Integral Regular Split Multigraphs

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May 8, 2019

Abstract

A split graph is a graph whose vertices can be partitioned into a clique and an independent set. Most results in spectral graph theory do not address multigraph concerns. Exceptions are [2] and [4], but these papers present results involving a special class of underlying split graphs, threshold graphs, in which all pairs of nodes exhibit neighborhood nesting, and all multiple edges are confined to the clique. We present formulas for the eigenvalues of some infinite families of regular split multigraphs in which all multiple edges occur betweeen the clique nodes and cone nodes, multiplicity of multiple edges $\mu > 1$ fixed, and which have integer eigenvalues for the adjacency, Laplacian and signless Laplacian matrices.

Keywords: split graphs, regular graphs, integer eigenvalues, Adjacency matrix, multigraphs

1 Introduction and Preliminaries

Let $G = (V, E)$ denote a graph with a set on v vertices and e edges. For all graph terminology not described we refer to [6] and for all matrix theory definitions and operations we refer to [1]. A graph G is called a *split graph* if its vertices can be partitioned into a clique and an independent set [Figure 1]. A *clique* is a maximal connected complete subgraph of G and an *independent set* is a subset of V(G) such that no two vertices in the subset are adjacent. The vertices in the independent set are called *cones*. In this paper, we only consider split graphs whose cone vertices all have the same degree, or a *proper* split graphs. Further, a split graph is *ideal* if every node in the clique is adjacent to the same number of cones. We define an x-Ideal Proper Split Graph, $x - IPS(c; d; b)$, as split graphs satisfying the following conditions:

- n nodes in the graph
- c cones of underlying degree d
- x-grouping of cone nodes adjacent to the same d clique nodes
- b clique nodes not adjacent to any cones
- $\frac{cd}{x}$ number of clique nodes
- $\frac{cd}{x} > x$
- $d = \frac{c}{x}$

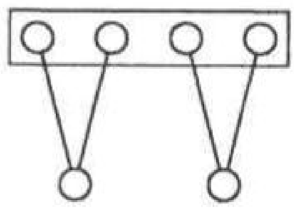

Figure 1: Ideal Proper Split Graph $1-IPS(2, 2, 0)$. Note that all the nodes in the box are adjacent to each other.

Formulas for Laplacian eigenvalues for infinite families of these graphs appear in [?], and when when $(x + d + \dfrac{cd}{x})^2 - 4\dfrac{cd^2}{x}$ is an odd perfect square, these are Laplacian integral, the first known Laplacian Integral non-threshold split graphs.

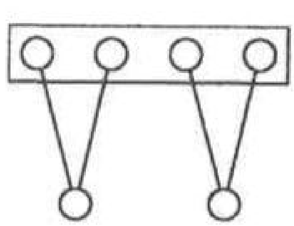

Figure 2: Ideal Proper Split Graph $1-IPS(2, 2, 0)$. Note that all the nodes in the box are adjacent to each other.

In this paper, all edges within the cliques are single. Multiply edges only appear between clique nodes and cone nodes. In order to generate a regular split-multi graph, one must ensure that each node is of the same degree. To do this, we define $\mu \in \mathbb{Z}^+$ to be the multiplicity of each cone edge, where $\mu = \frac{cd-x}{dx-x^2}$. Then our degree of regularity is $d(\mu)$.
Figure 1 can be modified to become a split multigraph by including μ.

$$\mu = \frac{2(2)-1}{2(1)-1^2} = 3$$

Figure 3 shows this graph when $\mu = 3$:

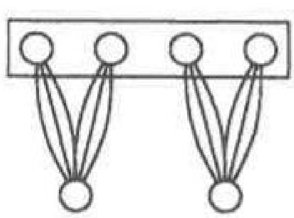

Figure 3: Ideal Proper Split Multigraph 1- IPSM(2,2(3))

Once the graph is created it can be classified using the general form:

$$x - IPSM(c, d(\mu), 0)$$

Investigation of the eigenvalues of IPSM has resulted in the classification of those with integer eigenvalues into five families. In the first three families, the cone nodes are adjacent to all clique nodes, meaning that the underlying graph is a special type of threshold graph called *complete split*.

- The first family is of underlying threshold graphs of the form $x - IPSM(x, (x+1)(x), 0)$.

  Example $4 - IPSM(4, 5(4), 0)$:

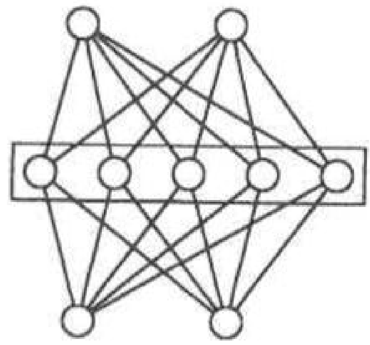

Figure 4: Ideal Proper Split Multigraph 4- IPSM(4,5(4))–each edge shown is of multiplicity 4

- The second family is of underlying threshold graphs of the form $x - IPSM(x, (2x-1)(2), 0)$.

