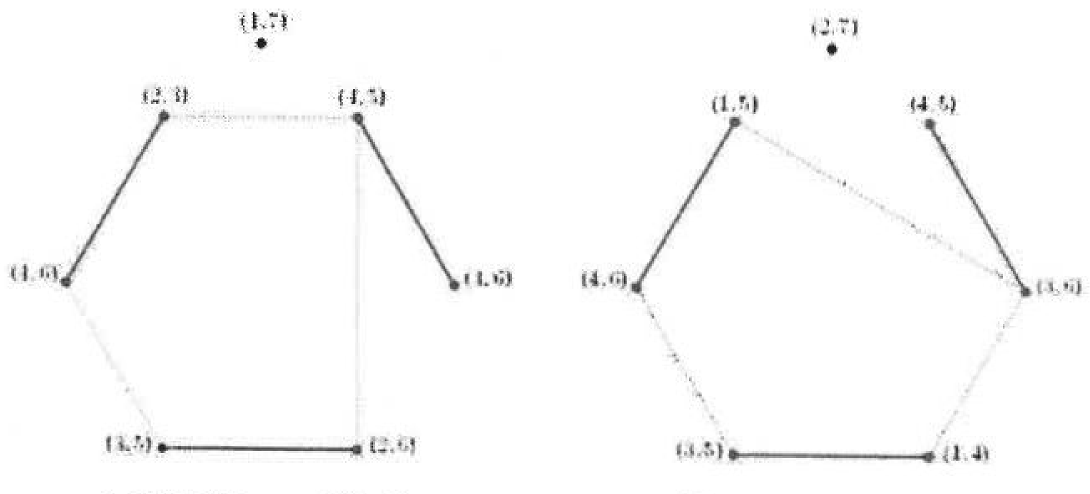


Figure 4: Arbitrary selecting thirteen K_3s

Therefore, without loss of generality, we can assume each edge of these 1-factorizations of the $KG_{5,2}$ neighborhood subgraphs form the triple with their corresponding vertex. Hence we have thirteen triples of the K_3-decomposition. Now we must consider $KG_{7,2}$ with the edges of these triples removed and once again look at the neighborhoods of the remaining vertices. The remaining graph is still symmetric in a sense, however, it is no longer vertex or edge transitive. There are six vertices of degree 6, twelve of degree 12 and three vertices of degree zero. We will focus on the vertices of degree 6. Specifically, consider vertices labelled $(1,7)$ and $(2,7)$. Figure 5 shows the induced subgraphs of their remaining neighbors. Here we notice each graph has a unique perfect matching, denoted with the solid edges. Hence these edges must be in a triple of the decomposition with the corresponding neighbor vertex. However, in these two neighborhoods, edge $(3,6)-(4,5)$ is in both perfect matchings. Therefore, we have reached a contradiction since this edge must be in a K_3 with both vertex $(1,7)$ and vertex $(2,7)$. *A similar contradiction is found in the neighborhoods of* $(3,7)$ *and* $(4,7)$ *and the neighborhoods of* $(5,7)$ *and* $(6,7)$.



(a) Neighbors of $(1,7)$ (b) Neighbors of $(2,7)$

Figure 5: Edge Belonging to Two Triples

□

4 Conclusion

As noted in the introduction, the symmetric nature of generalized Johnson graphs makes them ripe for H-decompositions. Theorems 2.2 and 3.2 highlight the use of symmetry in constructing such decompositions. However, Theorem 3.3 shows that a highly symmetric graph satisfying the combinatorial necessities still may not be decomposable. Thus we are left with the open question, "For what 4-tuples, (n,k,i,H) is $J(n,k,i)$ H-decomposable?"

References

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