  Example $4 - IPSM(4, 7(2), 0)$:

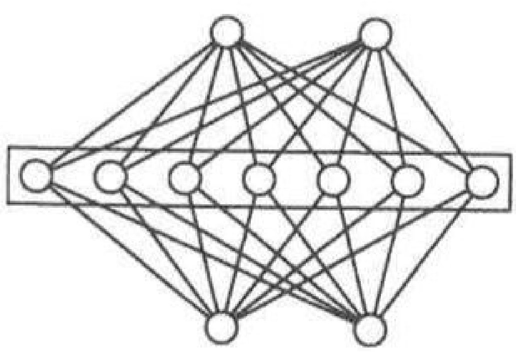

Figure 5: Ideal Proper Split Multigraph 4- IPSM(4,7(2))–each edge shown is of multiplicity 2

- The third family is of underlying threshold graphs of the form $x - IPSM(x, (x+2)((x+1)/2), 0)$.

  Example $5 - IPSM(5, 7(3), 0)$:

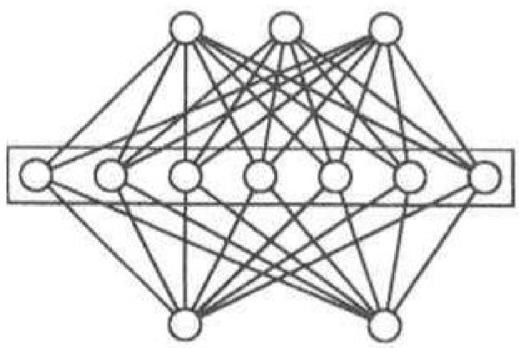

Figure 6: Ideal Proper Split Multigraph 5- IPSM(5 ,7(3))–each edge shown is of multiplicity 3

There are two other integral regular split multigraphs that do not fit into either of the three preceding families:

- The single member of this family is not underlying threshold. It is of the form, $2 - IPSM(3, 6(2), 0)$ and eigenvalues $12, -4^{(3)}, 3, 3, -1^{(6)}$.
- The single member of this family is not underlying threshold either. It is of the form $2 - IPSM(4, 5(3), 0)$ and eigenvalues $15, -10, 9, -6, -1^{(8)}, 0, 0$.

Remark 1: There is one graph that is a member of both family 1 and family 2. This graph is $2 - IPSM(2, 3(2), 0)$ which has eigenvalues 6, -4, -1, -1, 0.

Remark 2: The discovery and generalization of families 1 and 2 allows for the creation of infinitely many underlying threshold graphs with integer eigenvalues.

We will prove in the sequel that families 1,2 and three have adjacency eigenvalues $(0)^{(x-1)}$, $(-1)^{(d-1)}$, $[(d(\mu)]^{(1)}$, $[-x(\mu)]^{(1)}$.

The following is an important theorem:

Theorem 1.1 *For any r-regular multigraph, if the Adjacency matrix,* $\mathbf{A}$ *has integer eigenvalues, then we can say the graph is triply integral. This means its Adjacency, Laplacian, and Signless Laplacian Matrices all have integer eigenvalues. The eigenvalues of* $\mathbf{D} \pm \mathbf{A}$ *are:*

$$r \pm r, r \pm \lambda_2, ..., r \pm \lambda_n$$

where $\mathbf{D}$ *is the degree matrix and* $\mathbf{A}$ *is the adjacency matrix of* $\mathbf{M}$.

- For Family 1 we have Eigenvalues:

  $r = x^2 + x$

  Adjacency: $(0)^{(x-1)}, (-1)^{(x)}, (x^2+x)^{(1)}, (-x^2)^{(1)}$

  Laplacian: $(x^2+x)^{(x-2)}, (x^2+x+1)^{(x)}, (0)^{(2)}, (2x^2+x)^{(1)}$

  Signless Laplacian: $(x^2+x)^{(x-2)}, (x^2+x-1)^{(x)}, (2x^2+2x)^{(2)}, (x)^{(1)}$

- For Family 2 we have Eigenvalues:

  $r = 4x - 2$

  Adjacency: $(0)^{(x-1)}, (-1)^{(2x-2)}, (4x-2)^{(1)}, (-2x)^{(1)}$

  Laplacian: $(4x-2)^{(x-2)}, (4x-1)^{(2x-2)}, (0)^{(2)}, (6x-2)^{(1)}$

  Signless Laplacian: $(4x-2)^{(x-2)}, (4x-3)^{(2x-2)}, (8x-4)^{(2)}, (2x-2)^{(1)}$

- For Family 3 we have Eigenvalues:

  $r = \frac{(x+1)(x+2)}{2}$

  Adjacency: $(0)^{(x-1)}, (-1)^{(x+1)}, (\frac{(x+1)(x+2)}{2})^{(1)}, (\frac{-x(x+1)}{2})^{(1)}$

  Laplacian: $(0)^{(2)}, (\frac{(x+1)(x+2)}{2})^{(x-2)}, (\frac{(x+1)(x+2)+2}{2})^{(x+1)}, [(x+1)^2]^{(1)}$

  Signless Laplacian: $[(x+1)(x+2)]^{(2)}, (\frac{(x+1)(x+2)}{2})^{(x-2)}, (\frac{x(x+3)}{2})^{(x+1)}, (x+1)^{(1)}$

2 Three Families of Integral $x - IPSM(c, d, 0)$ Multigraphs

We present the theorems relating the parameters for the regular split multigraphs in these families to their adjacency eigenvalues, and the proof of the third family; families one and two are proven simiarly.

Theorem 2.1 *The adjacency matrix for family 1 of integral regular split multigraphs, $x - IPSM(x, (x+1)(x), 0)$, has eigenvalues* $(0)^{(x-1)}, (-1)^{(x)}, (x^2+x)^{(1)}, (-x^2)^{(1)}$.

Theorem 2.2 *The adjacency matrix for family 2 of integral regular split multigraphs, $x - IPSM(x, (2x-1)(2), 0)$, has eigenvalues* $(0)^{(x-1)}, (-1)^{(2x-2)}, (4x-2)^{(1)}, (-2x)^{(1)}$.

In the third family of integral regular split multigraphs, the underlying simple graph is threshold, with all cones adjacent to all clique nodes. The underlying split graph is sometimes called complete split, and they are not always adjacency integral [5]. An example of this family is in Figure 5, $5 - IPSM(5, 7(3), 0)$.

Theorem 2.3 *The adjacency matrix for family 3 of integral regular split multigraphs, $x-IPSM(x,(x+2)(\frac{x+1}{2}),0)$, has eigenvalues*

$$(0)^{(x-1)},(-1)^{(x+1)},((x+2)(\frac{x+1}{2}))^{(1)},((-x)(\frac{x+1}{2}))^{(1)}.$$

Proof. We observe that, in this family of regular split multigraphs, $c = x, d = x+2, \mu = \frac{x+1}{2}$. Without loss of generality, we set up the adjacency matrix in a specific form. Rows 1 through c will be the c cone nodes, and the remaining $n-c$ rows will be the cones. The matrix will take the form:

$$\begin{bmatrix} M_1 & | & M_2 \\ - & & - \\ M_3 & | & M_4 \end{bmatrix}$$

where M_1 is a $x \times x$ submatrix consisting of all zeros, M_2 a $x \times (x+2)$ submatrix of all $\frac{x+1}{2}$, M_3 an $(x+2) \times x$ submatrix of all μ, and M_4 an $(x+2) \times (x+2)$ submatrix consistng of zeros on the main diagonal and ones off it. To compute the eigenvalues, we consider the determinant of the matrix $\mathbf{A} - \lambda\mathbf{I} =$

$$\left[\begin{array}{cccc|ccccc} -\lambda & 0 & \cdots & 0 & \frac{x+1}{2} & \cdots & \cdots & \cdots & \frac{x+1}{2} \\ 0 & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & -\lambda & \frac{x+1}{2} & \cdots & \cdots & \cdots & \frac{x+1}{2} \\ \hline \frac{x+1}{2} & \cdots & \cdots & \frac{x+1}{2} & \lambda & 1 & \cdots & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & 1 & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 1 \\ \frac{x+1}{2} & \cdots & \cdots & \frac{x+1}{2} & 1 & \cdots & \cdots & 1 & -\lambda \end{array}\right]$$

The proof proceeds by a series of three matrix row operations to create an upper triangular matrix. This matrix has the same form as the adjacency matrix, so we refer to the four quadrants. The first is to transform submatrix M_3 into one having all zeros by taking $\frac{\frac{x+1}{2}}{\lambda}$ times rows 1 through x and adding each to rows $x+1$ through $2x+2$, putting the respective results in rows $x+1$ through $2x+2$. The matrix takes on this form:

$$\left[\begin{array}{cccc|ccccc}
-\lambda & 0 & \cdots & 0 & \frac{x+1}{2} & \cdots & \cdots & \cdots & \frac{x+1}{2} \\
0 & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & -\lambda & \frac{x+1}{2} & \cdots & \cdots & \cdots & \frac{x+1}{2} \\
\hline
0 & \cdots & \cdots & 0 & A & B & \cdots & \cdots & B \\
\vdots & \vdots & \vdots & \vdots & B & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & B \\
0 & \cdots & \cdots & 0 & B & \cdots & \cdots & B & A
\end{array}\right]$$

where $A = \dfrac{x(\frac{x+1}{2})^2}{\lambda} - \lambda, B = \dfrac{x(\frac{x+1}{2})^2}{\lambda} + 1$. The next step is to mutiply the bottom row by -1 and add the result to each of the rows from row $x+1$ to row $2x+1$, putting the result in rows $x+1$ to $2x+1$. The result is

$$\left[\begin{array}{cccc|ccccc}
-\lambda & 0 & \cdots & 0 & \frac{x+1}{2} & \cdots & \cdots & \cdots & \frac{x+1}{2} \\
0 & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & -\lambda & \frac{x+1}{2} & \cdots & \cdots & \cdots & \frac{x+1}{2} \\
\hline
0 & \cdots & \cdots & 0 & A-B & 0 & \cdots & 0 & B-A \\
\vdots & \vdots & \vdots & \vdots & 0 & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & 0 & \vdots & \vdots & A-B & B-A \\
0 & \cdots & \cdots & 0 & B & \cdots & \cdots & B & A
\end{array}\right]$$

observing that $A - B = -\lambda - 1$, and $B - A = \lambda + 1$. The final row operation is to multiply rows $c+1$ through $2x+1$ by $\dfrac{B}{B-A}$, adding each result to the last row. This creates the upper triangular matrix

$$\left[\begin{array}{cccc|ccccc}
-\lambda & 0 & \cdots & 0 & \frac{x+1}{2} & \cdots & \cdots & \cdots & \frac{x+1}{2} \\
0 & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & -\lambda & \frac{x+1}{2} & \cdots & \cdots & \cdots & \frac{x+1}{2} \\
\hline
0 & \cdots & \cdots & 0 & A-B & 0 & \cdots & 0 & B-A \\
\vdots & \vdots & \vdots & \vdots & 0 & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & 0 & \vdots & \vdots & A-B & B-A \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 & A+(d-1)B
\end{array}\right]$$

The characteristic polynomial is the product of the diagonal entries, which is $(-\lambda)^x(\lambda+1)^{x+1}(-1)^{2x+1}(-\lambda-1+(x+2)(\frac{x(\frac{x+1}{2})^2}{\lambda}+1) = (-\lambda)^x(\lambda+1)^{x+1}(-1)^{2x+1}(\frac{-\lambda^2+(x+1)\lambda+(x)(x+2)(\frac{x+1}{2})^2}{\lambda} = (-\lambda)^{x-1}(\lambda+1)^{x+1}(-1)(-\lambda^2+(x+1)\lambda+(x)(x+2)(\frac{x+1}{2})^2)$. Applying the quadratic formula to $(-\lambda^2+(x+1)\lambda+(x)(x+2)(\frac{x+1}{2})^2)$ yields the result. ■

3 Future Work

Since there are two integral regular split multigraphs that do not fit into the three existing infinite families, finding if there are others is one direction for future study. Also, there exist some integral regular split multigraphs that have varied multiplicites on the cone edges. This type of Regular Split Multigraphs are neither ideal nor proper; instead, these graphs have more than one value for d and μ. The cones are incident on all the clique nodes but a different number of times. We denote these graphs: $x - RSM(c, d_1(\mu_1) + \cdots + d_i(\mu_i), b)$, where:

- x is the number of cone multiedges incident on each clique node
- d_i is the number of clique nodes each cone nodes μ_i times
- $\sum_{i=1}^{j} d_i(\mu_i)$ is the degree of regularity
- $n = \sum_{i=1}^{j} d_i + c$
- $(x-1)^2$ is the number of clique nodes

- b is the number of clique nodes not adjacent to any cones

Category:
$\mathbf{x - RSM(x-1, (x-1)(2), [(x-1)(x-2)](1))}$ where $x \geq 3$

Examples

$3 - RSM(2, 2(2) + 2(1), 0)$
Eigenvalues: 6, -3, -2, -1, -1, -1

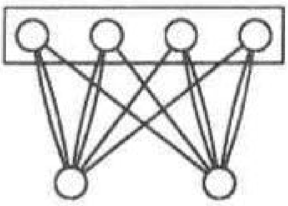

Figure 7: $3 - RSM(2, 2(2) + 2(1), 0)$.

$4 - RSM(3, 3(2) + 6(1), 0)$
Eigenvalues: 12, -4, $(\frac{-1\pm\sqrt{3}}{2})$, $(\frac{-1\pm\sqrt{3}}{2})$, $-1^{(6)}$
$5 - RSM(4, 4(2) + 12(1), 0)$
Eigenvalues: 20, -5, $(\frac{-1\pm\sqrt{17}}{2})^{(3)}$, $-1^{(12)}$.

These multigraphs need to be studied futher to determine their parameters, and under what conditions they are integral.

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